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Numerical study for Solving Bernoulli Differential Equations by using Runge-Kutta Method and "Newton's Interpolation and Aitken's Method"

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Abstract: *In Mathematics, one of important problem is solving differential equations by analytic methods and numerical methods. Most of researchers treated numerical methods to solve first order ordinary differential equations. These methods such as Taylor series method, Runge-Kutta method and Euler's method, etc. Faith Chelimo Kosgei studied this problem by combined the newton's interpolation and Lagrange method, Nasr Al Din IDE also studied this problem by Using Newton's Interpolation and Aitken's Method for Solving Riccati First Order Differential equation. This study will use Runge-Kutta method and Newton's interpolation and Aitken's method to solve Bernoulli Differential Equations, some examples treated to illustrate the efficiency of this method.*

Keywords: *Differential equation, Bernoulli Differential Equations, Analytic method, Numerical method, Runge-Kutta method, newton's interpolation method, Aitken Methods*

1. INTRODUCTION

Many problems can be formulated in Mathematics to form the ordinary differential equation, specially Bernoulli differential equations of first order, here we study and solve Bernoulli differential equations, numerical method is used to solve numerical problems using Runge Kutta Newton's Interpolation and Aitken's Methods. The differential equation problem [1-10], consists of at least one differential equation and at least one additional equation such that the system together have one and only one solution called the analytic or exact solution to distinguish it from the approximate numerical solutions that we shall consider In this paper of first order, Faith C. K [1] studied the problem of Riccati by using combination of newton's interpolation and Lagrange method, Nasr Al Din Ide [2] studied this problem also by using of Newton's Interpolation and Aitken's Method for Solving Riccati First Order Differential equations. In present study we will study Bernoulli Differential Equations by combined of Newton's interpolation and Aitken's method [4-10] and Runge-Kutta method. Finally we verified on a number of examples and numerical results obtained show the efficiency of the method given by present study in comparison with the exact solution. Let the Bernoulli differential equation which can be written in the following standard form:

$$y' + \mathrm{P(x)y} = \mathrm{Q(x)}y^{n} \tag{1}$$

where P and Q are functions of x, and n is a constant

n ≠ 1 (the equation is thus nonlinear).

Where y is a known function and the values in the initial conditions are also known numbers.

2. PRESENT AITKEN INTERPOLATION METHOD

2.1. Combined Newton's Interpolation and Lagrange Method [1, 2]

This study combine both Newton's interpolation method and Lagrange method. it used newton's interpolation method to find the second two terms then use the three values for y to form a quadratic equation using Lagrange interpolation method as follows;

2.1.1.Newton's interpolation method [1, 2, 9]

$$f_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+...+a_n(x-x_0)(x-x_1)..a_2(x-x_{n-1}) \quad (2)$$

Where

$$a_0=y_0\ ,\quad a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)}\ ,\quad a_2=\frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)} \quad (3)$$

Etc

2.1.2.Lagrang interpolation method [1, 8]

$$y_n=\frac{(x-x_1)-(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)-(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1+\frac{(x-x_0)-(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \quad (4)$$

3. DESCRIPTION OF THE METHOD

This method will combine both Newton's interpolation method and Lagrange method .it used newton's interpolation method to find the second two terms then use the three values for y to form a linear or quadratic equations using Lagrange interpolation method as follows;

$$f_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+...+a_n(x-x_0)(x-x_1)..a_2(x-x_{n-1}) \quad (5)$$

Where

$$a_0=y_0\ ,\quad a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)}\ ,\quad a_2=\frac{\frac{f(x_2)-f(x_1)}{(x_2-x_1)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)} \quad (6)$$

etc

$$y_1=a_0+a_1(x-x_0) \quad (7)$$

$$y_2=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1) \quad (8)$$

Forming quadratic interpolation of Lagrange, we have:

$$y_n=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}.y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}.y_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.y_2 \quad (9)$$

Note: we can use Newton's Forward Interpolation Formula instead of Newton's divided Interpolation method in (2.1).

3.1.Aitken interpolation method [3,8]

$$P_{o,k}(x)=\frac{1}{x_k-x_o}\begin{vmatrix} y_o & x_o-x \\ y_k & x_k-x \end{vmatrix} \quad (10)$$

$$P_{o,1,2}(x)=\frac{1}{x_2-x_1}\begin{vmatrix} P_{o,1}(x) & x_1-x \\ P_{o,2}(x) & x_2-x \end{vmatrix} \quad (11)$$

$$y_n=P_{o,1,2,...,n}(x)=\frac{1}{x_n-x_{n-1}}\begin{vmatrix} P_{o,1,...,(n-1)}(x) & x_{n-1}-x \\ P_{o,1,...,(n-2),n}(x) & x_n-x \end{vmatrix} \quad (12)$$

4. RUNGE-KUTTA METHOD [8]

For the equation $y'=f(x,y)$ and the initial condition $y(x_o)=y_o$

$$y(x+h)\sim y(x)+\frac{1}{6}(K_1+2K_2+2K_3+K_4) \quad (13)$$

$$K_1=h.f(x,y)$$

$$K_2=h.f\left(x+\frac{h}{2},y+\frac{K_1}{2}\right) \quad (14)$$

$$K_3=h.f\left(x+\frac{h}{2},y+\frac{K_2}{2}\right)$$

$$K_4=h.f(x+h,y+K_3)$$

5. EXAMPLES

In this section, we will check the effectiveness of the present technique (3). First numerical comparison for the following test examples taken in [3].

Example 1

Solve $y'=y+x.y^{\frac{1}{2}}$, *the exact solution of this problem is*

$y = (c.e^{\frac{x}{2}} - x - 2)^2$

For c=1, *the exact solution of this problem is* $y = (e^{\frac{x}{2}} - x - 2)^2$, *hence*, $y(0) = 1$

Now, by taking the step h=0.01

First by using Newton's interpolation, we have

$a_0 = 1 = y_0$

$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,1} = 0$

$y_1 = 1 + 0(0.01 - 0) = 1$

$a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} = \frac{[\frac{dy}{dx}]_{0.01,0.01} - [\frac{dy}{dx}]_{0,0}}{0.02 - 0} = 0.55$

$y_2 = 1 + 0(0.02 - 0) + 0.55(0.02 - 0)(0.02 - 0.01) = 1.000110000$

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = 1$

$P_{0,2}(x) = 0.0055x + 1$

$P_{0,1,2}(x) = 0.55x^2 - 0.0055x + 1$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, Table 1 gives the approximation solutions of Runge-Kutta method and Combined Newton's Interpolation and Aitken method with the exact solution of example 1 with the errors for :

x=0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1

Table1. *Solution of* $y' = y + x.y^{\frac{1}{2}}, \quad y(0) = 1$

x	Combined Newton's Interpolation and Aitken	Runge-Kutta Solution	exact Solution	Absolut error of Aitken and Exact Solutions	Absolut error of Runge-Kutta and Exact Solutions
0	1	1.010000000	1	0	0.010000000
0.01	1	1.020200499	1.009999833	0.009999833	0.010200666
0.02	1.000110000	1.030604514	1.019998665	0.019888665	0.010605849
0.03	1.000330000	1.041215115	1.029995492	0.029665492	0.011219623
0.04	1.000660000	1.052035426	1.039989307	0.039329307	0.012046119
0.05	1.001100000	1.063033143	1.049979102	0.048879102	0.013242123
0.06	1.001650000	1.074282095	1.059963867	0.058313867	0.012857083
0.07	1.002310000	1.085750449	1.069942587	0.067632587	0.015807862
0.08	1.003080000	1.097441548	1.089877829	0.076834247	0.007563719
0.09	1.003960000	1.109358794	1.090000000	0.086040000	0.019358794
0.1	1.004950000	1.121505643	1.100000000	0.095050000	0.021505643

Example 2

Solve $y' = 2xy + 2x^3.y^2$, *the exact solution of this problem is*

$y = 1/(c.e^{-x^2} + 1 - x^2)$

For c=0, *the exact solution of this problem is* $y = 1/(1 - x^2)$, *hence*, $y(0) = 1$

Now, by taking the step h=0.01

First by using Newton's interpolation, we have

$a_0 = 1 = y_0$

$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,1} = 0$

$y_1 = 1 + 0(0.01 - 0) = 1$

$$a_2 = \frac{\frac{f(x_2)-(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)} = 0.01000001$$

$y_2 = 1.000002$

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = 1$

$P_{0,2}(x) = 0.0001x + 1$

$P_{0,1,2}(x) = 0.01x^2 - 0.0001x + 1$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, Table 2 gives the approximation solution and the exact solution of example 1 with the error for :

x=0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1.

Table2. *Solution of* $y' = 2xy + 2x^3.y^2, \quad y(0) = 1$

x	Combined Newton's Interpolation and Aitken	Runge-Kutta Solution	exact Solution	Absolut error of Aitken and Exact Solutions	Absolut error of Runge-Kutta and Exact Solutions
0	1	1	1	0	0
0.01	1.000009000	1.000200023	1.000100010	0.000099110	0.000100013
0.02	1.000002000	1.000433443	1.000400610	0.000381600	0.000032833
0.03	1.000006000	1.001034244	1.000900811	0.000394160	0.000133433
0.04	1.000012000	1.001836355	1.001602564	0.001590564	0.000233791
0.05	1.002495000	1.002840706	1.002506266	0.002566760	0.000334440
0.06	1.000093000	1.004048471	1.003613007	0.003583007	0.000435464
0.07	1.000042000	1.005461083	1.004924128	0.004882128	0.004968670
0.08	1.000056000	1.007080229	1.006441224	0.006385224	0.000639005
0.09	1.000072000	1.008907866	1.008166146	0.008094146	0.00724606
0.1	1.000090000	1.010946221	1.010101010	0.010011010	0.000845211

Example 3

Solve $\quad y' = \mathrm{x}^3.\mathrm{y}^3 - xy,$ *the exact solution of this problem is*

$y = 1/(c.\mathrm{e}^{\mathrm{x}^2} + 1 + \mathrm{x}^2)$

For c=0, *the exact solution of this problem is* $y = 1/(1+\mathrm{x}^2)$, *hence*, $y(0) = 1$

Now, by taking the step h=0.01

First by using Newton's interpolation, we have

$a_0 = 1 = y_0$

$$a_1 = \frac{f(x_1)-f(x_0)}{(x_1-x_0)} = [\frac{dy}{dx}]_{0,1} = 0$$

$y_1 = 1 + 0(0.01-0) = 1$

$$a_2 = \frac{\frac{f(x_2)-(x_1)}{(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)} = -0.005$$

$y_2 = 0.999999$

Now, forming linear and quadratic using Aitken Method

$P_{0,1}(x) = 1$

$P_{0,2}(x) = -0.00005x + 1$

$P_{0,1,2}(x) = -0.005x^2 + 0.00005x + 1$

Hence, we can take the approximation solution of linear and quadratic using Aitken Method, if we take quadratic using Aitken Method, Table 3 gives the approximation solution and the exact solution of example 1 with the error for :

x=0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1.

Table3. *Solution of* $y' = x^3.y^3 - xy, \quad y(0) = 1$

x	Combined Newton's Interpolation and Aitken	Runge-Kutta Solution	exact Solution	Absolute error of Aitken and Exact Solutions	Absolute error of Runge-Kutta and Exact Solutions
0	1	1	1	0	0
0.01	1	0.9990001000	0.999900010	0.000099990	0.000899910
0.02	0.999999000	0.9970010997	0.999600160	0.000398840	0.002599060
0.03	0.999997000	0.9940046968	0.999100809	0.000896191	0.005096112
0.04	0.999994000	0.9990013484	0.998402556	0.001591444	0.000598792
0.05	0.999990000	0.9985030939	0.997506234	0.002483766	0.000996860
0.06	0.999985000	0.9979061424	0.996412914	0.003572086	0.001493228
0.07	0.999979000	0.9972110166	0.995123893	0.004855107	0.002087124
0.08	0.999972000	0.9641832507	0.993640700	0. 004855107	0.029457449
0.09	0.999964000	0.9955287605	0.991965083	0.007998917	0.003563678
0.1	0.999950500	0.9945430982	0.990099010	0.009851490	0.004444088

6. Conclusion

In this paper, we have been applied Runge-Kutta method and combined Newton's interpolation and Aitken method to solve nonlinear Bernoulli differential equation of first order, we find through some examples showing that that the method of Combined Newton's Interpolation and Aitken method is better than Runge-Kutta method compared to the exact solution.

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