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Topological Medial Semigroups

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Abstract: *In this paper, we prove some results for topological medial semigroups. Throghout, a semigroup will mean a topological semigroup, i.e., a Hausdorff space with a continuous associative multiplication. A medial semigroup is a semigroup satisfaying the medial low.*

Keywords: *Semigroup,medial, topological, homomorphism, quotient topology.*

1. Introduction

John B. and Pan, S.J. [2] proved some theorems for topological semigroups. Pettis, B.J.[5] proved some theorems for continuity and openness of homomorphism in topological groups. Paul S., Moster [6]have investigated the structure of topological semigroups. The purpose of this paper is to generalize some of their results to topological medial semigroups.

Let X and Y be topological space. Let $p:X\to Y$ be a surjective (onto) map. The map p is a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X. Let X and Y be topological space. Let $p:X\to Y$ be a surjective (onto) map. Set $C\subseteq X$ is saturated with respect to p if for all $y\in T$ such that $p^{-1}(\{y\})\cap C\neq\varnothing$ we have $p^{-1}(\{y\})\subseteq C$. If C saturated with respect to p, then for some $A\subseteq Y$ we have $p^{-1}(A)=C$. Let X and Y be topological space. Then $p:X\to Y$ is a quotient map if and only if p is continuoius and maps saturated open sets of X to open sets of Y. The map $f:X\to Y$ is an open map if for each open set $U\subseteq X$ the set $f(U)$ is open in Y. If $p:X\to Y$ is continuous and surjective and p is either open or closed map if for each closed set $A\subseteq X$ the set $f(A)$ is closed in Y. If X is a space, A is a set, and $p:X\to A$ is surjective (onto) map, then there exists exactly one topology T on A relative to which p is a quotient map . This topology is called the quotient topology induced by p.

2. Theorems for Topological Homomorphism

A semigroup S is medial if $xaby=xbay$ for all $x,a,b,y\in S$. Such a semigroup S satisfies $(xy)^n=x^ny^n$ and $(SxS)^n=S^nx^nS^n$ for all $x,y\in S$ and $n\in\mathbb{N}$.

A topological semigroup is a system consisting of a set S , an operation " · " and a topology T satisfaying the following conditions:

1) for any $x,y\in S'$, $xy\in S$;

2) for $x,y,z\in S$, $(xy)z=x(yz)$;

3) the operation "·" is continuous in the topology T.

A topological subsemigroup H of a semigroup S is a topological subspace of S and also a subsemigroup of S.

An equivalence relation R defined on a semigroup S is called homomorphic is for any $a,b,c,d\in S$, aRb and cRd imply $acRbd$.

Given an homomorphic equivalence relation R on S, we call the set of equivalence classes mod R the quotient set and we denote it by S/R .

The mapping from S into S/R defined by $n(x)=$ the class mod R to which x belongs is called the natural mapping from S into S/R.

The family U of all subsets U^* of S/R such that $n^{-1}(U^*)$ is open in S is a topology for S/R and is called quotient topology for S/R.

In general use the term homomorphism to mean continuous homomorphism, the terms mapping, function to mean continuous mapping, continuous functions .

Let S be a semigroup , R be a homomorphic equivalence relation on S and let S/R be the quotient set. We define an operation on S/R in the following manner. Suppose that A and B are two arbitrary elements in S/R, then $AB=C$ if for any $a\in A$ and $b\in B$ we have $ab\in C$. This operation is well-defined because R is a homomorphic equivalence relation. Also it is associative, because the semigroup S is associative.Therefore the quotient set S/R with the operation just defined is a semigroup. We call it the quotient semigroup.

We say a semigroup S statisfies the condition A if for every open set U of S, the subset $n^{-1}(n(U))$ is also open where n is the natural mapping from S onto S/R.

Theorem1: If the medial semigroup S statisfies the condition A, then the quotient set S/R is a topological medial semigroup with the quotient topology, and the natural mapping n from S into S/R is an open topological homomorphism.

Proof: We have shown that S/R is an abstract semigroup. Now we wish to show that the natural mapping n from S to S/R is an abstract homomorphism.

Let X and Y be two equivalence classes mod R and let $XY=Z$. Then by definition of the operation in S/R, for any $x\in X$ and $y\in Y$, $xy\in Z$. Since the natural mapping n assigns each element to the class it belongs, we have $n(X)=X$, $n(Y)=Y$, and $n(xy)=n(z)=Z$ .These equations together with the equation $XY=Z$ imply that $n(xy)=n(x)n(y)$.

This shows that the natural mapping n is an abstract homomorphism from S into S/R.

Now let U^* be an open set in S/R. By the definition of the quotient topology for S/R, $n^{-1}(U^*)$ is open. Hence n is continuous. Let U be an open set in S. Since S satisfies the condition A , $n^{-1}[n(U)]$ is open. Then by definition of the quotient topology, $n(U)$ is open.

Now we wish to show that the semigroup operation in S/R is continuous. Let A and B be two orbitrary elements in S/R such that $AB=C$. Suppose that W^* is an open neighborhood of C. Then $W=n^{-1}(W^*)$ is an open neighborhood of C, considered as a subset of S. Since the semigroup operation in S is continuous, for every $a\in A$ and every $b\in B$ such that $ab=c$, there is an open neighborhood U_a of a and an open neighborhood V_b of b such that $U_aV_b\subset W$ .Choose such a neighborhood V_b for every $b\in B$. Then $\underset{\substack{a\in A\\ b\in B}}{U} U_aV_b=\left[\underset{a\in A}{U}U_a\right]\left[\underset{b\in B}{U}V_b\right]\subset W$.

Now $\underset{a\in A}{U}U_a$ is an open neighborhood of A in S, and n is an open mapping. It follows that $n\left[\underset{a\in A}{U}U_a\right]$ is an open neighbordood of the element A in S/R. Similarly $n\left[\underset{b\in B}{U}V_b\right]$ is an open neighborhood of the element B in S/R. Since $\left[\underset{a\in A}{U}U_a\right]\left[\underset{b\in B}{U}V_b\right]\subset W$, we have

$$n\left[\underset{a\in A}{U}U_a\right]n\left[\underset{b\in B}{U}V_b\right]=n\left[\underset{a\in A}{U}U_a\underset{b\in B}{U}V_b\right]\subset n(W)=W^*$$

Hence we have found an open neighborhood $n\left[\underset{a\in A}{U} U_A\right]$ of A and an open neighborhood $n\left[\underset{b\in B}{U} V_b\right]$ of B such that $n\left[\underset{a\in A}{U} U_a\right] n\left[\underset{b\in B}{U} V_b\right] \subset W^*$. This shows that the semigroup operation in S/R is continuous.

Theorem2: If S and T are two medial semigroups and g is a homomorphism from S into T then g induces a homomorphic equivalence relation R_g on S.

Proof: We define a relation R_g on S in the following manner. Suppose that a and a^* are two element of S, then : $a = a^*$ if and only if $g(a) = g(a^*)$. Evidently, R_g is an equivalence relation.

We show that R_g is homomorphic, *i.e.* , if $a, a^*, b, b^* \in S$ such that $a = a^* \bmod R_g$ and $b = b^* \bmod R_g$, then $ab = a^*b^* \bmod R_g$. Now $a = a^* \bmod R_g$ implies $g(a) = g(a^*)$ and $b = b^* \bmod R_g$ implies $g(b) = g(b^*)$. These two equations imply that $g(a)g(b) = g(a^*)g(b^*)$.

Since g is a homomorphism we have $g(a)g(b) = g(ab)$ and $g(a^*)g(b^*) = g(a^*b^*)$.

Hence $g(ab) = g(a^*b^*)$.

Theorem3: Let S and T be two topological medial semigroups and let g be an open homomorphism from S onto T. Then

a) S/R_g is a topological medial semigroup with the quotient topology;

b) the natural mapping n from S onto S/R_g is an open homomorphism;

c) the mapping h from S/R_g onto T defined by $h(A) = g(a)$ for any $a \in A$ as a subset of S and $A \in S/R_g$ is a topological isomorphism.

Proof: By theorem2, g includes a homomorphic equivalence relation R_g on S. Let S/R_g be the quotient set. Then S/R_g is a medial semigroup. Let n be the natural mapping from S onto S/R_g. We show that the medial semigroup S satisfies the condition A.

Let U be an open subset in S. Since g is an open map, $g(U)$ is open in T. Also g is continuous. Hence the subset $g^{-1}[g(U)]$ is open in S. But $g^{-1}[g(U)] = \{x \in s / g(x) = g(y)$ for some $y \in U\}$ and $n^{-1}[n(U)] = \{x \in S / g(x) = g(y)$ for some $y \in U\}$ thence $n^{-1}[n(U)] = g^{-1}[g(U)]$ and $n^{-1}[n(U)]$ is open. This shows that S satisfies the condition A. Since S satisfies the condition A, the parts a) and b) follow from theorem1. Before proving part c), we wish to show that the mapping h defined in the theorem is well-defined.

Let A be any element of S/R_g and let a^* and a^{**} be any two elements of A as a subset of S. Then $a^* = a^{**} \bmod R_g$. This implies $g(a^*) = g(a^{**})$. Hence $h(A) = g(a^*) = g(a^{**})$.

This shows that h is well-defined. Also h is a one to one mapping.

For each $A \in S/R_g$ there corresponds a unique value $h(A) = g(a)$ in T as shown above.

Now since g is a mapping from S onto T, for each $t \in T$ there is an element $a \in S$ such that $t = g(a)$, by definition of Rg, $a = b \bmod R_g$ if and only if $g(a) = g(b)$.

It follows that for each $g(a) = t$, there is one and only one equivalence class $A \bmod R_g$ such that $h(A) = g(a) = t$. Hence h is a one to one mapping.We further show that h is an algebraic homomorphism. Let A and B be any two elements in S/R_g. Then

$h(AB) = g(ab) = g(a)g(b) = h(A)h(B)$, where a and b are orbitrary elements of A and B respectively. This shows that h is an algebraic homomorphism.

We show also that h is continuous. Let A be an element in S/R_g such that $h(A) = t$ and let W be an open neighborhood of t.

Since $h(A) = g(a)$ for every $a \in A$,and since g is continuous, for every $a \in A$, there is an open neighborhood U_a of a such that $g(U_a) \subset W$.

Choose such an open neighborhood U_a for every $a \in A$. Then $\bigcup_{a \in A}(U_a)$ is a neighborhood of A in S and $n\left[\bigcup_{a \in A}(U_a)\right]$ is an open neighborhood of the element A in S/R_g. But

$$g\left[\bigcup_{a \in A}(U_a)\right] = h\left\{n\left[\bigcup_{a \in A}(U_a)\right]\right\} \subset W.$$

So for any neighborhood w of $h(A)$, we have found a neighborhood $n\left[\bigcup_{a \in A}(U_a)\right]$ of A such that $h\left\{n\left[\bigcup_{a \in A}(U_a)\right]\right\} \subset W$. This shows that h is continuous.

Finally we show that h is open. Let U^* be an open subset of S/R_g. Since the natural mapping n from S onto S/R_g is continuous, $n^{-1}(U^*)$ is an open subset in S into T. So $g\left[n^{-1}(U^*)\right]$ is open on T.

But $g\left[n^{-1}(U^*)\right] = h\left\{n\left[n^{-1}(U^*)\right]\right\} = h(U^*)$. Hence $h(U^*)$ is open in T. This shows that h is an open mapping. This completes the proof.

If the medial semigroup S satisfies the condition A, then the quotient set S/R is a topological medial semigroup with the quotient topology, and the natural mapping n from S onto S/R is an open topological homomorphism.

Conversely, if g is an open homomorphism from S into a medial semigroup T, then T is topologically isomorphic to the quotient semigroup S/R_g, where R_g is homomorphic equivalence relation defined by aR_gb if and only if $g(a) = g(b), a, b \in S$.

3. Fundamental Theorem of Homomorphism of the Topological Medial Semigroups

Theorem4: Let S and T be two topological medial semigroups both satisfying the condition A.

Let g be an open homomorphism from S onto T and let R^* be a homomorphic equivalence relation defined on T. Then there is a homomorphic equivalence relation R on S and there is a mapping h from S/R onto T/R^* which is a topological isomorphism.

Proof: Since R^* is a homomorphic equivalence relation on T, by theorem1, T/R^* is a topological medial semigroup and the natural mapping n from T onto T/R^* is an open topological homomorphism. Since the mapping g from T onto T/R^* is an open topological homomorphism.

Since the mapping g from S onto T is also a homomorphism, it follows that the product mapping ng from S onto T/R^* is also a homomorphism. We show that ng is open.

Let U be an open set in S. Since g is open $g(U)$ is open in T. Also, n is an open map;

So $ng(U)$ is open in T/R^*. This shows that ng is an open topological homomorphism.

Now S and T/R^* are two topological medial semigroups. S satisfies the condition A, and ng is an open topological homomorphism from S onto T/R^*. Hence, by theorem2 ng induces a homomorphic equivalence relation R_{ng} and T/R^*. Denote R_{ng} by R. Then we have $S/R \cong T/R^*$.

We call this isomorphism h.

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