



## A Weakly Majorization Inequality for Singular Values

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**Abstract:** For the study of inequalities for singular values, it has been a research hotspot for a long time. In this paper, on the basis of  $2 \times 2$  block matrix, we will study the positive semidefinite matrix  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ , and get a weakly majorization inequality for singular values about positive semidefinite block matrix.

**Keywords:** Inequality for Singular Values Positive Semidefinite Matrix Majorization

### 1. INTRODUCTION

In the field of matrix analysis, the techniques of block matrices play an important role in the derivation of matrix inequalities. In recent years, the study of  $2 \times 2$  block matrices, especially on the study of singular value inequalities about block matrices, has attracted extensive attention from scholars, and some very good results have also been obtained. For example, X. Zhan in reference [1] proved that  $s_j(A - B) \leq s_j(A \oplus B)$ ,  $j = 1, 2, \dots, n$ , for positive semidefinite matrices  $A, B \in M_n(C)$ . In reference [2], Bhatia and Kittaneh say that  $2s_j(AB^*) \leq s_j(A^*A + B^*B)$ ,  $j = 1, 2, \dots, n$ , for any  $A, B \in M_n(C)$ .

And recently, Audeh and Kittaneh [3] proved that if  $A, B, C \in M_n(C)$  and satisfy that

$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , then  $s_j(B) \leq s_j(A \oplus C)$ ,  $j = 1, 2, \dots, n$ . In this paper, our purpose is to study some

properties of  $2 \times 2$  block matrix which mentioned above. Thus, we will get a new inequality for singular values.

### 2. PRELIMINARIES

The matrices considered in this paper are complex matrices, and we use  $M_{m,n}(C)$  to denote the space of all  $m \times n$  complex matrices, when  $m = n$ , we use  $M_n(C)$  to represent  $M_{m,n}(C)$ . For the  $n \times n$  positive semi definite matrix  $A$ , we denote their eigen values and singular values respectively to be  $\lambda_i(A)$ ,  $i = 1, 2, \dots, n$  and  $s_j(A)$ ,  $j = 1, 2, \dots, n$ , and arrange them in decreasing order by  $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$  and  $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$  respectively. For the matrices  $A, B$ , we use the symbol  $A \geq B$  to represent that  $A - B$  is positive semi definite. If  $A \geq 0$  then we write  $\lambda_i(A) = s_i(A)$ .

Let the two real vectors  $x = (x_1, x_2, \dots, x_n)^T$ ,  $y = (y_1, y_2, \dots, y_n)^T$  with their components in decreasing

order as  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$  and  $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$ . If  $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, k = 1, 2, \dots, n$ , then we say that  $x$  is weakly majorized by  $y$ , and write  $x \prec_w y$ .

Finally, the symbol  $A^* = (A^{-})^T$  denotes the conjugate transpose of matrix  $A$ ,  $|A|$  denotes the matrix be operated as  $(A^*A)^{\frac{1}{2}}$ .

**3. MAIN RESULTS**

It is obviously that if  $A, B, C \in M_n(C)$  and satisfy  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , then  $A$  and  $C$  are positive semi

definite matrices. We hope that when  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , the inequality  $s_j(B) \leq s_j(A - C)$  holds. But in fact,

this proposition is not necessarily true, let's look at the following example:

Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ , and we can verify that  $s_2(B) > s_2(A - C)$ .

In spite of that, we can still get a slightly weakened conclusion. For proving convenience, we'll give the following lemmas first:

**Lemma 1.1**(Mirsky) If  $A, B \in M_n(C)$ , then

$$\|diag(s(A) - s(B))\| \leq \|A - B\|,$$

and it holds for any unitary invariant norm.

**Lemma 1.2** If  $A, B \in M_n(C)$ , then  $s(A) - s(B) \prec_w s(A - B)$ .

**Proof** We notice that lemma 1.1 can be equivalently rewritten as

$$|s(A) - s(B)| \prec_w s(A - B),$$

at the same time, the following obvious facts hold true

$$s(A) - s(B) \prec_w |s(A) - s(B)|,$$

so the lemma can be easily to get.

**Lemma 1.3** If  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , then there exists a contractive matrix  $W$  such that  $B = A^{\frac{1}{2}}WC^{\frac{1}{2}}$ .

**Lemma 1.4** If  $A, B, C \in M_n(C)$ , then we have

$$s(ABC) \prec_w s(A) \circ s(B) \circ s(C).$$

**Theorem 1** Let  $A, B, C \in M_n(C)$  and satisfy that  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , if

$$\lambda_i(A) \geq \frac{3 + \sqrt{5}}{2} \lambda_i(C),$$

then we have  $s(B) \prec_w s(A - C)$ .

**Proof** From lemma 1.2 we can know that  $s(A) - s(C) \prec_w s(A - C)$ .

Due to  $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ , from lemma 1.3 we can know that there exists a contractive matrix  $W$  such that

$B = A^{\frac{1}{2}}WC^{\frac{1}{2}}$ . And then, by using lemma 1.4, we conclude that

$$s(B) = s(A^{\frac{1}{2}}WC^{\frac{1}{2}}) \prec_w s(A^{\frac{1}{2}}) \circ s(W) \circ s(C^{\frac{1}{2}}) \prec_w s(A^{\frac{1}{2}}) \circ s(C^{\frac{1}{2}}).$$

At the same time, we notice that  $A, C$  are positive semi definite matrices, so

$$s(A) = \lambda(A), s(C) = \lambda(C).$$

When  $\lambda_i(A) \geq \frac{3+\sqrt{5}}{2} \lambda_i(C)$ , we can get that  $\lambda_i(A) - \lambda_i(C) \geq \sqrt{\lambda_i(A)\lambda_i(C)}$ ,

so we can know that  $s(A^{\frac{1}{2}}) \circ s(C^{\frac{1}{2}}) \prec_w s(A) - s(C)$ ,

and then  $s(B) \prec_w s(A^{\frac{1}{2}}) \circ s(C^{\frac{1}{2}}) \prec_w s(A) - s(C) \prec_w s(A - C)$ .

So, the theorem is proved

#### 4. REMARK

In Theorem 1, if we choose  $A = C, B \neq 0$ , and we can see that the theorem is not true. This shows that it

is necessary to add the condition  $\lambda_i(A) \geq \frac{3+\sqrt{5}}{2} \lambda_i(C)$ .

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