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On the Equality of Rank of a Fifth-Idempotent Matrix

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Abstract: *The equality of rank a fifth-idempotent matrix is established by means of elementary transformation and properties of idempotent matrix.*

Keywords: *fifth-idempotent matrix, rank, equality.*

1. Introduction

Proposition 1 $A^5 = A \Leftrightarrow rank(A) + rank(E - A^4) = n$

Proof: Since the elementary transformation of a matrix does not change the rank of the matrix, the following equality can be obtained.

$$rank\begin{bmatrix} A & \\ & E-A^4 \end{bmatrix} = rank\begin{bmatrix} A & \\ A^4 & E-A^4 \end{bmatrix} = rank\begin{bmatrix} A & A \\ A^4 & E \end{bmatrix} = rank\begin{bmatrix} A-A^5 & 0 \\ A^4 & E \end{bmatrix} = rank\begin{bmatrix} A-A^5 & 0 \\ 0 & E \end{bmatrix}$$

Therefore $A^5 = A \Leftrightarrow rank(A) + rank(E - A^4) = n$

Proposition 2 $A^5 = A \Rightarrow rank(A^a) + rank(E - A^4)^b = n$, $\forall a, b \in N^+$

Proof: On the one hand, by $A^5 = A$, we have $A(E - A^4) = 0$, So for every positive integer, we have $A^a(E - A^4) = 0$.With the help of the property of matrix multiplication operation, we can get $rank(A^a) + rank(E - A^4) \le n$.

On the other hand, The minimum polynomial of matrix A obtained from $A^5 = A$ is the factor of polynomial $\lambda^5 - \lambda$. Therefore, the minimum polynomial of **A** has no multiple roots, so A can be diagonalized.

For every positive integer a, b, there exists an invertible matrix P such that the following equation holds.

$$P[A^a + (E - A^3)^b]P^{-1} = PA^aP^{-1} + P(E - A^3)^b P^{-1} = (PAP^{-1})^a + [E - (PAP^{-1})^3]^b$$

It is not hard to get $rank[A^a + (E - A^4)^b] = n$.

Hence it follows that $n = rank[A^a + (E - A^4)^b] \le rank(A^a) + rank(E - A^4)^b \le n$

Therefore $rank(A^a) + rank(E - A^4)^b = n$

Proposition 3 $A^5 = A \Rightarrow rank(A) + rank(E - A^4 + A^3) = n + rank(A^4)$

Proof The following equation can be obtained from elementary transformation.

$$rank\begin{bmatrix} A & \\ & E - A^4 + A^3 \end{bmatrix} = rank\begin{bmatrix} A & \\ A & E - A^4 + A^3 \end{bmatrix} = rank\begin{bmatrix} A & A^4 - A^3 \\ A & E \end{bmatrix}$$
$$= rank\begin{bmatrix} A - A^5 + A^4 & A^4 - A^3 \\ 0 & E \end{bmatrix} = rank\begin{bmatrix} A - A^5 + A^4 & 0 \\ 0 & E \end{bmatrix}$$

Substituting $A^5 = A$, $rank\begin{bmatrix} A - A^5 + A^4 & 0 \\ 0 & E \end{bmatrix} = rank\begin{bmatrix} A^4 & 0 \\ 0 & E \end{bmatrix}$,

Therefore $rank\begin{bmatrix} A & \\ & E - A^4 + A^3 \end{bmatrix} = rank\begin{bmatrix} A^4 & 0 \\ 0 & E \end{bmatrix}$.

That is to say $rank(A) + rank(E - A^4 + A^3) = n + rank(A^4)$

Conversely, it does not necessarily hold true. For example, $A = (1 + \sqrt{3})E$

$rank(A) + rank(E - A^4 + A^3) = 2n = n + rank(A^4)$, but obviousiy there are $A^5 \ne A$.

Proposition 4 $A^5 = A \Rightarrow rank(E - A^4 + A^3) = rank(A^4) + rank(E - A^4)$

Since $A^5 = A$, from proposition 1 and 2, $rank(A) + rank(E - A^4) = n$, $rank(A) + rank(E - A^4 + A^3) = n + rank(A^4)$.

So we can get $rank(E - A^4 + A^3) = rank(A^4) + rank(E - A^4)$.

But the same, based on $rank(E - A^4 + A^3) = rank(A^4) + rank(E - A^4)$, we can not get $A^5 = A$.

For example,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A^3 = A^4 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rank(E - A^4 + A^3) = 3 = rank(A^4) + rank(E - A^4)$$

But $A^5 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ne \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A$

According to the definition of the fifth-idempotent matrix and its operation, the following properties of the fifth-idempotent matrix can be given.

Proposition 4

(1)If the fifth-idempotent matrices A, B are commutative, then AB is also a fifth-idempotent matrix.

(2) If A is a fifth-idempotent matrix, then A^4 is an idempotent matrix.

(3) If A is a fifth-idempotent matrix, $E - A^4$ is an idempotent matrix.

(4) If A is a fifth-idempotent matrix, then for any positive integer, there are

$$A^n = \begin{cases} A, 4 \mid n-1 \\ A^2, 4 \mid n-2 \\ A^3, 4 \mid n-3 \\ A, 4 \mid n \end{cases}$$

Proposition 5 If A, B are all fifth-idempotent matrices, the following equality is satisfied

(1) $rank(A^4 + B^4) = rank\begin{bmatrix} A^4 & B^4 \\ B^4 & 0 \end{bmatrix} - rankB^4 = rank\begin{bmatrix} B^4 & A^4 \\ A^4 & 0 \end{bmatrix} - rankA^4$

(2) $rank(A^4 + B^4) = rank[A^4 - A^4B^4, B^4] = rank[B^4 - B^4A^4, A^4]$

(3) $rank(A^4 + B^4) = rank(A^4 - A^4B^4 - B^4A^4 + B^4A^4B^4) + rankB^4$

(4) $rank(A^4 + B^4) = rank(A^4 - A^4B^4 - B^4A^4 + A^4B^4A^4) + rankA^4$

(5) $rank(A^4 + B^4) = rank\begin{bmatrix} A^4 & B^4 & 0 \\ B^4 & 0 & A^4 \end{bmatrix} = rank\begin{bmatrix} A^4 & B^4 \end{bmatrix}$

(6)If a_1, a_2 are two non-zero real numbers and $a_1 + a_2 \neq 0$, then

$rank(a_1A^4 + a_2B^4) = rank(A^4 + B^4)$.

Theorem 1 $A \in P^{n\times n}$, $f(x), g(x) \in P[x]$ is a polynomial with any number greater than 1. Let

$d(x) = (f(x), g(x))$, $m(x) = [f(x), g(x)]$, then

$rankf(A) + rankg(A) = rankd(A) + rankm(A)$.

Corollary $A \in P^{n\times n}$, $f(x) \in P[x]$ is a polynomial with any number greater than 1. Let

$d(x) = (f(x), x - x^5)$ and $m(x) = [f(x), x - x^5]$, then

$rankf(A) + rank(A - A^5) = rankd(A) + rankm(A)$.

With the help of corollary we can get if ***A*** is a fifth-idempotent matrix, then $rankf(A) = rankd(A) + rankm(A)$.

This corollary shows that there are also many rank eigenvalues of ffifth-idempotent matrices.

Theorem 2

$A \in P^{n\times n}$, $t \geq 1 \in N^{+}$, $rank(A) + rank(A^{t} - A^{t+4}) = rank(A^{t}) + rank(A - A^{5})$.

Proof: When t = 1, the equation clearly holds.

Let $t > 1$, $f(x) = x^{t}$, $g(x) = x - x^{5}$, By simple calculation we get $(f(x), g(x)) = x$ $[f(x), g(x)] = x^{t} - x^{t+4}$. By the above we can get the following equation.

$$rank(A) + rank(A^{t} - A^{t+4}) = rank(A^{t}) + rank(A - A^{5})$$

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