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On the Equality of Rank of a Fourth-Idempotent Matrix

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Abstract: *The equality of rank a fourth-idempotent matrix is established by means of elementary transformation and properties of idempotent matrix.*

Keywords: *fourth-idempotent matrix, rank, equality.*

1. Introduction

Proposition 1 $A^4 = A \Leftrightarrow rank(A) + rank(E - A^3) = n$

Proof: Since the elementary transformation of a matrix does not change the rank of the matrix, the following equality can be obtained.

$$rank\begin{bmatrix} A & \\ & E-A^3 \end{bmatrix} = rank\begin{bmatrix} A & \\ A^3 & E-A^3 \end{bmatrix} = rank\begin{bmatrix} A & A \\ A^3 & E \end{bmatrix} = rank\begin{bmatrix} A-A^4 & 0 \\ A^3 & E \end{bmatrix} = rank\begin{bmatrix} A-A^4 & 0 \\ 0 & E \end{bmatrix}$$

Therefore $A^4 = A \Leftrightarrow rank(A) + rank(E - A^3) = n$

Proposition 2 $A^4 = A \Rightarrow rank(A^a) + rank(E - A^3)^b = n$

Proof: On the one hand, by $A^4 = A$, we have $A(E - A^3) = 0$, So for every positive integer, we have $A^a(E - A^3) = 0$.With the help of the property of matrix multiplication operation, we can get $rank(A^a) + rank(E - A^3) \le n$.

On the other hand, The minimum polynomial of matrix A obtained from $A^4 = A$ is the factor of polynomial $\lambda^4 - \lambda$. Therefore, the minimum polynomial of A has no multiple roots, so A can be diagonalized.

For every positive integer a, b, there exists an invertible matrix P such that the following equation holds. $P[A^a + (E - A^3)^b]P^{-1} = PA^aP^{-1} + P(E - A^3)^b P^{-1} = (PAP^{-1})^a + [E - (PAP^{-1})^3]^b$

It is not hard to get $rank[A^a + (E - A^3)^b] = n$. Hence it follows that

$$n = rank[A^a + (E - A^3)^b] \le rank(A^a) + rank(E - A^3)^b \le n$$

Therefore $rank(A^a) + rank(E - A^3)^b = n$

Conversely, it does not necessarily hold true. Here's an example. $A = \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$, when $a = 4, b = 4$,

$rank(A^a) + rank(E - A^3)^b = 3$, but $A^4 \ne A$

Proposition 3 $A^4 = A \Rightarrow rank(A) + rank(E - A^3 + A^2) = n + rank(A^3)$

Proof The following equation can be obtained from elementary transformation.

$$rank\begin{bmatrix} A & \\ & E-A^3+A^2 \end{bmatrix} = rank\begin{bmatrix} A & \\ A & E-A^3+A^2 \end{bmatrix} = rank\begin{bmatrix} A & A^3-A^2 \\ A & E \end{bmatrix}$$

$$= rank\begin{bmatrix} A-A^4+A^3 & A^3-A^2 \\ 0 & E \end{bmatrix} = rank\begin{bmatrix} A-A^4+A^3 & 0 \\ 0 & E \end{bmatrix},$$

Substituting $A^4 = A$, $rank\begin{bmatrix} A-A^4+A^3 & 0 \\ 0 & E \end{bmatrix} = rank\begin{bmatrix} A^3 & 0 \\ 0 & E \end{bmatrix}$, Therefore

$rank\begin{bmatrix} A & \\ & E-A^3+A^2 \end{bmatrix} = rank\begin{bmatrix} A^3 & 0 \\ 0 & E \end{bmatrix}$. That is to say

$$rank(A)+rank(E-A^3+A^2) = n+rank(A^3)$$

Conversely, it does not necessarily hold true. For example, $A = 3E$,

$rank(A)+rank(E-A^3+A^2) = 2n = n+rank(A^3)$, but $A^4 = 81E \neq 3E = A$.

Proposition 4 $A^4 = A \Rightarrow rank(E-A^3+A^2) = rank(A^3)+rank(E-A^3)$

$A^4 = A$, $rank(A)+rank(E-A^3+A^2) = n+rank(A^3)$, $rank(A)+rank(E-A^3) = n$,

So we can get $rank(E-A^3+A^2) = rank(A^3)+rank(E-A^3)$.

But the same, based on $rank(E-A^3+A^2) = rank(A^3)+rank(E-A^3)$, we can not get $A^4 = A$. For example, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = A^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$rank(E-A^3+A^2) = 3 = rank(A^3)+rank(E-A^3)$, but $A^4 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A$

According to the definition of the fourth-idempotent matrix and its operation, the following properties of the fourth-idempotent matrix can be given.

Proposition 4

(1)If the fourth-idempotent matrices A, B are commutative, then AB is also a fourth-idempotent matrix.

(2) If A is a fourth-idempotent matrix, then A^3 is an idempotent matrix.

(3) If A is a fourth-idempotent matrix, $E-A^3$ is an idempotent matrix.

(4) If A is a fourth-idempotent matrix, then for any positive integer, there are $A^n = \begin{cases} A, 3|n-1 \\ A^2, 3|n-2 \\ A^3, 3|n \end{cases}$

Proposition 5 If A, B are all fourth-idempotent matrices, the following equality is satisfied

1) $rank(A^3+B^3) = rank\begin{bmatrix} A^3 & B^3 \\ B^3 & 0 \end{bmatrix} - rankB^3 = rank\begin{bmatrix} B^3 & A^3 \\ A^3 & 0 \end{bmatrix} - rankA^3$

(2) $rank(A^3+B^3) = rank[A^3-A^3B^3, B^3] = rank[B^3-B^3A^3, A^3]$

(3) $rank(A^3+B^3) = rank(A^3-A^3B^3-B^3A^3+B^3A^3B^3)+rankB^3$

(4) $rank(A^3+B^3) = rank(A^3-A^3B^3-B^3A^3+A^3B^3A^3)+rankA^3$

(5) $rank(A^3+B^3)=rank\begin{bmatrix} A^3 & B^3 & 0 \\ B^3 & 0 & A^3 \end{bmatrix}=rank\begin{bmatrix} A^3 & B^3 \end{bmatrix}$

(6)If a_1, a_2 are two non-zero real numbers and $a_1+a_2\neq 0$, then $rank(a_1A^3+a_2B^3)=rank(A^3+B^3)$.

Theorem 1 $A\in P^{n\times n}$, $f(x)\in P[x]$ is a polynomial with any number greater than 1. Let $d(x)=(f(x),x-x^4)$ and $m(x)=[f(x),x-x^4]$, then

$rankf(A)+rank(A-A^4)=rankd(A)+rankm(A)$.

With the help of theorem 1 we can get if ***A*** is a fourth-idempotent matrix, then

$rankf(A)=rankd(A)+rankm(A)$

This theorem shows that there are also many rank eigenvalues of fourth-idempotent matrices.

Theorem 2

$A\in P^{n\times n}$, $t\geq 1\in N^+$, $rank(A)+rank(A^t-A^{t+3})=rank(A^t)+rank(A-A^4)$.

Proof: When t = 1, the equation clearly holds.

Let $t>1$, $f(x)=x^t$, $g(x)=x-x^4$, By simple calculation we get $(f(x),g(x))=x$ $[f(x),g(x)]=x^t-x^{t+3}$. By the above we can get the following equation.

$rank(A)+rank(A^t-A^{t+3})=rank(A^t)+rank(A-A^4)$

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