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Inequalities for α-Conformable Partial Derivatives

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Abstract: *In the paper, we introduce two concepts of α-conformable partial derivatives and α -conformable fractional integrals, and some new properties are listed. As applications, we establish Opial type inequalities for α -conformable partial derivatives and α-conformable fractional integrals. The new inequalities in special cases yield some of the recent results on inequality of this type.*

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1. Introduction

In 1960, Opial [1] established the following interesting and important inequality:

Theorem A Suppose $f \in C^1[0, a]$ satisfies $f(0) = f(a) = 0$ and $f(x) > 0$ for all $x \in (0, a)$.

Then the inequality holds

$$\int_0^a |f(x)f'(x)|\,dx \le \frac{a}{4}\int_0^a (f'(x))^2 dx, \tag{1.1}$$

where this constant a/4 is best possible.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations ([2-6]). The inequality (1.1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature ([7-18]).

Recently, some new Opial's inequalities for the conformable fractional integrals were established (see [19-22]). In the paper, we introduce two new concepts of α-conformable partial derivatives and α-conformable fractional integrals. Some properties of these new concepts are proved. As applications, we establish some Opial type inequalities for α-conformable partial derivatives and α-conformable fractional integrals.

2. α - Conformable Partial Derivatives

We recall the well-known Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0, 1]$ and $t \in [0, \infty)$, given by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon} \tag{2.1}$$

and

$$D_\alpha(f)(0) = \lim_{t \to 0} D_\alpha(f)(t) \tag{2.2}$$

provided the limits exist. If f is fully differentiable at t, then

$$D_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t).$$

A function f is α-differentiable at a point $t \geq 0$, if the limits in (2.1) and (2.2) exist and are finite. Inspired by this, we propose a new concept of α-conformable partial derivative. In the way of (2.1), we define α-conformable partial derivative.

Deftnition 2.1 (α-conformable partial derivative) Let $\alpha \in (0,1]$ and $s,t \in [0,\infty)$. Suppose $f(s,t)$ is a continuous function and has partial derivatives, the α-conformable partial derivative at a point $t \geq 0$, denoted by $\frac{\partial}{\partial t}(f)_\alpha(s,t)$, defined by

$$\frac{\partial}{\partial t}(f)_\alpha(s,t) = \lim_{\varepsilon \longrightarrow 0}\frac{f(s,te^{\varepsilon t^{-\alpha}}) - f(s,t)}{\varepsilon} \tag{2.3}$$

provided the limits exist, and call α-conformable partial differentiable.

To generalize Definition 2.1, we give the following definition.

Remark 2.2 Let α ∈ (0, 1] and s, t ∈ [0, ∞). Suppose f (s, t) and $\frac{\partial}{\partial t}$ (f) α (s, t) are continuous functions and have partial derivatives, we define a bivariate partial derivative, denoted by $\frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t)$, defined by

$$\frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}(f)_\alpha(s,t)\right) \tag{2.4}$$

And

$$\frac{\partial^2}{\partial s \partial t}(f)_\alpha(0,0) = \lim_{s\longrightarrow 0, t \longrightarrow 0}\frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t),$$

provided the limits exist, and call α-conformable partial differentiable.

Theorem 2.3 Let α ∈ (0, 1], s, t ∈ [0, ∞) and f (s, t), g (s, t) be α-conformable partial differentiable, then

(i)

$$\frac{\partial^2}{\partial s \partial t}(a \cdot f + b \cdot g)_\alpha(s,t) = a \cdot \frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t) + b \cdot \frac{\partial^2}{\partial s \partial t}(g)_\alpha(s,t) \tag{2.5}$$

for all a, b ∈R.

(ii)

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t}(fg)_\alpha(s,t) &= f(s,t) \cdot \frac{\partial^2}{\partial s \partial t}(g)_\alpha(s,t) + g(s,t) \cdot \frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t) \\ &+ \frac{\partial}{\partial t}(f)_\alpha(t) \cdot \frac{\partial g(s,t)}{\partial s} + \frac{\partial}{\partial t}(g)_\alpha(t) \cdot \frac{\partial f(s,t)}{\partial s}.\end{aligned} \tag{2.6}$$

Proof Here, we only prove (2.6). Let

$$v = tu \quad \text{and} \quad u = te^{\varepsilon t^{-\alpha}}$$

From (2.3), (2.4), and in view of L'Hospital rule, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t}(f)_\alpha(s,t) &= \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}(f)_\alpha(s,t)\right) \\ &= \frac{\partial}{\partial s}\left(\lim_{\varepsilon \longrightarrow 0}\frac{f(s,tu) - f(s,t)}{\varepsilon}\right) \\ &= \frac{\partial}{\partial s}\left(\lim_{\varepsilon \longrightarrow 0}\frac{\partial}{\partial \varepsilon}f(s,tu)\right) \\ &= \frac{\partial}{\partial s}\left(\lim_{\varepsilon \longrightarrow 0}\frac{\partial}{\partial v}f(s,v) \cdot t^{1-\alpha}u\right) \\ &= \frac{\partial}{\partial s}\left(t^{1-\alpha}\frac{\partial}{\partial t}f(s,t)\right) \\ &= t^{1-\alpha}\frac{\partial^2}{\partial s \partial t}f(s,t).\end{aligned} \tag{2.7}$$

From (2.3), (2.4) and (2.7), we have

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}(fg)_\alpha(s,t) &= t^{1-\alpha}\frac{\partial^2}{\partial s\partial t}(f(s,t)g(s,t))\\ &= t^{1-\alpha}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}(f(s,t)g(s,t))\right)\\ &= t^{1-\alpha}\frac{\partial}{\partial s}\left(f(s,t)\frac{\partial g(s,t)}{\partial t}+g()s,t\frac{\partial f(s,t)}{\partial t}\right)\\ &= f(s,t)\cdot\frac{\partial^2}{\partial s\partial t}(g)_\alpha(s,t)+g(s,t)\cdot\frac{\partial^2}{\partial s\partial t}(f)_\alpha(s,t)\\ &+ \frac{\partial}{\partial t}(f)_\alpha(t)\cdot\frac{\partial g(s,t)}{\partial s}+\frac{\partial}{\partial t}(g)_\alpha(t)\cdot\frac{\partial f(s,t)}{\partial s}.\end{aligned}$$

This completes the proof.

Theorem 2.4 Let α ∈ (0, 1], s, t ∈ [0, ∞) and f (s, t), g(s, t) be α-conformable partial differentiable, then

$$\frac{\partial^2}{\partial s\partial t}(f\circ g)_\alpha(s,t)=\frac{\partial^2}{\partial s\partial u}(f)_\alpha(u)\cdot\frac{\partial(g(s,t))}{\partial t}+\frac{\partial^2}{\partial s\partial t}(g)_\alpha(s,t)\cdot\frac{\partial(f(u))}{\partial u}, \tag{2.8}$$

where u = g(s, t), and f is partial derivative at g(s, t).

Proof From Definitions 2.1 and 2.2, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}(f\circ g)_\alpha(s,t) &= t^{1-\alpha}\frac{\partial^2(f\circ g)}{\partial t^2}(s,t)\\ &= t^{1-\alpha}\frac{\partial^2(f(g(s,t)))}{\partial s\partial t}\\ &= t^{1-\alpha}\frac{\partial}{\partial s}\left(\frac{\partial(f(g(s,t)))}{\partial t}\right)\\ &= t^{1-\alpha}\frac{\partial}{\partial s}\left(\frac{\partial(f(u))}{\partial u}\cdot\frac{\partial(g(s,t))}{\partial t}\right)\\ &= \frac{\partial(g(s,t))}{\partial t}\cdot\frac{\partial^2}{\partial s\partial u}(f)_\alpha(u)+\frac{\partial(f(u))}{\partial u}\cdot\frac{\partial^2}{\partial s\partial t}(g)_\alpha(s,t),\end{aligned}$$

where u = g(s, t).

3. Inequalities For α -Conformable Partial Derivatives

Deftnition 3.1 (*α*-conformable fractional integral) Let *α* ∈ (0, 1] and 0 ≤ *a* < *b*. A function *f*(*x, y*) : [*a*, *b*] × [*a*, *b*] → R is *α*-fractional integrable on [*a*, *b*] × [*a*, *b*], if the integral

$$\int_a^b\int_c^d f(x,y)d_\alpha x d_\alpha y := \int_a^b\int_c^d y^{\alpha-1}f(x,y)dxdy \tag{3.1}$$

exists and is finite.

Theorem 3.2 Let α ∈ (0, 1], 0 ≤ s ≤ c, 0 ≤ t ≤ d, and p(s, t) be nonnegative and continuous function on [0, c]×[0, d]. Let u(s, t) be a α-conformable partial differentiable function on [0, c]×[0, d] with u(s, d) = u(b, t) = u(b, d) = 0, then

$$\int_0^b\int_0^d p(s,t)\left|u(s,t)\right|^2 d_\alpha s d_\alpha t\le\frac{(bd)^\alpha}{4\alpha^2}\left(\int_0^b\int_0^d p(s,t)d_\alpha s d_\alpha t\right)\left(\int_0^b\int_0^d\left|\frac{\partial^2}{\partial s\partial t}(u)_\alpha(s,t)\right|^2 d_\alpha s d_\alpha t\right). \tag{3.2}$$

Proof Let

$$y(s,t)=\int_0^s\int_0^t\left|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_\alpha(\sigma,\tau)\right|d_\alpha\sigma d_\alpha\tau,$$

and

$$z(s,t)=\int_s^b\int_t^d\left|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_\alpha(\sigma,\tau)\right|d_\alpha\sigma d_\alpha\tau.$$

Then

$$\frac{\partial^2}{\partial s\partial t}(y)_\alpha(s,t)=\left|\frac{\partial^2}{\partial s\partial t}(u)_\alpha(s,t)\right|=\frac{\partial^2}{\partial s\partial t}(z)_\alpha(s,t) \tag{3.3}$$

and for all (s, t) ∈ [0, b] × [0, d],

$$u(s,t)\le y(s,t),\ \ u(s,t)\le z(s,t) \tag{3.4}$$

Hence

$$u(s,t)\le\frac{y(s,t)+z(s,t)}{2}=\frac{1}{2}\int_0^b\int_0^d\left|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_\alpha(\sigma,\tau)\right|d_\alpha\sigma d_\alpha\tau. \tag{3.5}$$

from (3.5) and in view of Cauchy-Schwarz inequality for α-conformable fractional integral, we obtain

$$\int_0^b\int_0^d p(s,t)\left|u(s,t)\right|^2 d_\alpha s d_\alpha t$$

$$\begin{aligned}
&\le \frac{1}{4}\int_0^b\int_0^d p(s,t)\left(\int_0^b\int_0^d\left|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_\alpha(\sigma,\tau)\right|d_\alpha\sigma d_\alpha\tau\right)^2 d_\alpha s d_\alpha t\\
&\le \frac{1}{4}\left(\int_0^b\int_0^d p(s,t)d_\alpha s d_\alpha t\right)\left(\int_0^b\int_0^d d_\alpha\sigma d_\alpha\right)\left(\int_0^b\int_0^d\left|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_\alpha(\sigma,\tau)\right|^2 d_\alpha\sigma d_\alpha\tau\right)\\
&= \frac{(bd)^\alpha}{4\alpha^2}\left(\int_0^b\int_0^d p(s,t)d_\alpha s d_\alpha t\right)\left(\int_0^b\int_0^d\left|\frac{\partial^2}{\partial s\partial t}(u)_\alpha(s,t)\right|^2 d_\alpha s d_\alpha t\right).
\end{aligned}$$

This completes the proof.

Remark 3.3 Taking for α = 1 in (3.2), we have

$$\int_0^b\int_0^d p(s,t)\left|u(s,t)\right|^2 dsdt\le\frac{bd}{4}\left(\int_0^b\int_0^d p(s,t)dsdt\right)\left(\int_0^b\int_0^d\left|\frac{\partial^2}{\partial s\partial t}u(s,t)\right|^2 dsdt\right) \tag{3.6}$$

Let p(s, t) and u(s, t) reduce to p(t) and u(t), respectively, and with suitable modifications, (3.6) becomes the following result. Let p(t) be a nonnegative and continuous function on [0, h]. Let u(t) be an absolutely continuous function on [0, h] with u(0) = u(h) = 0, then

$$\int_0^h p(t)\left|u(t)\right|^2 dsdt\le\frac{h}{4}\left(\int_0^h p(t)dt\right)\left(\int_0^h\left|u'(t)\right|^2 dt\right)$$

This is just an inequality which was established in [20].

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