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Weak Insertion of a Perfectly Continuous Function

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Abstract: *A sufficient condition in terms of lower cut sets are given for the insertion of a perfectly continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets are open.*

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [10]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i. e., sets that are intersection of open sets are called $\Lambda-$sets [10].

Recall that a real-valued function f defined on a topological space X is called *$A-$continuous* [15] if the preimage of every open subset of R belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [2, 5].

Hence, a real-valued function f defined on a topological space X is called *perfectly continuous*[14] (resp. *contra-continuous* [3]) if the preimage of every open subset of R is a clopen (i. e., open and closed) (resp. closed) subset of X.

We have a function is perfectly continuous if and only if it is continuous and contra-continuous.

Results of Katˇetov [6, 7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give a necessary and sufficient conditions for the insertion of a perfectly continuous function between two comparable realvalued functions on the topological spaces that $\Lambda-$sets are open [10].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [8].

A property P defined relative to a real-valued function on a topological space is a *$pc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any perfectly continuous function also has property P. If $P1$ and $P2$ are $pc-$property, the following terminology is used: A space X has the *weak $pc-$insertion property* for ($P1$,$P2$) if and only if for any functions g and f on X such that $g \leq f$, g has property $P1$ and f has property $P2$, then there exists a perfectly continuous function h such that $g \leq h \leq f$.

In this paper, is given a sufficient condition for the weak $pc-$insertion property. Also, several insertion theorems are obtained as corollaries of these results. In addition, the insertion and strong insertion of a contracontinuous function between two comparable contra-precontinuous (contrasemi-continuous) functions have also recently considered by the author in [11, 12].

2. The Main Result

Before giving a sufficient condition for insertability of a perfectly continuous function, the necessary definitions and terminology are stated.

Let (X,τ) be a topological space, the family of all open, closed and clopen will be denoted by $O(X,\tau)$, $C(X,\tau)$ and $Clo(X,\tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X,τ). We define the subsets A^{Λ} and A^{V} as follows:

$A^{\Lambda} = \cap\{O : O \supseteq A, O \in O(X,\tau)\}$ and $A^{V} = \cup\{F : F \subseteq A, F \in C(X,\tau)\}$.

In [4, 9, 13], A^{Λ} is called the *kernel* of A.

Definition 2.2. Let A be a subset of a topological space (X,τ). Respectively, we define the *closure, interior, clo-closure* and *clo-interior* of a set A, denoted by $Cl(A), Int(A), clo(Cl(A))$ and $clo(Int(A))$ as follows:

$Cl(A) = \cap\{F : F \supseteq A, F \in C(X,\tau)\}$, $Int(A) = \cup\{O : O \subseteq A, O \in O(X,\tau)\}$, $clo(Cl(A)) = \cap\{F : F \supseteq A, F \in Clo(X,\tau)\}$ and $clo(Int(A)) = \cup\{O : O \subseteq A, O \in Clo(X,\tau)\}$.

If (X,τ) be a topological space whose $\Lambda-$sets are open, then respectively, we have $A^{V}, clo(Cl(A))$ are closed, clopen and $A^{\Lambda}, clo(Int(A))$ are open, clopen.

The following first two definitions are modifications of conditions considered in [6, 7].

Definition 2.3. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho}\, y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.4. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

- If $A_i \rho B_j$ for any $i \in\{1,\ldots,m\}$ and for any $j \in\{1,\ldots,n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in\{1,\ldots,m\}$ and any $j \in\{1,\ldots,n\}$.
- If $A \subseteq B$, then $A \bar{\rho} B$.
- If $A \rho B$, then $clo(Cl(A)) \subseteq B$ and $A \subseteq clo(Int(B))$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition 2.5. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X, in which $\Lambda-$sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t1 < t2$ then $A(f,t1)\ \rho\ A(g,t2)$, then there exists a perfectly continuous function h defined on X such that $g \leq h \leq f$. **Proof.** Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t1 < t2$ then $A(f,t1)\ \rho\ A(g,t2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If $t1$ and $t2$ are any elements of Q with $t1 < t2$, then $F(t1)\ \bar{\rho}\ F(t2), G(t1)\ \bar{\rho}\ G(t2)$, and $F(t1)\ \rho\ G(t2)$. By Lemmas 1 and 2 of [7] it follows that there exists a function H mapping Q into the power set of X such that if $t1$ and $t2$ are any rational numbers with $t1 < t2$, then $F(t1)\ \rho\ H(t2), H(t1)\ \rho\ H(t2)$ and $H(t1)\ \rho\ G(t2)$.

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t1$ and $t2$ with $t1 < t2$, we have $h^{-1}(t1,t2) = clo(Int(H(t2))) \setminus clo(Cl(H(t1)))$. Hence $h^{-1}(t1,t2)$ is a clopen subset of X, i. e., h is a perfectly continuous function on X.

The above proof used the technique of proof of Theorem 1 of [6].

3. Applications

The abbreviations c, pc and cc are used for continuous, perfectly continuous and contra-continuous, respectively.

Before stating the consequences of theorems 2.1, we suppose that X is a topological space that $\Lambda-$sets are open.

Corollary 3.1. If for each pair of disjoint closed (resp. open) sets $F1,F2$ of X , there exist clopen sets $G1$ and $G2$ of X such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then X has the weak $pc-$insertion property for (c,c) (resp.

(cc,cc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are c (resp. cc), and $g \leq f$.If a binary relation ρ is defined by $A \rho B$ in case $Cl(A) \subseteq Int(B)$ (resp. $A^{\Lambda} \subseteq B^{V}$), then by hypothesis ρ is a strong binary relation in the power set of X. If $t1$ and $t2$ are any elements of Q with $t1 < t2$, then

$A(f,t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g,t2)$;

since $\{x \in X : f(x) \leq t1\}$ is a closed (resp. open) set and since $\{x \in X : g(x) < t2\}$ is an open (resp. closed) set, it follows that $Cl(A(f,t1)) \subseteq Int(A(g,t2))$ (resp. $A(f,t1)^{\Lambda} \subseteq A(g,t2)^{V}$). Hence $t1 < t2$ implies that $A(f,t1)$ ρ $A(g,t2)$. The proof follows from Theorem 2.1.

Corollary 3.2. If for each pair of disjoint closed (resp. open) sets $F1,F2$, there exist clopen sets $G1$ and $G2$ such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then every continuous (resp. contra-continuous) function is perfectly continuous.

Proof. Let f be a real-valued continuous (resp. contra-continuous) function defined on the X. By setting $g = f$, then by Corollary 3.1, there exists a perfectly continuous function h such that $g = h = f$.

Corollary 3.3. If for each pair of disjoint closed (resp. open) sets $F1,F2$ of X , there exist clopen sets $G1$ and $G2$ of X such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then X has the $pc-$insertion property for (c,c) (resp. (cc,cc)). **Proof.** Let g and f be real-valued functions defined on the X, such that f and g are c (resp. cc), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since g and f are perfectly continuous functions hence h is a perfectly continuous function.

Corollary 3.4. If for each pair of disjoint subsets $F1,F2$ of X , such that $F1$ is closed and $F2$ is open, there exist clopen subsets $G1$ and $G2$ of X such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ then X have the weak $pc-$insertion property for (c,cc) and (cc,c).

Proof. Let g and f be real-valued functions defined on the X, such that g is c (resp. cc) and f is cc (resp. c), with $g \leq f$.If a binary relation ρ is defined by $A \rho B$ in case $A^{\Lambda} \subseteq Int(B)$ (resp. $Cl(A) \subseteq B^{V}$), then by hypothesis ρ is a strong binary relation in the power set of X. If $t1$ and $t2$ are any elements of Q with $t1 < t2$, then

$A(f,t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g,t2)$;

since $\{x \in X : f(x) \leq t1\}$ is an open (resp. closed) set and since $\{x \in X : g(x) < t2\}$ is an open (resp. closed) set, it follows that $A(f,t1)^{\Lambda} \subseteq Int(A(g,t2))$ (resp. $Cl(A(f,t1)) \subseteq A(g,t2)^{V}$). Hence $t1 < t2$ implies that

$A(f,t1)$ ρ $A(g,t2)$. The proof follows from Theorem 2.1.

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