International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 6, Issue 9, 2018, PP 16-24
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0609003
www.arcjournals.org

Star–in–Coloring of Some Splitting Graphs

A. Sugumaran[1], P. Kasirajan[2]

Department of Mathematics, Government Arts College, Tiruvannamalai Tamil Nadu, India

Corresponding Author:** ***P.Kasirajan, *Department of Mathematics, Government Arts College, Tiruvannamalai Tamil Nadu, India*

Abstract: *A proper coloring of a graph $G = (V, E)$ is a mapping $f: V \rightarrow \{1,2,3, \dots\}$such that if $e = v_i v_j \epsilon E$, then $f(v_i) \neq f(v_j)$. A proper coloring of a directed graph (digraph) G is said to admit star–in–coloring, if the graph has to satisfy the following two conditions: (i) no path of length threeis bicolored (ii) if any path of length twowith terminal vertices are of the same color, then the edges must be oriented towards the middle vertex. The minimum number of colors required to color a graph G as star–in–coloring, is called the star–in–chromatic number of the graph G and it is denoted by $\chi_{si}(G)$.In this paper, we consider the splitting graphs of the graphs such as cycle, gear, regular cyclic, a complete binary tree and web graph, we investigatethe star–in–chromatic number of these graphs*

Keywords: *Coloring; Splitting graph; Star–in–coloring;Star–in–chromatic number.*

1. Introduction

Let $G = (V, E)$ be a simple, connected digraph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E, each element of E is a directed edge. An orientation of a graph G is obtained by applying an orientation for each edge $e = v_i v_j \epsilon E$ either from v_i to v_j or v_j tov_i. A proper *coloring* of a graph G is a mapping $f: V \rightarrow \{1,2,3, \dots\}$ such that if $e = v_i v_j \epsilon E$, then $f(v_i) \neq f(v_j)$.A *star–coloring* of a graph G is a proper coloring of the graph with the condition that no path of length three (P_4) is bicolored. The concept of star–coloring of graphs was introduced by Grunbaum [2]. The star–coloring of graphs have been investigated by Fertin et al. [1] and Nesetril et al. [3].

A digraph G is said to be in–coloring if any path of length two with end vertices are of same color, then the edges are always directed towards the middle vertex. Motivated by the concepts of star–coloring and in–coloring, Sudha and Kanniga [5,6] have introduced a new concept known asstar–in–coloring of graphs. A graph G is said to admit *star–in–coloring* if it satisfies the following two conditions.

- No path of length three$(P_4)$is bicolored.
- If any path of length two $(P_3)$with end vertices are of the same color, then the edges of $P_3$are directed towards the middle vertex.

Sugumaran and Kasirajan [7] have found the lower, upper bounds and star–in–chromatic number of the graphs such as cycle, regular cyclic, gear, fan, double fan, web and complete binary tree.

Definition 1.1 The minimum number of colors required for the star–in–coloring of a graph G is called the *star–in–chromatic number* ofG and it is denoted by $\chi_{si}(G)$.The simplest star–in–coloringof a cycle C_4 is shown in Fig.1.

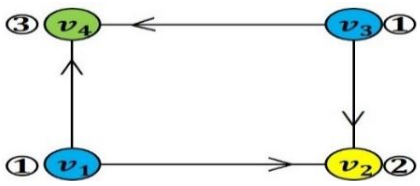

Fig1. *Star – in – coloring of Cycle C_4*

The star–in–chromatic number of the above graph is 3.

Definition 1.2 A cycle $C_p$is said to be a *regular cyclic* graph ifmaximum number of chords are drawn without forming a triangleand the resulting graph is regular. Then this regular cyclic graph is denoted by RC(p, n), where n is the degree of each vertex in this graph.

Definition 1.3 A connected acyclic graph is called a *tree*. A binary tree is a tree in which only one vertex is of degree two and each of the remaining vertices are of degree one or three. A vertex of degree two in a binary tree is called its root vertex. In a binary tree, a vertex v is said to be at level l if v is at a distance l from the root vertex.

Definition 1.4 A binary tree with level n is said to be *complete* if each level l of the binary tree contains exactly 2^l vertices, where $0 \leq l \leq n$. A *complete binary tree* with level n is denoted by BT_n.

Note that the complete binary tree BT_n contains $|V| = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ vertices and $|E| = |V| - 1$ edges.

Defininition 1.5 For any graph G, the *splitting graph* $S(G)$ is obtained by adding to each vertex v_i in G a new vertex $v_i^{'}$ such that$N(v_i) = N(v_i^{'})$, where $N(v_i)$is the set of neighbours of the vertex v_i.

Splitting graph was defined by Sampathkumar and Walikar [4].

2. Main Results

In this section, we find the lower and upper bounds of the star–in–chromatic number of splitting graphs of some standard graphs. First we find out the star–in–chromatic number of a splitting graph of cycleC_n.

Theorem 1. The graph $S(C_n)$ admits star–in–coloring and its star–in–chromaticnumber satisfies the inequality $5 \leq \chi_{si}[S(C_n)] \leq 7$, where n is even and $n \geq 4$.

Proof. Let $G = S(C_n)$ and let V, E be the vertex set and edge set of G respectively. Then $|V| = 2n$ and $|E| = 3n$. For each $i = 1,2, \ldots, n$, let $v_i$be a vertex of cycle C_n and let $v_i^{'}$ be a new vertex (corresponding to a vertex v_i) added in G.

We define a function $f: V \rightarrow \{1, 2, 3, \ldots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$.

The star–in–coloring pattern of G we need to consider two different cases:

Case1:$For\ n \equiv 0\ (mod\ 4)$

In a cycle C_n, a path of length 3 is not bi-colored. So we need at least three colors.

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4) \end{cases}$$

Further,

$$f(v_i^{'}) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 4, & if\ i \equiv 2\ (mod\ 4) \\ 5, & if\ i \equiv 0\ (mod\ 4) \end{cases}$$

Case 2:$For\ n \equiv 2\ (mod\ 4)$

We assign $f(v_0) = 6, f(v_0^{'}) = 7$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 4, & if\ i \equiv 2\ (mod\ 4) \\ 5, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

From the above cases, we conclude that $\chi_{si}[S(C_n)] = \begin{cases} 5, & if\ n \equiv 0\ (mod 4) \\ 7, & if\ n \equiv 2\ (mod 4) \end{cases}$

Illustration: The star–in–coloring of $S(C_n)$ is shown for $n = 6, 8$ in Fig. 2 and Fig. 3 respectively. Note that the star–in–chromaticnumber$\chi_{si}[S(C_6)] = 7$ and $\chi_{si}[S(C_8)] = 5$. In general $\chi_{si}[S(C_{4n})] =$ 5and $\chi_{si}[S(C_{4n+2})] = 7, n \in N$.

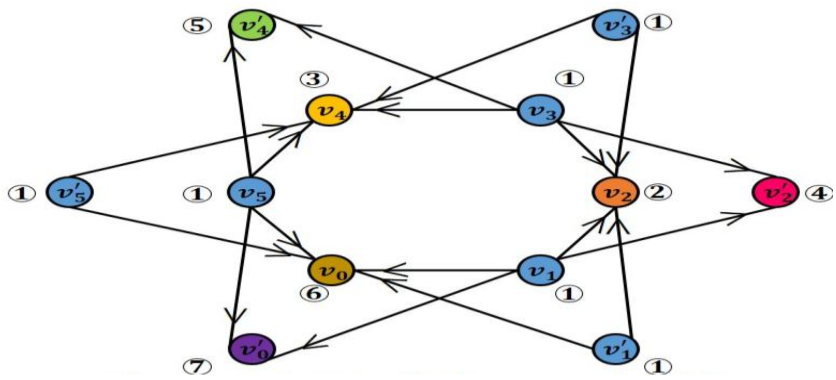

Fig2. *Splitting graph of* C_6

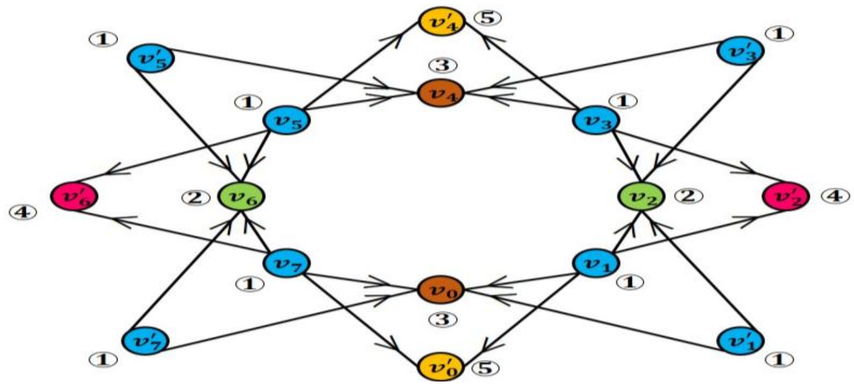

Fig3. *Splitting graph of* C_8

Theorem 2. The splitting graph of Gear graph G_n admits star–in–coloring and its star–in–chromaticnumber satisfies the inequality $7 \leq \chi_{si}[S(G_n)] \leq 9$, where $n \geq 3$.

Proof. Let $G = S(G_n)$ and let V, E be the vertex set and edge set of G respectively. Then$|V| = 4n + 2$ and $|E| = 9n$. For each $i = 1,2, \dots, n$, let $v_i$be a vertex of Gear G_n and let $v_i^{'}$ be a new vertex (corresponding to the vertex v_i) added in G.

We define a function $f: V \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$.

The star–in–coloring pattern of G we need to consider two different cases:

Case 1: For$n$ is even

We assign $f(v_0) = 6, f(v_0^{'}) = 7$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 2, & if\ i \equiv 2\ (mod\ 4) \\ 3, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ 4, & if\ i \equiv 2\ (mod\ 4) \\ 5, & if\ i \equiv 0\ (mod\ 4)\ and\ i > 0 \end{cases}$$

Case 2: For n is odd

We assign $f(v_0) = 8, f(v_0^{'}) = 9$ and

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1 (mod 2) \\ 2, & if\ i \equiv 2(mod4)\ and\ i < 2n - 2 \\ 3, & if\ i \equiv 0(mod4)\ and\ i > 0 \\ 4, & if\ i = 2n - 2. \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 1, & if\ i \equiv 1(mod2) \\ 5, & if\ i \equiv 2(mod4)\ and\ i < 2n-2 \\ 6, & if\ i \equiv 0(mod4)\ and\ i > 0 \\ 7, & if\ i\ = 2n-2. \end{cases}$$

From the above cases, we conclude that $\chi_{si}[S(G_n)] = \begin{cases} 7, & if\ n\ is\ even \\ 9, & if\ n\ is\ odd \end{cases}$

Illustration: The star–in–coloring of $S(G_n)$ is hown for $n = 3, 4$ in Fig. 4 and Fig. 5 respectively. Note that the star–in–chromaticnumber$\chi_{si}[S(G_3)] = 9$ and $\chi_{si}[S(G_4)] = 7$.In general $\chi_{si}[S(G_{2n+2})] = 7$and $\chi_{si}[S(G_{2n+1})] = 9, n \in N$.

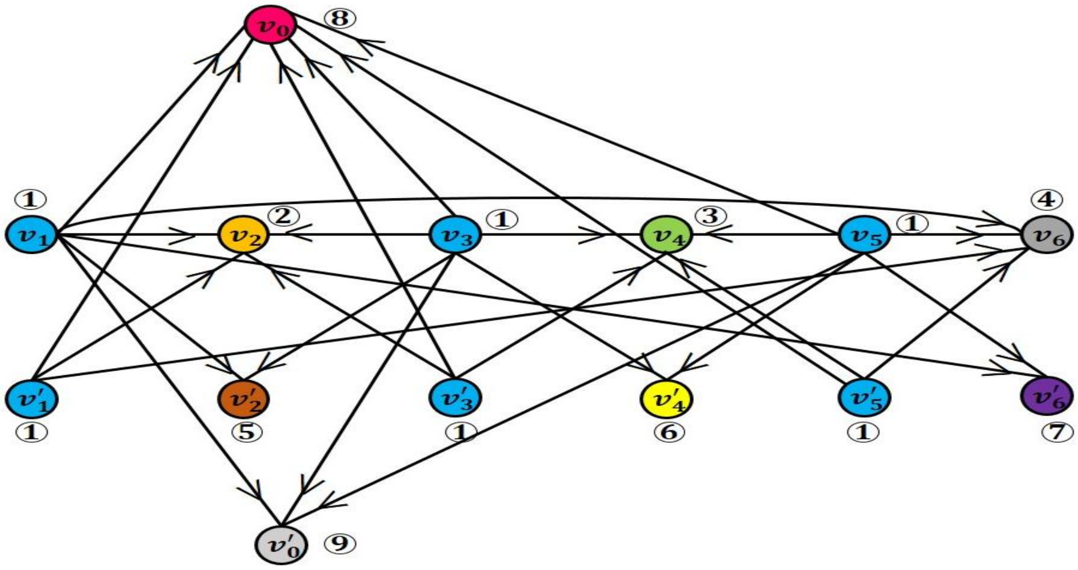

Fig4. *Splitting graph of G_3*

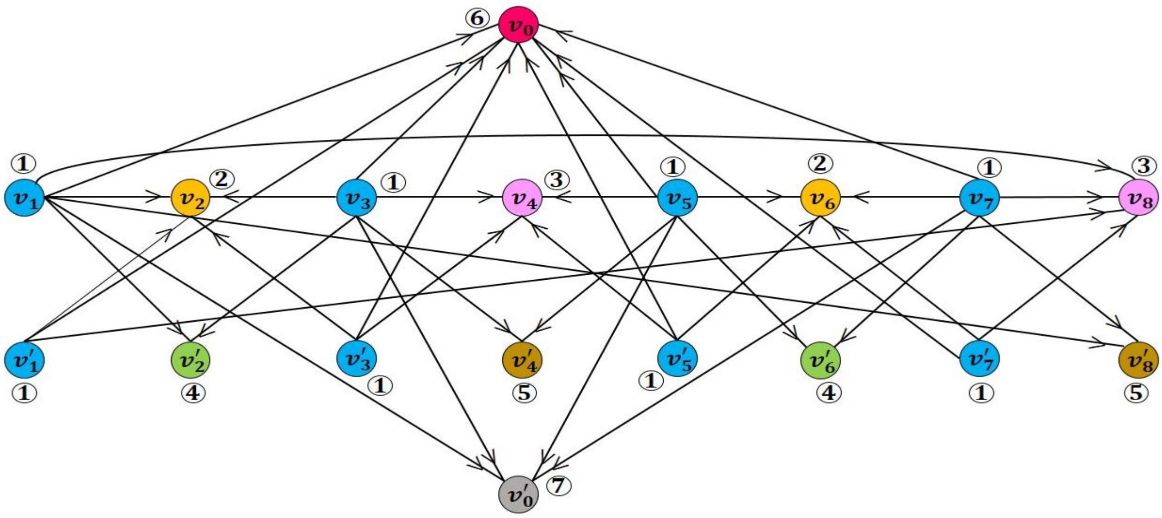

Fig5. *Splitting graph of G_4*

Theorem 3. The splitting graph of a regular cyclic graph RC(p, n)admits star–in–coloring and its star–in–chromaticnumber is $2n + 1$, where p is an even integer and $p > 3$.

Proof. Let $G = S(\text{RC}(p, n))$ and let V, E be the vertex set and edge set of G respectively. Then $|V| = 4n$ and $|E| = 3n^2$. For each $i = 1,2, \dots, n$, let $v_i$be a vertex of cycle G_n and let $v_i^{'}$ be a new vertex (corresponding to the vertex v_i) added in G.

We define a function $f: V \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$.

The star–in–coloring pattern is as follows:

We assign,

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ \frac{i}{2} + 1, & if\ i \equiv 2\ (mod\ 2) \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ \frac{i}{2} + n + 1, & if\ i \equiv 2\ (mod\ 2) \end{cases}$$

By using the above pattern of coloring the splitting graph of regular cyclic graph is star–in–colored and$\chi_{si}\,[S(\mathrm{RC}(p,n))] = 2n + 1$.

Illustration: The star–in–coloring of $S(\mathrm{RC}(8,4))$ is shown in Fig. 6. Note that the star–in–chromaticnumber$\chi_{si}\,[S(\mathrm{RC}(8,4))] = 9$.

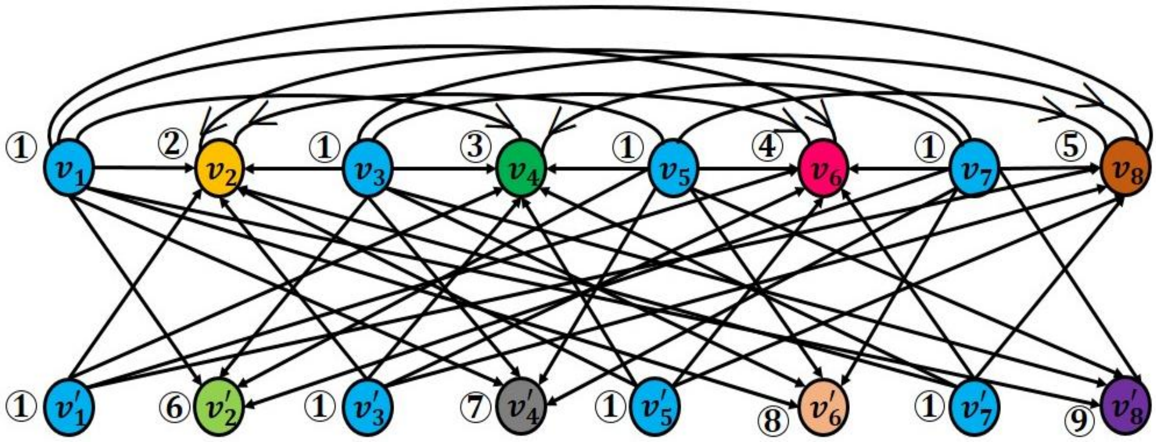

Fig6. *Splitting graph of* $RC(8,4)$

Remark: The graph $\mathrm{RC}(p,n)$is not star–in–coloring, when p is odd, since at least one of the edges is left without orientation.

Theorem 4. The splitting graph of complete binary tree BT_n for all $n \geq 2$ admits star–in–coloring and $\chi_{si}\,[S(BT_n)] = 4$.

Proof. Consider a complete binary tree $BT_n$with $|V| = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ vertices and $|E| = |V| - 1$ edges. The root vertex (degree 2) is denoted by $v_0$and the other vertices are denoted by $v_1^1, v_1^2, v_2^1, v_2^2, \ldots, v_n^{2^n}$.

The graph $S(BT_n)$ consists of $|V| = 2(1 + 2 + 2^2 + \cdots + 2^n)$ vertices and $|E| = 3(|V| - 1)$ edges.

We define a function $f: V \to \{\,1, 2, 3, \ldots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon\, E$.

The star–in–coloring pattern is as follows:

We assign $f(v_0) = 1$,$f(u_0) = 3$and for each $j = 1,2, \ldots, 2^i$

$$f(v_i^j) = \begin{cases} 1, & if\ i \equiv 0(mod3) \\ 2, & if\ i \equiv 1(mod3) \\ 3, & if\ i \equiv 2(mod3) \end{cases}$$

$f(u_i^j) = 4$,for all $1 \leq i \leq n$.

Note that $f(u_1^1) \neq 1$ and 3, since by definition of proper coloring. Also if we assign $f(u_1^1) = 2$, then the star – coloring condition for the path connecting the vertices $u_1^1, v_2^2, v_1^1, v_2^1$ is affected. Hence a new color 4 is assigned to the vertex u_1^1.

With this pattern of coloring, the splitting graph of complete binary tree BT_n can be star–in–colored and$\chi_{si}\,[S(BT_n)] = 4$ for all n ≥ 2.

Illustration: The star–in–coloring of $S(BT_2)$ is shown in Fig. 7. Note that the star–in–chromaticnumber$\chi_{si}\,[S(BT_2)] = 4$.

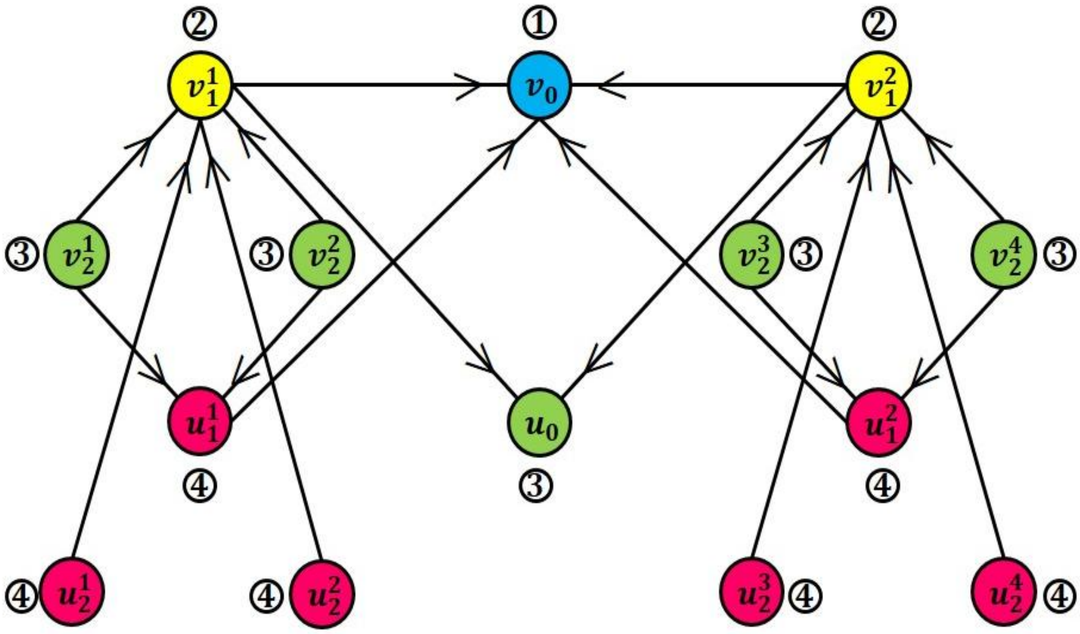

Fig7. *Splitting graph of BT_2*

Theorem 5. The splitting graph of web graph $W_{n,r}$ admits star–in–coloring and its star–in–chromatic number satisfies the inequality $9 \leq \chi_{si}[S(W_{n,r})] \leq 11$, for all even n.

Proof. The graph $S(W_{n,r})$ consists of $2nr$ vertices and $3n(2r-1)$ edges. The vertex sets V and U in $W_{n,r}$ are partioned into r vertex sets denoted by $V^1, V^2, V^3, \dots, V^r$ and $U^1, U^2, U^3, \dots, U^r$ where each vertex set consists of n vertices. Assume that the vertex set V^j consists of the vertices $v_1^j, v_2^j, \dots, v_n^j$ and the vertex set U^j consists of the vertices $u_1^j, u_2^j, \dots, u_n^j$ for all $1 \leq j \leq r$.

The general pattern of coloring has been grouped into two cases: One for $n \equiv 0 \ (mod\ 4)$ and other for $n \equiv 2 \ (mod\ 4)$.

Case 1: $For\ n \equiv 0 \ (mod\ 4)$

$f(v_i^j) = 1\ and\ f(u_i^j) = 1\ if\ i + j$ is even.

For all other values of i and j, we consider four subcases as follows:

Subcase 1.1: $For\ j \equiv 1 \ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 2, & if\ i \equiv 2(mod4) \\ 3, & if\ i \equiv 0(mod4) \end{cases}$$

$$f(u_i^j) = \begin{cases} 6, & if\ i \equiv 2(mod4) \\ 7, & if\ i \equiv 0(mod4) \end{cases}$$

Subcase 1.2: $For\ j \equiv 2 \ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 4, & if\ i \equiv 1(mod4) \\ 5, & if\ i \equiv 3(mod4) \end{cases}$$

$$f(u_i^j) = \begin{cases} 8, & if\ i \equiv 1(mod4) \\ 9, & if\ i \equiv 3(mod4) \end{cases}$$

Subcase 1.3: $For\ j \equiv 3 \ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 3, & if\ i \equiv 2(mod4) \\ 2, & if\ i \equiv 0(mod4) \end{cases}$$

$$f(u_i^j) = \begin{cases} 7, & if\ i \equiv 2(mod4) \\ 6, & if\ i \equiv 0(mod4) \end{cases}$$

Subcase 1.4: $For\ j \equiv 0 \ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 5, & if\ i \equiv 1(mod4) \\ 4, & if\ i \equiv 3(mod4) \end{cases}$$

$$f(u_i^j) = \begin{cases} 9, & if\ i \equiv 1(mod4) \\ 8, & if\ i \equiv 3(mod4) \end{cases}$$

By using the above pattern of coloring the web graph is star–in–colored. According to Case 1 the star–in–chromatic number of$W_{n,r}$is $\chi_{si}(W_{n,r}) = 9$.

Case 2: $For\ n \equiv 2\ (mod\ 4)$

$$f(v_i^j) = 1\ and\ f(u_i^j) = 1\ if\ i + j \text{ is even.}$$

For all other values of i and j, we consider four subcases as follows:

Subcase 2.1:$For\ j \equiv 1\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 2, & if\ i \equiv 2(mod4)\ and\ i < n \\ 3, & if\ i \equiv 0(mod4) \\ 4, & if\ i = n. \end{cases}$$

$$f(u_i^j) = \begin{cases} 8, & if\ i \equiv 2(mod4)\ and\ i < n \\ 9, & if\ i \equiv 0(mod4) \\ 10, & if\ i = n. \end{cases}$$

Subcase 2.2:$For\ j \equiv 2\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 5, & if\ i \equiv 1(mod4)\ and\ i < n-1 \\ 6, & if\ i \equiv 3(mod4) \\ 7, & if\ i = n-1. \end{cases}$$

$$f(u_i^j) = \begin{cases} 11, & if\ i \equiv 1(mod4)\ and\ i < n-1 \\ 12, & if\ i \equiv 3(mod4) \\ 13, & if\ i = n-1. \end{cases}$$

Subcase 2.3:$For\ j \equiv 3\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 4, & if\ i \equiv 2(mod4)\ and\ i < n \\ 2, & if\ i \equiv 0(mod4) \\ 3, & if\ i = n. \end{cases}$$

$$f(u_i^j) = \begin{cases} 10, & if\ i \equiv 2(mod4)\ and\ i < n \\ 8, & if\ i \equiv 0(mod4) \\ 9, & if\ i = n. \end{cases}$$

Subcase 2.4:$For\ j \equiv 0\ (mod\ 4)$

$$f(v_i^j) = \begin{cases} 7, & if\ i \equiv 1(mod4)\ and\ i < n-1 \\ 5, & if\ i \equiv 3(mod4) \\ 6, & if\ i = n-1. \end{cases}$$

$$f(u_i^j) = \begin{cases} 13, & if\ i \equiv 1(mod4)\ and\ i < n-1 \\ 11, & if\ i \equiv 3(mod4) \\ 12, & if\ i = n-1. \end{cases}$$

From the above cases, we conclude that $\chi_{si}[S(W_{n,r})] = \begin{cases} 9, & if\ n \equiv 0\ (mod4) \\ 13, & if\ n \equiv 2\ (mod4) \end{cases}$

Illustration: The star–in–coloring of $S(W_{n,r})$ for $n = 4, 6$ and $r = 4$ are shown in Fig. 8 and Fig. 9 respectively. Note that the star–in–chromaticnumber$\chi_{si}[S(W_{4,4})] = 9$ and $\chi_{si}[S(W_{6,4})] = 13$. In general $\chi_{si}[S(W_{4n,r})] = 9$and $\chi_{si}[S(W_{4n+2,r})] = 13, n \in N$.

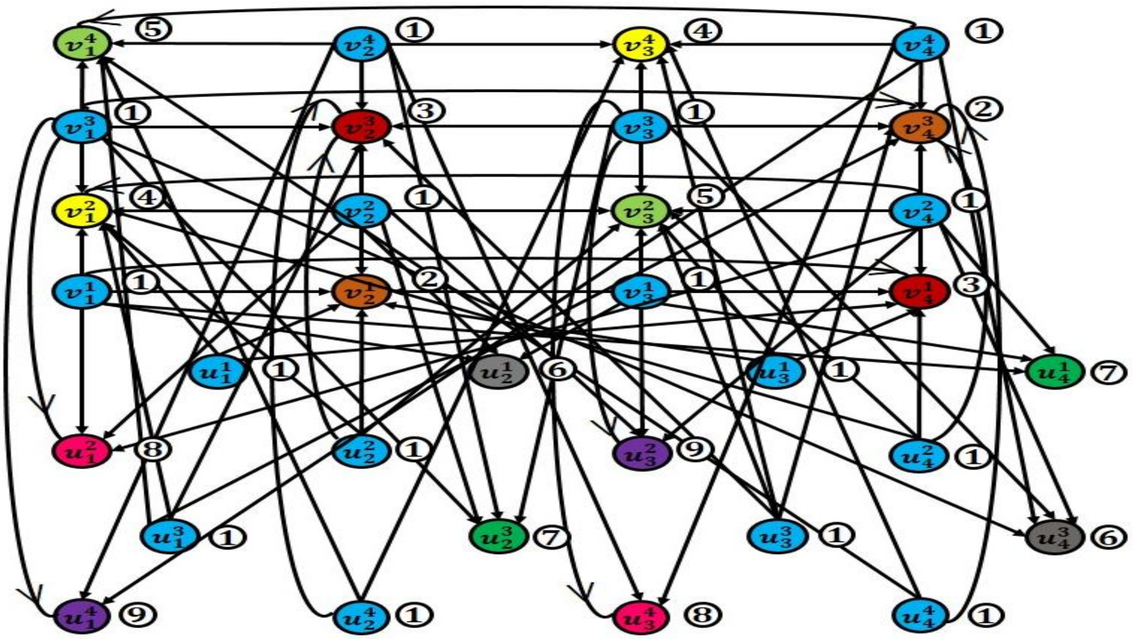

Fig8. *Splitting graph of* $W_{4,4}$

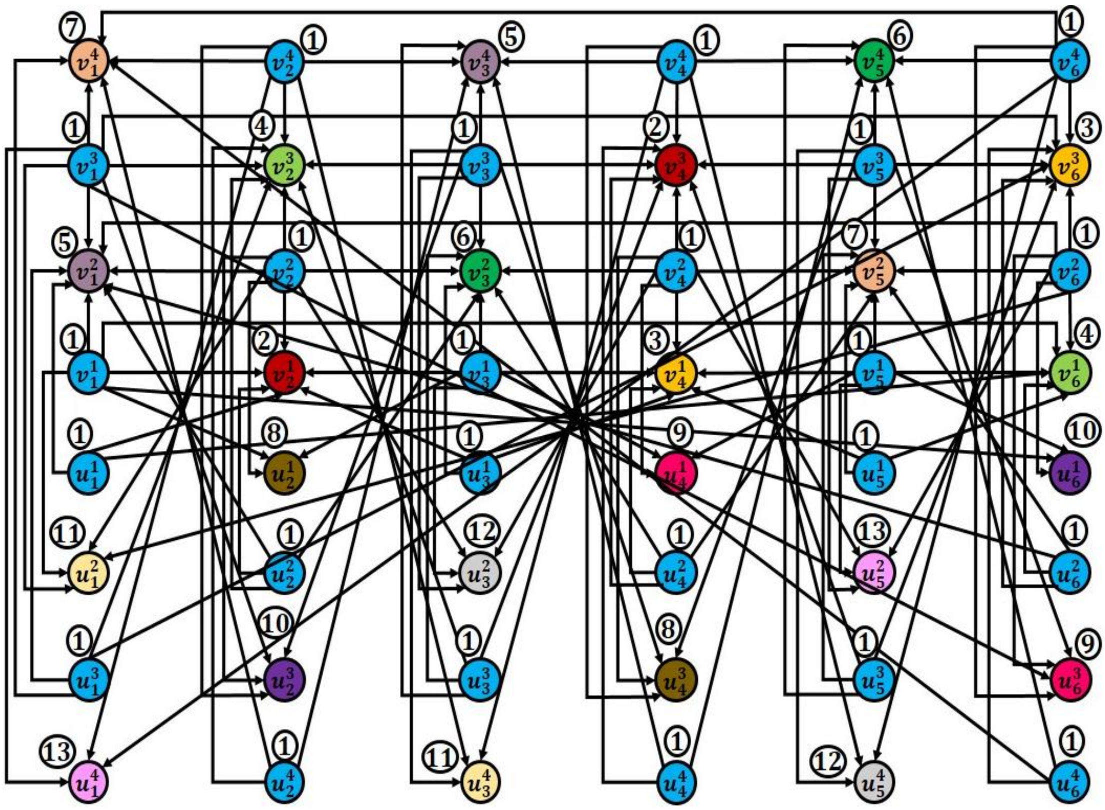

Fig9. *Splitting graph of* $W_{6,4}$

3. Conclusion

In this paper, we have shown that the lower and upper bounds for star–in–chromatic number of some of the standard graphs are as given below.

1. $5 \leq \chi_{si}[S(C_n)] \leq 7$ $n$is even.
2. $7 \leq \chi_{si}[S(G_n)] \leq 9$.
3. $\chi_{si}[S(\mathrm{RC}(p,n))] = 2n+1,\ p > 3$.
4. $\chi_{si}[S(BT_n)] = 4, n \geq 2$.
5. $9 \leq \chi_{si}[S(W_{n,r})] \leq 13$.

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Citation: *A. Sugumaran & P. Kasirajan (2018). Star–in–Coloring of Some Splitting GraphsInternational Journal of Scientific and Innovative Mathematical Research (IJSIMR), 6(9), pp.16-24. http://d x.doi.org/10. 20431/2347-3142.0609003*