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On the Existence of Nontrivial Nucleus and on the Nilpotence of Moufang Loops of Odd Order

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Abstract: *We give a necessary and sufficient condition for the existence of non- trivial nucleus in Moufang loops of odd order. Assuming the property* $[A, B] \le Z$ *(*$\mathrm{Inn}\ Q$*) we study the relation of the nilpotence of Moufang loops of odd order with the existence of nontrivial nucleus.*

Keywords: *Moufang loop, nucleus, multiplication group, inner mapping group. 2000 Mathematics Subject Classification: 20N05, 08A05.*

1. INTRODUCTION

A quasigroup Q that possesses an element 1 satisfying $1x = x1 = x$ for every $x \in Q$ is called a *loop* with neutral element 1. The mappings $L_a(x) = ax$ (*left translation*) and $R_a(x) = xa$ (*right translation*) are permutations on the elements of Q for every $a \in Q$. The permutation group generated by left and right translations $\mathrm{Mlt}\, Q = \langle L_a, R_a \mid a \in Q \rangle$ is called the *multiplication group* of Q. The *inner mapping group*, $\mathrm{Inn}\, Q$ is defined as the stabilizer of 1 in $\mathrm{Mlt}\, Q$.

In this paper we focus on Moufang loops satisfying any one of the three equivalent Moufang identities: $((xy)x)\,z = x(y(xz))$; $((xy)z)\,y = x(y(zy))$; $(xy)(zx) = x(yz)x$.

A Moufang loop can be viewed as a "group with weakened associativity". The links between algebra (alternating rings, octonions), geometry (Moufang planes), and group theory (triality, representation of exceptional Lie groups) explain the importance of this class.

Phillips' problem (Wikipedia, Moufang loops, Open problems): Does there exist a Moufang loop of odd order with trivial nucleus?

In this paper we give a necessary and sufficient condition for the existence of nontrivial nucleus in Moufang loops of odd order. Assuming the property $[A, B] \le Z\,(\mathrm{Inn}\, Q)$ we study the relation of the nilpotence of Moufang loops of odd order with the existence of nontrivial nucleus.

For the proof of our statements we need the structural properties of the multiplication group of Moufang loops of odd order.

Glauberman [11] and Doro [3] studied the structure of Moufang loops of odd order. Glauberman [11] proved that Feit–Thompson's Theorem can be extended to Moufang loops, namely every Moufang loop of odd order is solvable. It turned out that the multiplication group G of a Moufang loop of odd order with trivial nucleus is a group with triality, i.e., $S_3 \le \mathrm{Aut}\, G$ with special identities.

In paper [8] applying the theory of connected transversals we constructed two automorphisms of order 2 of the multiplication group of a Moufang loop of odd order with trivial nucleus. These automorphisms generate S_3.

In such a way we get the classical triality group of Glauberman [11] and Doro [3]. The reader can find more details about the connection between the normalizers of the transversals, between pseudoautomorphisms and between autotopisms in Pflugfelder's book [15] and in Mikheev's paper [13].

In paper [10] we showed that the nilpotence of Moufang loop of odd order is equivalent to the property that the elements of coprime order commute. In the proof first we verified the existence of nontrivial nucleus by using the properties of these automorphisms, too.

In paper [8] by using the properties of these automorphisms we proved that the existence of nontrivial commutant implies the existence of nontrivial nucleus.

Our proofs are completely group theoretical relying on the theory of connected transversals. This concept was introduced by Niemenmaa and Kepka [14]. Using their characterization theorem we can transform loop theoretical problems into group theoretical problems.

2. Basic Definition and Results

For the basic concepts of loop theory we refer to Bruck [2]. Here we review some definitions, notations and results. Standard group theory notation is being used (see e.g. [12]).

Let Q be a loop. Set $A = \{L_c \mid c \in Q\}$, $B = \{R_d \mid d \in Q\}$. Then A and B are left transversals to $\mathrm{Inn}\, Q$ in $\mathrm{Mlt}\, Q$, $\langle A, B\rangle = \mathrm{Mlt}\, Q$, $[A, B] \le \mathrm{Inn}\, Q$ and $\mathrm{core}_{\mathrm{Mlt}(Q)}\mathrm{Inn}\,(Q) = 1$ (i.e. the largest normal subgroup of $\mathrm{Mlt}\, Q$ in $\mathrm{Inn}\, Q$ is trivial). As a consequence $A \cap \mathrm{Inn}\, Q = B \cap \mathrm{Inn}\, Q = 1$ holds.

Conversely, consider a group G with the following properties: H is a subgroup of G, A and B are left transversals to H in G. A and B are *H-connected transversals* by definition if $[A, B] \le H$.

By a result of Kepka and Niemenmaa [14], the above two situations are equivalent:

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there is a subgroup H, for which there exist H-connected transversals A and B such that $\langle A, B\rangle = G$ and* $\mathrm{core}_G H = 1$.

Let Q be a loop and S be a normal subloop of Q. Put $\mathrm{M}(S) = \langle L_c, R_c \mid c \in S\rangle$. Then $\mathrm{M}(S)\mathrm{Inn}\, Q \le \mathrm{Mlt}\, Q$ (this is a standard fact). Put $K(S) = \mathrm{core}_{\mathrm{Mlt}\, Q}\, (\mathrm{M}(S)\mathrm{Inn}\, Q)$. Denote by f the natural homomorphism of $\mathrm{Mlt}\, Q$ onto $\mathrm{Mlt}\, Q/K(S)$. Then $f(A)$ and $f(B)$ are $f(\mathrm{Inn}\, Q)$-connected transversals in $\mathrm{Mlt}\, Q/K(S)$ and $\mathrm{Mlt}\, Q/K(S) \cong \mathrm{Mlt}\,(Q/S)$.

The permutation group generated by all left translations is called the left multiplication group and we shall denote it by $\mathscr{L} = \mathscr{L}(Q) = \langle A\rangle$. In a similar way the right multiplication group $\mathcal{R} = \mathcal{R}(Q) = \langle B\rangle$ is generated by all right translations. Let $\mathscr{L}_1 = \mathscr{L} \cap \mathrm{Inn}\, Q$, and $\mathcal{R}_1 = \mathcal{R} \cap \mathrm{Inn}\, Q$.

Denote

$$L(x,y) = L_{xy}^{-1} L_x L_y, \quad R(x,y) = R_{yx}^{-1} R_x R_y.$$

Proposition 2.2.

$$\mathscr{L}_1 = \langle L(x,y) \mid x, y \in Q,$$

$$\mathcal{R}_1 = \langle R(x,y) \mid x, y \in Q,$$

and $\mathrm{Inn}\, Q$ *is generated by* $\mathscr{L}_1 \cup \mathcal{R}_1 \cup \{T_x \mid x \in Q\}$ *where* $T_x = R_x^{-1} L_x$ *for all* $x \in Q$.

We say that Q is an A_l-loop (A_r-loop) if $\mathscr{L}_1 \le \mathrm{Aut}\, Q$ ($\mathcal{R}_1 \le \mathrm{Aut}\, Q$). A loop Q is an $A_{r,l}$-loop if it is both an A_r-loop and an A_l-loop.

The *left*, *middle* and *right* nucleus of a loop Q are defined, respectively, by

$N_\lambda = N_\lambda(Q) := \{a \in Q \mid a(xy) = (ax)y \text{ for all } x, y \in Q\}$,

$N_\mu = N_\mu(Q) := \{a \in Q \mid x(ay) = (xa)y \text{ for all } x, y \in Q\}$,

$N_\rho = N_\rho(Q) := \{a \in Q \mid x(ya) = (xy)a \text{ for all } x, y \in Q\}$.

The intersection $N = N(Q) = N_\lambda \cap N_\mu \cap N_\rho$ is called the *nucleus* of Q.

The *center* of Q is defined by $Z(Q) = \{a \in N \mid xa = ax \text{ for every element } x \in Q\}$.

By putting $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, where $Z(Q)$ denotes the center of Q, we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q but $Z_n = Q$, then Q is centrally nilpotent of class n.

The connection between the center of the loop Q and the center of the multiplication group $\mathrm{Mlt}\,Q$ is the following [1]:

$Z(\mathrm{Mlt}\,Q) = \{L_x \mid x \in Z(Q)\} = \{R_x \mid x \in Z(Q)\}$.

The commutant of Q is defined by $C(Q) = \{x \in Q \mid L_x = R_x\}$.

Proposition 2.3. *Let Q be a loop. Then:*

i) *Centralizer of* $\mathcal{R}$ *in* $\mathrm{Mlt}\,Q$

$C_{\mathrm{Mlt}\,Q}(\mathcal{R}) = \{L_c \mid c \in N_\lambda\}$,

where N_λ is the left nucleus of Q. Similarly,

$C_{\mathrm{Mlt}\,Q}(\mathcal{L}) = \{R_d \mid d \in N_\rho\}$,

where N_ρ is the right nucleus of Q.

ii) *If* $\mathcal{R} \trianglelefteq \mathrm{Mlt}\,Q$*, then* $C_{\mathrm{Mlt}\,Q}(\mathcal{R}) \trianglelefteq \mathrm{Mlt}\,Q$ *and* $N_\lambda \trianglelefteq Q$.

iii) *If* $\mathcal{L} \trianglelefteq \mathrm{Mlt}\,Q$*, then* $C_{\mathrm{Mlt}\,Q}(\mathcal{L}) \trianglelefteq \mathrm{Mlt}\,Q$ *and* $N_\rho \trianglelefteq Q$.

iv) $A_0A = A$, $B_0B = B$, *where* $A_0 = C_{\mathrm{Mlt}\,Q}(\mathcal{R})$, $B_0 = C_{\mathrm{Mlt}\,Q}(\mathcal{L})$.

Proof. i), ii), iii): see [4, Lemma 1.7]. iv) is trivial. □

The *associator subloop*, $A(Q)$ of Q is the least normal subloop of Q such that $Q/A(Q)$ is a group.

Lemma 2.4 ([7, Lemma 2.4]). $\mathrm{Inn}\,Q \cap N_{\mathrm{Mlt}\,Q}(A) = \mathrm{Inn}\,Q \cap N_{\mathrm{Mlt}\,Q}(B) = \mathrm{Aut}\,Q \cap \mathrm{Inn}\,Q$.

3. Moufang Loops

Let Q be a Moufang loop. Denote $G = \mathrm{Mlt}\,Q$, $H = \mathrm{Inn}\,Q$, $A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$. In the language of H-connected transversals the definition of Moufang loops is the following (some of them are equivalent):

$$aAa = A, \quad bBb = B \quad \text{for every} \quad a \in A,\ b \in B.$$
$$aA^{b^{-1}} = A, \quad bB^{a^{-1}} = B, \quad A^b a = A, \quad B^a b = B \text{ for every } a \in A,\ b \in B \cap aH.$$

As usual, we have $[A,B] \leq H$, $\langle A,B\rangle = G$, $\mathrm{core}_G H = 1$ and $A \cap H = B \cap H = e$,

$$A_0 = \{L_x \mid x \in N\},\ B_0 = \{R_x \mid x \in N\},$$

where $N = N(Q)$ denotes the nucleus of Q.

Proposition 3.1 ([7, Proposition 3.1]).

i) *For arbitrary $a \in A$, $b \in B \cap aH$, and for all integers k*

$$ab = ba,\ a^k \in A,\ b^k \in B \cap a^kH \text{ and } \sigma(a) = \sigma(b).$$

ii) $\langle A\rangle \cap H = \langle B\rangle \cap H = [A,B]$ *and*

$$H = \langle [A,B],\ a^{-1}b \mid a \in A,\ b \in B \cap aH\rangle$$

iii) $\langle A\rangle$, $\langle B\rangle$, $C_G(A)$, *and* $C_G(B)$ *are normal subgroups in G.*

iv) $N_\lambda = N_\rho = N_\mu = N$ *and* $N \trianglelefteq Q$.

$$B_0 = C_G(A) = \{R_x \mid x \in N\},$$
$$A_0 = C_G(B) = \{L_x \mid x \in N\}.$$

Lemma 3.2 ([7, Lemma 3.2]). i) $[a,b]^{a^j} \in H$, $[a,b]^{b^j} \in H$ *for arbitrary $a \in A$, $b \in B$ and for all integers j.*

ii) *Let* $a \in A, b \in B$ *such that* $[a,b] \in H \cap N_G(A)$. *Then* $a \in C_G([a, b])$ *and* $b \in C_G([a, b])$.

iii) *Let* $a \in A, b \in B$ *such that the order of a is relatively prime to the order of b,* $a \in C_G([a, b])$, *and* $b \in C_G([a, b])$, *then* $[a, b] = e$.

Proof. i), ii): see [6, Lemma 3.2]. i) is the consequence of diassociativity of Moufang loops.

iii) By $[A, B] \le H$ it follows $[a, b] = h \in H$. We have $a \in C_G(h)$, $b \in C_G(h)$. Hence $a^{(b^i)} = ah^i$. Let $o(b) = t$, then $a = ah^t$. As $o(a) = o(a^{(b^i)})$ and $a \in C_G(h)$, using $(o(a), o(b)) = 1$ we get $(o(h), o(b)) = 1$, whence it follows $[a, b] = e$. □

Proposition 3.3 ([7, Lemma 3.4]). *For every* $a, a_1 \in A$, $b \in B \cap aH$, $b_1 \in B \cap a_1H$

$$a{a_1}^{b^{-1}} = a^{b_1}a_1, \quad b{b_1}^{a^{-1}} = b^{a_1}b_1 \text{ hold.}$$

Proposition 3.4 ([7, Proposition 3.7]). *For every* $a_1, a_2 \in A$, $b_1 \in B \cap a_1H$, $b_2 \in B \cap a_2H$ *the following statements are true:*

i) $b_1b_2b_1^{-1}(b_2^{-1})^{a_1} \in N_G(A)$.

ii) $a_1a_2a_1^{-1}(a_2^{-1})^{b_1} \in N_G(B)$.

Theorem 3.5 ([7, Theorem 4.1]). *Let Q be a finite Moufang loop.* $N = N(Q)$ *is the nucleus of Q. Assume* $Z(Q/N) \neq 1$, *then* $Z(Q) \neq 1$.

Theorem 3.6 ([8, Theorem 5.1]). *Let Q be a Moufang loop of odd order. Suppose that the commutant of Q is not trivial. Then Q has nontrivial nucleus.*

Theorem 3.7 ([10, Theorem 5.2]). *Let Q be a Moufang loop of odd order. Then Q is centrally nilpotent if and only if the elements of coprime order of Q commute.*

Theorem 3.8 ([9, Theorem 4.3]). *Let Q be a Moufang loop with nucleus N. Suppose* $xN \in Z(Q/N)$ *for some* $x \in Q \setminus N$. *Then* $xZ(Q) \in N(Q/Z(Q))$ *if and only if* $L(y, x) \in Z(\mathrm{Inn}\, Q)$ *for every* $y \in Q$.

4. The Properties of the Multiplication Group of Moufang Loops of Odd Order with Trivial Nucleus

In this section Q is a Moufang loop of odd order with trivial nucleus.

Denote $G = \mathrm{Mlt}\, Q$, $H = \mathrm{Inn}\, Q$, $A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$. A and B are left (and right) transversals to H. We have $[A, B] \le H$, $\mathrm{core}_G H = 1$, $\langle A, B\rangle = G$, $A \cap H = B \cap H = 1$. By the definition of Moufang loops we have $a_1A^{b_1^{-1}} = A$, $b_1B^{a_1^{-1}} = B$ for every $a_1 \in A$, $b_1 \in B \cap a_1H$.

As Q has trivial nucleus, by Prop. 3.1 iv) we have $C_G(A) = C_G(B) = 1$.

Glauberman proved [11]: If K is a solvable π-subloop of a Moufang loop of odd order, then $\mathrm{Mlt}\, K$ the multiplication group of K is a solvable π-group. Then he showed that the Moufang loops of odd order are solvable.

Thus our Q is a solvable loop, $\mathrm{Mlt}\, Q$ is a solvable group and $|\mathrm{Mlt}\, Q|$ has the same prime divisors as $|Q|$, consequently $G = \mathrm{Mlt}\, Q$ is of odd order.

Q has trivial commutant, i.e. $A \cap B = 1$ (see Theorem 3.6).

Lemma 4.1 ([8, Lemma 4.1]). i) *B is a right (and left) transversal to* $N_G(A)$ *in G and* $N_G(A) \cap B = 1$.

ii) *A is a right (and left) transversal to* $N_G(B)$ *in G and* $N_G(B) \cap A = 1$.

Lemma 4.2 ([8, Lemma 4.2]). i) *A is a right (and left) transversal to* $N_G(A)$ *in G and* $A \cap N_G(A) = 1$.

ii) *B is a right (and left) transversal to* $N_G(B)$ *in G and* $B \cap N_G(B) = 1$.

Denote $C = \{ab \mid a \in A, b \in B \cap aH\}$.

Lemma 4.3 ([8, Lemma 4.7]). *C is a left (and right) transversal to H in G,* $C \cap H = 1$ *and* $H = N_G(C)$.

Lemma 4.4 ([8, Lemma 4.3]). $|H| = |N_G(A)| = |N_G(B)|$.

We define a mapping $\phi : G \to G$ in the following way:

Let $g \in G$ be arbitrary. By Lemma 4.2 g can be written uniquely in the form $g = na$ with $n \in N_G(A)$, $a \in A$. Define $\phi(g) = na^{-1}$.

Lemma 4.5 ([8, Lemma 4.4]).

i) $\phi \in \mathrm{Aut}\, G$, *the order of* ϕ, *i.e.* $o(\phi) = 2$.

ii) $C_G(\phi) = N_G(A)$.

iii) A *is the subset of all elements of G transformed into inverses by* ϕ, *i.e.,* $\phi(a) = a^{-1}$ *where* $a \in A$.

iv) $\phi(b) = ab$ *where* $b \in B$ *and* $a \in A \cap bH$, *i.e.* $\phi(B) = C$.

In a similar way we can define a mapping $\psi : G \to G$ in the following way:
Let g be arbitrary. As B is a right transversal to $N_G(B)$ (see Lemma 4.2) g can be written uniquely in the form $g = nb$ with $n \in N_G(B)$ and $b \in B$. Define $\psi(g) = nb^{-1}$.

Lemma 4.6 ([8, Lemma 4.5]).

i) $\psi \in \mathrm{Aut}\, G$, *the order of* ψ, *i.e.* $o(\psi) = 2$.

ii) $C_G(\psi) = N_G(B)$.

iii) B *is the subset of all elements of G transformed into inverses by* ψ, *i.e.,* $\psi(b) = b^{-1}$ *where* $b \in B$.

iv) $\psi(a) = ab$ *where* $a \in A$ *and* $b \in B \cap aH$, *i.e.* $\psi(A) = C$.

Lemma 4.7 ([8, Lemma 4.6]). i) $\phi(H) = N_G(B)$. ii) $\psi(H) = N_G(A)$.

Lemma 4.8 ([8, Lemma 4.8]). $\langle \phi, \psi \rangle \cong S_3$.

5. Main Results

In this section Q is a Moufang loop. Denote $G = \mathrm{Mlt}\, Q$, $H = \mathrm{Inn}\, Q$, $A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$. We have $\langle A, B \rangle = G$, $[A, B] \le H$, $\mathrm{core}_G H = 1$.

First we give a necessary and sufficient condition for the existence of non-

trivial nucleus in Moufang loops of odd order. This condition is in connection with $A_{e,r}$ property. In case of $A_{e,r}$ Moufang loop our earlier result concerning the nilpotence is the following:

Theorem 5.1 ([9, Theorem 4.15]). *Let Q be a Moufang loop. Assume Q is an* $A_{e,r}$ *-loop. Then Q is centrally nilpotent if and only if Q over the associator subloop i.e.* $Q/A(Q)$ *is nilpotent.*

Instead of $A_{e,r}$ -property we studied that case if for only one element, $x \in Q$, $x \neq 1$, $L(y, x) \in \mathrm{Aut}\, Q$ holds for every $y \in Q$:

Theorem 5.2. *Let Q be a Moufang loop of odd order. Then Q has nontrivial nucleus if and only if there exists* $1 \neq x \in Q$ *such that* $L(y, x) \in \mathrm{Aut}\, Q$ *for every* $y \in Q$.

For the proof of this theorem we need the following lemma:

Lemma 5.3. *Let Q be a Moufang loop,* $x, y \in Q$. *Then* $L(x,y) = (L_x^{-1})^{R_y^{-1}} L_x$.

Proof of Lemma 5.3. Denote $a_1 = L_x$, $b_1 = R_x$, $a_2^{-1} = L_y$, $b_2^{-1} = R_y$. By definition $a = a_2^{-1} a_1^{b_2} \in A$. As $[A, B] \le H$, $a_1^{b_2} = a_1 h_0$ with $h_0 \in H$. Since $L(y, x) = L_{yx}^{-1} L_y L_x \in H$ and $a^{-1} a_2^{-1} a_1 = h_0^{-1} \in H$ we get $L_{yx} = a$. Hence $L(y, x) = a^{-1} a_2^{-1} a_1 = (a_1^{-1})^{b_2} a_1 = (L_x^{-1})^{R_y^{-1}} L_x$. □

Proof of Theorem 5.2. First suppose Q has a nontrivial nucleus N. Let $x \in N$, $x \neq 1$, then $L_x \in C_G(B)$, $R_x \in C_G(A)$ (see Proposition 3.1). Let $y \in Q$ be arbitrary. By Lemma 5.3 $L(y, x) = (L_x^{-1})^{R_y^{-1}} L_x$. Using $L_x \in C_G(B)$, we get $L(y, x) = 1$, consequently $L(y, x) \in \mathrm{Aut}\, Q$ for every $y \in Q$.

Conversely, suppose there exists $1 \neq x \in Q$ such that $L(y, x) \in \mathrm{Aut}\, Q$ for every $y \in Q$.

Assume our Q has trivial nucleus, so we can apply the statements of Section 4 for our Q. Denote $a_1 = L_x$, $b_1 = R_x$, $a_2^{-1} = L_y$, $b_2^{-1} = R_y$. By $[A, B] \le H$ $a_1^{b_2} = a_1h_0$ with $h_0 \in H$. Using our Lemma 5.3 and Lemma 2.4 we get

$$L(y,x) = (a_1^{-1})^{b_2}a_1 = h_0^{-1} \in \mathrm{Aut}\, Q \cap N_G(B)$$

for every $b_2 \in B$. Hence $a_1^{b_2} = a_1 h_0$. Lemma 3.2 ii) gives $a_1^{b_2} = h_0 a_1$.

By Proposition 3.4 $a_1a_2a_1^{-1}(a_2^{-1})^{b_1} \in N_G(B)$. Hence $h_0 a_1a_2 a_1^{-1}(a_2^{-1})^{b_1} = a_1^{b_2}a_2a_1^{-1}(a_2^{-1})^{b_1} \in N_G(B)$. Denote $a_3 = a_1^{b_2}a_2$, $a_4 = a_1^{-1}(a_2^{-1})^{b_1}$. Using Proposition 3.3 and the definition we get $a_3, a_4 \in A$. Thus $a_3, a_4 \in N_G(B)$. By Lemma 4.7 we get $N_G(B) = H^{\varphi}$. Applying φ we get $a_3^{-1} a_4^{-1} \in H$ (see Lemma 4.5).

As A is left transversal to H, it follows $a_3^{-1} a_4^{-1} = 1$, i.e. $a_3 = a_4^{-1}$. Thus $a_1^{b_2}a_2 = a_2^{b_1}a_1$. Since $a_1^{b_2}a_2 = a_1a_2^{b_1^{-1}}$ (see again Proposition 3.3), then $a_1a_2 = (a_2 a_1)^{b_1^2}$ ($a_1 \in C_G(b_1)$) whence $a_1b_1^2 \in C_G(a_2a_1)$ for every $a_2 \in A$ and we can conclude $a_1b_1^2 \in C_G(A)$. Applying ψ it follows $a_1b_1^{-1} \in C_G(C)$ (see Lemma 4.6). As C is a left transversal to H (see Lemma 4.3) and $a_1b_1^{-1} \in H$ we can conclude $a_1b_1^{-1} \in \mathrm{core}_G H$. But $\mathrm{core}_G H = 1$, whence $a_1 = b_1$ i.e. $A \cap B \ne 1$, which means the commutant of Q is not trivial. Then by Theorem 3.6 Q has nontrivial nucleus. A contradiction. □

In the following we study the influence of the property $[A, B] \le Z$ (Inn Q) on the existence of nontrivial nucleus and to the nilpotence.

Our earlier results in connection with this property $[A, B] \le Z$ (Inn Q) are the following:

Theorem 5.4 ([9, Corollary 4.5]). *Let Q be a Moufang loop of odd order. Assume Q over the nucleus Q/N is an abelian group, then* $[A,B] \le Z$ (Inn Q).

Theorem 5.5 ([9, Theorem 4.6]). *Let Q be a Moufang loop. Assume $Q/Z(Q)$ is a group. Then the factor loop over the nucleus $Q/N(Q)$ is an abelian group if and only if* $[A,B] \le Z$ (Inn Q).

Theorem 5.6 ([9, Corollary 4.8]). *Let Q be a Moufang loop such that the associator subloop does not contain element of order three. Assume $Q/Z(Q)$ is a group. Then* $[A, B] \le Z$ (Inn Q).

We give a sufficient condition for the existence of nontrivial nucleus in Moufang loops of odd order.

Theorem 5.7. Let Q be a Moufang loop of odd order. Suppose $[A, B] \le Z$ (Inn Q). *Then Q has nontrivial nucleus.*

Proof. Let Q be a counterexample, i.e. Q has trivial nucleus. So we can apply the statements of Section 4 for our Q.

Let $a \in A$, $b^* \in B$ such that $\sigma(a)$ is relatively prime to the order of b^*. We show a and b^* commute. As $[A, B] \le H$, $[a, b^*] = h \in H$. Let $b \in B \cap aH$. Clearly $ba^{-1} \in H$, hence our condition $[A, B] \le Z(\mathrm{Inn}\, Q) = Z(H)$ gives $h^{ba^{-1}} = h$, i.e. $h^b = h^a$.

We show $h^a = h^{b^{-1}}$. By Lemma 3.2 $h^a = [a, b^*]^a \in H$. We have $h^{\varphi} \in N_G(B)$ (see Lemma 4.7). By Lemma 4.6 $C_G(\psi) = N_G(B)$ we get $h^{\phi\psi} = h^{\phi}$. Hence using the fact ϕ is of order 2, it follows $h^{\phi\psi\phi} = h$. Since $h^a \in H$, then similarly we get $(h^a)^{\phi\psi\phi} = h^a$. Using Lemma 4.5 and Lemma 4.6 $a^{\phi\psi\phi} = (a^{-1})^{\psi\phi} = (a^{-1}b^{-1})^{\varphi} = b^{-1}$. Thus $(h^a)^{\phi\psi\phi} = h^{b^{-1}} = h^a$.

As $h^a = h^b$ we can conclude $h^{b^2} = h$. Using b is of odd order, it follows that $b \in C_G(h)$, whence $a \in C_G(h)$. Similarly we can show $b^* \in C_G(h)$, i.e. $a \in C_G([a, b^*])$, $b^* \in C_G([a, b^*])$. Since we have the order of a is relatively prime to the order of b^*, using Lemma 3.2 iii) we get $[a, b^*] = e$, i.e. a and b^* commute.

Denote $L_x = a$, $R_y = b^*$. Then we have $L_xR_y = R_y L_x$. Hence $L_x (R_y (1)) = R_y L_x (1)$ which means $x(1y) = y(1x)$, consequently $xy = yx$. We have $\sigma(L_x) = \sigma(x)$, $\sigma(R_y) = \sigma(y)$. As $\sigma(L_x)$ and $\sigma(R_y)$ are relatively prime, further x and y are arbitrary elements, by Theorem 3.7 Q is centrally nilpotent, consequently $Z(Q) \ne 1$, but $1 \ne Z(Q) \le N(Q)$ which gives a contradiction.

Theorem 5.8. *Let Q be a Moufang loop of odd order. Assume* $[A, B] \le Z$ (Inn Q). *Then Q is centrally nilpotent if and only if $Q/A(Q)$ is centrally nilpotent.*

Proof. Clearly the centrally nilpotence of Q implies the centrally nilpotence of $Q/A(Q)$.

Conversely, let Q be a counterexample of smallest order. By Theorem 5.7 we have $N(Q) \neq 1$. Suppose first $N(Q) = Q$, then Q is a group and $A(Q) = 1$, consequently Q is centrally nilpotent, a contradiction. Thus $N(Q) \neq Q$. Clearly the factor loop $Q/N(Q)$ satisfies every condition of our theorem. Since $|Q/N(Q)| < |Q|$ the minimality of Q implies the centrally nilpotence of $Q/N(Q)$. Hence $Z(Q/N(Q)) \neq 1$. Theorem 3.5 results $Z(Q)$ is not trivial. Clearly $Q^* = Q/Z(Q)$ satisfies every condition of our theorem. The minimality of Q gives the centrally nilpotence of Q^*, consequently Q is centrally nilpotent. This is the final contradiction. □

Studying the nilpotency class of Moufang loops of odd order with the property $[A, B] \leq Z(Q)$ we get the following result:

Corollary 5.9. *Let Q be a Moufang loop of odd order such that* $[A, B] \leq Z(\mathrm{Inn}\, Q)$. *Let N be the nucleus of Q. Assume the nilpotency class* $\mathrm{cl}\, Q/N = k$. *Then $A(Q)$ is centrally nilpotent and* $\mathrm{cl}\, A(Q) \leq k$.

Proof. Let Q be a minimal counterexample. By Theorem 5.7 we have $N(Q) \neq 1$. Suppose first, $N(Q) = Q$. Then Q is a group, and $A(Q) = 1$, i.e. the statement holds, a contradiction. Thus $N(Q) \neq Q$. Clearly the factor loop $Q/N(Q)$ satisfies every condition of our theorem. Since $|Q/N(Q)| < |Q|$ the minimality of Q implies the centrally nilpotence of $Q/N(Q)$. Hence $Z(Q/N(Q)) \neq 1$. Theorem 3.5 results that $Z(Q) \neq 1$.

Denote $Q_1 = Q/Z(Q)$. The property $[A, B] \leq Z(\mathrm{Inn}\, Q)$ implies $L(x, y) \in Z(\mathrm{Inn}\, Q)$ for every $x, y \in Q$. By using Theorem 3.8 it follows $\mathrm{cl}\, Q_1/N(Q_1) \leq k - 1$. Clearly Q_1 satisfies every condition of our theorem. The minimality of Q implies $\mathrm{cl}\, A(Q_1) \leq k - 1$, consequently $\mathrm{cl}\, A(Q) \leq k$. A contradiction.

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