International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 6, Issue 8, 2018, PP 42-53
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0608004
www.arcjournals.org

Existence, Uniqueness and Stability of Hilfer Type Neutral Pantograph Differential Equations with Nonlocal Conditions

D. Vivek[1], E. M. Elsayed[2*], K. Kanagarajan[3]

[1]*Department of Mathematics CA, Sri Ramakrishna College of Arts and Science, Coimbatore-641006, India.*

[2]*Department of Mathematics, Faculty of Science,King Abdulaziz University, Jeddah, Saudi Arabia. Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt.*

[3]*Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science,Coimbatore India*

***Corresponding Author:** *E. M. Elsayed, Department of Mathematics, Faculty of Science,King Abdulaziz University, Jeddah, Saudi Arabia. Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt.*

Abstract: *A study on dynamics and stability of nonlinear neutral pantograph equations with Hilfer fractional derivative is the main aim of this paper. The existence result is obtained by using Scheafer's fixed point theorem. In addition we discuss Ulam stability of the system by employing Banach contraction principle.*

Keywords: *Generalized Riemann-Lioville fractional derivative; nonlinear neutral pantograph equations; fixed point; Generalized Ulam-Hyers stability.*

1. Introduction

In recent years, the theory of fractional differential equations has played a very important role in a new branch of applied mathematics, which has been utilized for mathematical models in engineering, physics, chemistry, signal analysis, etc. There has been a tremendous development in the study of differential equations involving fractional derivatives (see [4, 12, 16], and the references therein). Recently, a study of Hilfer type of equation has received a significant amount of attention, we refer to [5, 6, 11, 12, 13, 20].

The Ulam stability of functional equation, which was invented by Ulam on a talk given to a conference at Wisconsin University in 1940, is one of the essential subjects in the mathematical analysis area. The finding of Ulam stability plays a pivotal role in regard to this subject. For extensive study on the advance of Ulam type stability, readers refer to [1, 9, 8] and the references therein. The credit of solving this problem partially goes to Hyers. To study Hyers-Ulam stability of fractional differential equations, different researchers presented their works with different methods, see [7, 18, 19].

The aim of this paper is to study pantograph equation with nonlocal conditions involving Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = f(t, x(t), x(\lambda t), D_{0^+}^{\alpha,\beta}x(\lambda t)), \quad 0<\lambda<1,\ t\in[0,T], \\ I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \alpha\le\gamma=\alpha+\beta-|\alpha\beta<1,\ \tau_i\in[0,T], \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $0 < \alpha < 1$, $0 \le \beta \le 1$, $0 < \lambda < 1$ and let X be a Banach space, $f : J \times X \times X \times X \to X$ is given continuous function.

In passing, we remark that the application of nonlocal condition $I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^{m} c_i x(\tau_i)$ in physical problems yields better effect than the initial condition $I_{0^+}^{1-\gamma}x(0) = x_0$.

The pantograph equations have been studied extensively (see, [2, 14, 15] and references there in) since they can be used to describe many phenomena arising in number theory, dynamical systems, probability, quantum mechanics, and electro dynamics. Recently, fractional pantograph differential equations have been studied by many researchers. One of interesting subjects in this area, is the investigation of the existence of solutions by fixed point theorems, we refer to [2].

The pantograph type is one of the special types of delay differential equations, and growing attention is given to its analysis and numerical solution. Pantograph type always has the delay term fall after the initial value but before the desire approximation being calculated. When the delay term of pantograph type involved with the derivative(s), the equation is named as neutral delay differential equation of pantograph type. Recently, Vivek et. al.[17] investigated the existence and Ulam-Hyers stability results for pantograph differential equations with Hilfer fractional derivative. The novelty of this paper is that existence and stability results devoted to neutral pantograph equations with Hilfer fractional derivative.

The outline of the paper is as follows. In Section 2, we give some basic definitions an results concerning the Hilfer fractional derivative. In Section 3, we present our main result by using Schaefer's fixed point theorem. In section 4, we discuss stability analysis.

2. Preliminaries

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel.

For more details, we refer to [5, 6, 12, 13, 20].

Definition 2.1. The left-sided Riemann-Liouville fractional integral of order $\alpha \in R+$ of function f(t) is defined by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad (t > 0),$$

where ℵ(•) is the Gamma function.

Definition 2.2. The left-sided Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z+$ of function f(t) is defined by

$$(D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{\alpha-1} f(s)ds, \ (t > 0),$$

Definition 2.3. The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function f(t) is defined by

$$D_{0+}^{\alpha,\beta} f(t) = \left(I_{0+}^{\beta(1-\alpha)} D \left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)\right)(t),$$

$$where\ D := \frac{d}{dx}.$$

The Hilfer fractional derivative is considered as an interpolator between the Riemann- Liouville and Caputo derivative, then the following remarks can be presented to show the relation with Caputo and Riemann-Liouville operators.

Remark 2.4. 1. The operator al$D_{0+}^{\alpha,\beta}$ε written as

$$D_{0+}^{\alpha,\beta} = I_{0+}^{\beta(1-\alpha)} D I_{0+}^{(1-\beta)(1-\alpha)} = I_{0+}^{\beta(1-\alpha)} D_{0+}^{\gamma}, \ \gamma = \alpha + \beta - \alpha\beta.$$

2. Let $\beta = 0$, the left-sided Riemann-Liouville fractional derivative can be presented as

$$D_{0+}^{\alpha} := D_{0+}^{\alpha,0}.$$

3.Let $\beta = 1$, the left-sided Caputo fractional derivative can be presented as

$${}^{c}D_{0+}^{\alpha} := I_{0+}^{1-\alpha} D.$$

Secondly, we need the following basic work spaces. Let C[J,X] be the Banach space of all continuous functions from [J,X] into X with the

norm $\|x\|_c = \max\{|x(t)| : t \in [0,T]\}$. For $0 \le \gamma < 1$, we denote the space $C_\gamma[J,X]$ as

$$C_\gamma[J,X] := \{f(t) : [0,T] \to X | t^\gamma f(t) \in C[J,X]\},$$

where C_γ [J, X] is the weighted space of the continuous functions f on the finite interval [0, T].

Obviously, $C\gamma$ [J, X] is the Banach space with the norm

$$\|f\|_{C_\gamma} = \|t^\gamma f(t)\|_C .$$

Meanwhile, $C_\gamma^n[J,X] := \{f \in C^{n-1}[J,X] : f^{(n)} \in C_\gamma[J,X]\}$ is the Banach space with the norm

$$\|f\|_{C_\gamma^n} = \sum_{i=0}^{n-1} \left\|f^k\right\|_C + \left\|f^{(n)}\right\|_{C_\gamma}, \ n \in \mathbb{N}.$$

Moreover, C_γ [J, X] := C_γ [J, X].

Lemma 2.5. If α > 0 and β > 0, there exist

$$\left[I_{0+}^\alpha s^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+1)} t^{\beta+\alpha-1}$$

And

$$\left[D_{0+}^\alpha s^{\alpha-1}\right](t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.6. If α > 0, β > 0, and $f \in L^1(R^+)$, for t ∈ [0, T] there exist the following properties

$$(I_{0^+}^{\alpha} I_{0^+}^{\beta} f)(t) = (I_{0^+}^{\alpha+\beta} f)\ (t)$$

and

$$(D_{0^+}^{\alpha} I_{0^+}^{\alpha} f)\ (t) = f\ (t).$$

In particular, if f ∈ C_γ [J, X] or f ∈ C[J, X], then these equalities hold at t ∈ [0, T].

Lemma 2.7. Let 0 < α < 1, 0 ≤ γ < 1. If f ∈ C_γ [J, X] and $I_{0^+}^{1-\alpha} f \in C_\gamma^1[J, X]$, then

$$I_{0+}^\alpha D_{0+}^\alpha f(t) = f(t) - \frac{I_{0+}^{1-\alpha} f(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \forall\ t \in [0,T].$$

Lemma 2.8. For 0 ≤ γ < 1 and f ∈ C_γ [J, X], then

$$I_{0+}^\alpha f(0) := \lim_{t \to 0+} I_{0+}^\alpha f(t) = 0, \quad 0 \le \gamma < \alpha.$$

In order to solve our problem, the following spaces are given

$$C_{1-\gamma}^{\alpha,\beta} = \left\{f \in C_{1-\gamma}[J,X],\, D_{0+}^{\alpha,\beta} f \in C_{1-\gamma}[J,X]\right\}$$

and

$$C_{1-\gamma}^{\gamma} = \{f \in C_{1-\gamma}[J,X],\, D_{0+}^{\gamma} f \in C_{1-\gamma}[J,X]\}.$$

It is obvious that

$$C_{1-\gamma}^{\gamma}[J,X] \subset C_{1-\gamma}^{\alpha,\beta}[J,X].$$

Lemma 2.9. Let α > 0, β > 0, and γ = α + β − αβ. If $f \in C_{1-\gamma}^{\gamma}$ [J, X], then

$I_{0^+}^{\gamma} D_{0^+}^{\gamma} f = I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f$, $D_{0^+}^{\gamma} I_{0^+}^{\alpha} f = D_{0^+}^{\beta(1-\alpha)} f$ (t).

Lemma 2.10. Let $f \in L^1(R_+)$ and $D_{0^+}^{\beta(1-\alpha)} f \in L^1(R_+)$ existed, then

$D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} f = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\beta(1-\alpha)} f.$

Lemma 2.11. [5] Let $f : J \times X \to X$ be a function such that $f \in C_{1-\gamma}[J, X]$ for any $x \in C_{1-\gamma}[J, X]$. A function $x \in C_{1-\gamma}^{\gamma}[J, X]$ is a solution of fractional initial value problem:

$$\begin{cases} D_{0+}^{\alpha,\beta} x(t) = f(t, x(t)), & 0 < \alpha < 1, \ 0 \leq \beta \leq 1, \\ I_{0+}^{1-\gamma} x(0) = x_0, & \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

if and only if x satisfies the following Volterra integral equation:

$$x(t) = \frac{x_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds.$$

Lemma 2.12. Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}[J, X]$ and $I_{0^+}^{1-\beta(1-\alpha)} f$ in $C_{1-\gamma}^{1}[J, X]$, then $I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f$ exists in J and

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f(t) = f(t).$$

Proof. By Lemma 2.9, we have

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f(t) = I_{0^+}^{\gamma} D_{0^+}^{\gamma} f(t),$$

and applying Lemma 2.8 and Lemma 2.7

$$I_{0+}^{\alpha} D_{0+}^{\alpha,\beta} f(t) = f(t) - \frac{I_{0+}^{1-\gamma} f(0)}{\Gamma(\gamma)} t^{\gamma-1}.$$

Finally, we get

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} f(t) = f(t).$$

Lemma 2.13. [3] Let $v : [0, T] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^{\alpha}} ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^{\alpha}} ds,$$

for every $t \in [0, T]$.

According to Lemma 2.11, a new and important equivalent mixed type integral equation for our system (1) can be established. We adopt some ideas in [20] to establish an equivalent mixed type integral equation:

$$x(t) = \frac{Zt^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) ds \tag{2}$$

Where

$$Z := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} c_i (\tau_i)^{\gamma-1}}, \quad \text{if} \quad \Gamma(\gamma) \neq \sum_{i=1}^{m} c_i (\tau_i)^{\gamma-1}, \tag{3}$$

$$K_x(t) := D_{0+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\alpha,\beta} x(\lambda t)). \tag{4}$$

Lemma 2.14. Let $f : J \times X \times X \times X \to X$ be a function such that $f \in C_{1-\gamma}[J, X]$ for any $x \in C_{1-\gamma}[J, X]$. A function $x \in C_{1-\gamma}^{\gamma}[J, X]$ is a solution of the system (1) if and only if x satisfies the mixed type integral (2).

Proof. According to Lemma 2.11, a solution of system (1) can be expressed by

$$x(t) = \frac{I_{0^+}^{1-\gamma}x(0)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)ds. \tag{5}$$

Next, we substitute $t = \tau_i$ into the above equation,

$$x(\tau_i) = \frac{I_{0^+}^{1-\gamma}x(0)}{\Gamma(\gamma)} (\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds, \tag{6}$$

by multiplying c_i to both sides of (6), we can write

$$c_i x(\tau_i) = \frac{I_{0^+}^{1-\gamma}x(0)}{\Gamma(\gamma)} c_i(\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds.$$

Thus we have

$$\begin{aligned} I_{0^+}^{1-\gamma}x(0) &= \sum_{i=1}^m c_i x(\tau_i) \\ &= \frac{I_{0^+}^{1-\gamma}x(0)}{\Gamma(\gamma)} \sum_{i=1}^m c_i(\tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \end{aligned}$$

which implies

$$I_{0^+}^{1-\gamma}x(0) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds. \tag{7}$$

Submitting (7) to (5), we derive that (2). It is probative that x is also a solution of the integral equation (2), when x is a solution of (1).

The necessity has been already proved. Next, we are ready to prove its sufficiency. Applying $I_{0^+}^{1-\gamma}$ to both sides of (2), we have

$$\begin{aligned} I_{0^+}^{1-\gamma}x(t) &= I_{0^+}^{1-\gamma} t^{\gamma-1} \frac{Z}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \\ &\quad + I_{0^+}^{1-\gamma} I_{0^+}^{\alpha} K_x(t), \end{aligned}$$

using the Lemmas 2.5 and 2.6,

$$\begin{aligned} I_{0^+}^{1-\gamma}x(t) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \\ &\quad + I_{0^+}^{1-\beta(1-\alpha)} K_x(t). \end{aligned}$$

Since $1 - \gamma < 1 - \beta(1 - \alpha)$, Lemma 2.8 can be used when taking the limits as $t \to 0$,

$$I_{0^+}^{1-\gamma}x(0) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds. \tag{8}$$

Substituting $t = \tau_i$ into (2), we have

$$\begin{aligned} x(\tau_i) &= \tfrac{Z}{\Gamma(\alpha)} (\tau_i)^{\gamma-1} \textstyle\sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \\ &\quad + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds. \end{aligned}$$

Then, we derive

$$\begin{aligned} \sum_{i=1}^m c_i x(\tau_i) &= \frac{Z}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \sum_{i=1}^m c_i(\tau_i)^{\gamma-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds \left(1 + Z\sum_{i=1}^m c_i(\tau_i)^{\gamma-1}\right) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds, \end{aligned}$$

that is

$$\sum_{i=1}^m c_i x(\tau_i) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i-s)^{\alpha-1} K_x(s)ds. \tag{9}$$

It follows (8) and (9) that

$$I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^m c_i x(\tau_i).$$

Now by applying $D0^{\gamma}_{+}$ to both sides of (2), it follows from Lemma 2.5 and 2.9 that

$$D^{\gamma}_{0+}x(t) = D^{\beta(1-\alpha)}_{0+}K_x(t). \tag{10}$$

Since $x \in C_1^{\gamma}{}_{-\gamma}$ [J, X] and by the definition of $C_1^{\gamma}{}_{-\gamma}$ [J, X], we have $D_0{}^{\gamma}{}_{+}\ x \in C_{1-\gamma}$ [J, X], then, $D_0{}^{\beta}{}_{+}{}^{(1-\alpha)}f = DI_0{}^{1}{}_{+}{}^{-\beta(1-\alpha)}f \in C_{1-\gamma}$ [J, X]. For $f \in C_{1-\gamma}$ [J, X], it is obivious that $I_0{}^{1}{}_{+}{}^{-\beta(1-\alpha)}f \in C_{1-\gamma}$ [J, X], then $I_0{}^{1}{}_{+}{}^{-\beta(1-\alpha)}f \in C_1{}^{1}{}_{-\gamma}$ [J, X]. Thus f and $I_0{}^{1}{}_{+}{}^{-\beta(1-\alpha)}f$ satisfy the conditions of Lemma 2.7.

Next, by applying $I_0{}^{\beta}{}_{+}{}^{(1-\alpha)}$ to both sides of (10) and using Lemma 2.7, we can obtain

$$D^{\alpha;\beta}_{0+}x(t) = K_x(t) - \frac{I^{1-\beta(1-\alpha)}_{0+}K_x(0)}{\Gamma(\beta(1-\alpha))}(t)^{\beta(1-\alpha)-1},$$

where $I_0{}^{\beta}{}_{+}{}^{(1-\alpha)}K_x(0) = 0$ is implied by Lemma 2.8.

Hence, it reduces to $D_0{}^{\alpha,\beta}{}_{+}\ x(t) = K_x(t) = f\ (t, x(t), x(\lambda t), D_0{}^{\alpha,\beta}{}_{+}\ x(\lambda t))$. The results are proved completely.

3. Existence Results

First we list the following hypothesis:

(H1) The function $f : J \times X \times X \times X \to X$ is continuous.

(H2) There exist l, p, q, $r \in C_{1-\gamma}$ [J, X] with $l^* = \sup_{t\in J} l(t) < 1$ such that

$|f(t, u, v, w)| \le l(t) + p(t)\,|u| + q(t)\,|v| + r(t)\,|w|$

for $t \in J$, $u, v, w \in X$.

Theorem 3.1. Assume the hypothesis (H1) and (H2) are satisfied. Then, the system (1) has at least one solution in $C_1{}^{\gamma}{}_{-\gamma}$ [J, X] ⊂ $C_1{}^{\alpha,\beta}{}_{-\gamma}$ [J, X].

Proof. The proof will be given in several steps.

Consider the operator $N : C_{1-\gamma}$ [J, X] → $C_{1-\gamma}$ [J, X].

$$(Nx)(t) = \frac{Z}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i \int_0^{\tau_i}(\tau_i - s)^{\alpha-1}K_x(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_x(s)ds \tag{11}$$

It is obvious that the operator N is well defined.

Claim 1: N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}$ [J, X]. Then for each $t \in J$,

$$\begin{aligned}
|((Nx_n)(t) - (Nx)(t))t^{1-\gamma}| &\le \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_0^{\tau_i}(\tau_i - s)^{\alpha-1}\,|K_{x_n}(s) - K_x(s)|\,ds \\
&\quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|K_{x_n}(s) - K_x(s)|\,ds \\
&\le \frac{|Z|\,B(\gamma,\alpha)\sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)}\,\|K_{x_n}(\cdot) - K_{x_n}(\cdot)\|_{C_{1-\gamma}} \\
&\quad + \frac{T^{\alpha}B(\gamma,\alpha)}{\Gamma(\alpha)}\,\|K_{x_n}(\cdot) - K_x(\cdot)\|_{C_{1-\gamma}} \\
&\le \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^{\alpha}\right)\|K_{x_n}(\cdot) - K_x(\cdot)\|_{C_{1-\gamma}}
\end{aligned}$$

where we use the formula

$$\begin{aligned}
\int_a^t (t-s)^{\alpha-1}\,|u(s)|\,ds &\le \left(\int_a^t (t-s)^{\alpha-1}(s-a)^{\gamma-1}ds\right)\|u\|_{C_{1-\gamma}} \\
&= (t-a)^{\alpha+\gamma-1}B(\gamma,\alpha)\,\|u\|_{C_{1-\gamma}}
\end{aligned}$$

since f is continuous, then we have

$$\|Nx_n - Nx\|_{C_{1-\gamma}} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2: N maps bounded sets into bounded sets in $C_{1-\gamma}$ [J, X]. Indeed, it is enough to show that for q > 0, there exists a positive constant l such that

$x \in B_q \{x \in C_{1-\gamma}[J, X] : \|x\| \leq q\}$, we have $\|N(x)\|_{C_{1-\gamma}} \leq l$.

$$|(Nx)(t)t^{1-\gamma}| \leq \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |K_x(s)| \, ds$$
$$:= A_1 + A_2.$$

and

$$\begin{aligned} |K_x(t)| &= |f(t, x(t), x(\lambda t), K_x(t))| \\ &\leq l(t) + p(t) |x(t)| + q(t) |x(\lambda t)| + r(t) |K_x(t)| \\ &\leq l^* + p^* |x(t)| + q^* |x(\lambda t)| + r^* |K_x(t)| \\ &\leq \frac{l^* + p^* |x(t)| + q^* |x(\lambda t)|}{1 - r^*}. \end{aligned} \tag{12}$$

Where

$$A_1 = \frac{|Z|}{(1 - r^*)} \sum_{i=1}^{m} c_i \left(\frac{l^*(\tau_i)^\alpha}{\Gamma(\alpha+1)} + (p^* + q^*) \frac{(\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)} B(\gamma, \alpha) \|x\|_{C_{1-\gamma}} \right),$$
$$A_2 = \frac{1}{1 - r^*} \left(\frac{l^* T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + (p^* + q^*) \frac{T^\alpha}{\Gamma(\alpha)} B(\gamma, \alpha) \|x\|_{C_{1-\gamma}} \right).$$

From (12), we have

$$\begin{aligned} |(Nx)(t)t^{1-\gamma}| \leq & \frac{l^*}{(1 - r^*)\Gamma(\alpha+1)} \left(|Z| \sum_{i=1}^{m} c_i(\tau_i)^\alpha + T^{\alpha+\gamma-1} \right) \\ & + \frac{(p^* + q^*)}{(1 - r^*)\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^\alpha \right) B(\gamma, \alpha) \|x\|_{C_{1-\gamma}} \\ := & \, l. \end{aligned}$$

Claim 3: N maps bounded sets into equicontinuous set of $C_{1-\gamma}$ [J, X].

Let $t_1, t_2 \in J$, $t_2 \leq t_1$ and $x \in B_q$. Using the fact f is bounded on the compact set

$J \times B_q$ (these $\sup_{(t,x) \in J \times B_r} \|K_x(t)\| := C_0 < \infty$) , we will get

$$\begin{aligned} |(Nx)(t_1) - (Nx)(t_2)| \leq & \frac{C_0 |Z| B(\gamma, \alpha) \sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \left| t_1^{\gamma-1} - t_2^{\gamma-1} \right| \\ & + \frac{C_0 B(\gamma, \alpha)}{\Gamma(\alpha)} \left| t_1^{\alpha+\gamma-1} - t_2^{\alpha+\gamma-1} \right| \\ \leq & \frac{C_0 |Z| B(\gamma, \alpha) \sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \left| \frac{t_2 - t_2}{t_1 t_2} \right|^{1-\gamma} \\ & + \frac{C_0 B(\gamma, \alpha)}{\Gamma(\alpha)} \left| t_1^{\alpha+\gamma-1} - t_2^{\alpha+\gamma-1} \right|. \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of claim 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $N : C_{1-\gamma} [J, X] \to C_{1-\gamma} [J, X]$ is continuous and completely continuous. Claim 4: A priori bounds.

Now it remains to show that the set

$$\omega = \{x \in C_{1-\gamma}[J, X] : x = \delta N(x), \quad 0 < \delta < 1\}$$

is bounded set.

Let $x \in \omega$, $x = \delta N(x)$ for some $0 < \delta < 1$. Thus for each $t \in J$. We have

$$x(t) = \delta \left[\frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} K_x(s) ds \right].$$

This implies by (H2)that for each $t \in J$, we have

$$\begin{aligned} |x(t)t^{1-\gamma}| \leq & \|(Nx)(t)t^{1-\gamma}\| \\ \leq & \frac{l^*}{(1 - r^*)\Gamma(\alpha+1)} \left(|Z| \sum_{i=1}^{m} c_i(\tau_i)^\alpha + T^{\alpha+\gamma-1} \right) \\ & + \frac{(p^* + q^*)}{(1 - r^*)\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i(\tau_i)^{\alpha+\gamma-1} + T^\alpha \right) B(\gamma, \alpha) \|x\|_{C_{1-\gamma}} \\ := & \, R. \end{aligned}$$

This shows that the set ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of problem (1).

4. STABILITY ANALYSIS

Next, we shall give the definitions and the criteria of Ulam-Hyers stability and Ulam-Hyers-Rassias stability for nonlinear neutral pantograph equations under Hilfer fractional derivative. We use some ideas from [10].

Definition 4.1. The equation (1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of the inequality

$$\left|D_{0+}^{\alpha;\beta}z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t))\right| \leq \epsilon, \quad t \in J,$$

there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, X]$ of equation (1) with

$$|z(t) - x(t)| \leq C_f \epsilon, \quad t \in J.$$

Definition 4.2. The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C_{1-\gamma}[J, X]$, $\psi(0) = 0$ such that for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of the inequality

$$\left|D_{0+}^{\alpha;\beta}z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t))\right| \leq \epsilon, \quad t \in J,$$

there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, X]$ of equation (1) with

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 4.3. The equation (1) is Ulam-Hyers-Rassias stable with respect to $\phi \in C_{1-\gamma}[J, X]$ if there exists a real number $C_f > 0$ such that for each $\varrho > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of the inequality

$$\left|D_{0+}^{\alpha;\beta}z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t))\right| \leq \epsilon\varphi(t), \quad t \in J,$$

there exists a solution $x \in C_{1-\gamma}^{\gamma}[J, X]$ *of equation* (1) *with*

$$|z(t) - x(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. A function $z \in C_{1-\gamma}^{\gamma}[J, X]$ is a solution of the inequality

$$\left|D_{0+}^{\alpha;\beta}z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t))\right| \leq \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma}^{\gamma}[J, X]$ such that

(i) $\quad |g(t)| \leq \epsilon, t \in J,$

(ii) $\quad D_{0+}^{\alpha;\beta}z(t) = f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t)) + g(t), t \in J,$

Lemma 4.6. If a function $z \in C_{1-\gamma}^{\gamma}[J, X]$ is a solution of the inequality

$$\left|D_{0+}^{\alpha;\beta}z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha;\beta}z(\lambda t))\right| \leq \epsilon, \quad t \in J,$$

then with $A_x = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) ds.$

$$\left|z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) ds\right| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}. \tag{13}$$

Proof. The proof directly follows from Remark 4.5 and Lemma 2.14

Remark 4.7. 1. Definition 4.1$\Rightarrow$ Definition 4.2.

2. Definition 4.3$\Rightarrow$ Definition 4.4.

We ready to prove our stability results for problem (1). The arguments are based on the Banach contraction principle. First we list the following hypothesis:

(H3) Let $f : J \times X \times X \times X \to X$ be a function such that $f \in C_{1-\gamma}^{\beta(1-\alpha)}[J, X]$ for any x in $C_{1-\gamma}^{\gamma}[J, X]$ and there exist positive constants $K > 0$ and $L > 0$ such that

$$|f(t, u, v, w) - f(t, \overline{u}, \overline{v}, \overline{w})| \leq K (|u - \overline{u}| + |v - \overline{v}|) + L |w - \overline{w}|$$

for any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in X$ and $t \in J$.

(H4) There exists an increasing function $\varphi \in C_{1-\gamma}[J, X]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{0+}^{\alpha}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 4.8. Assume that hypothesis (H1) and (H3) are fulfilled. If

$$\frac{2K}{(1-L)} \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^\alpha \right) < 1, \tag{14}$$

then the system (1) has a unique solution.

Proof. Let the operator $N : C_{1-\gamma}[J, X] \to C_{1-\gamma}[J, X]$.

$$(Nx)(t) = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) ds$$

By Lemma 2.14, it is clear that the fixed points of N are solutions of system (1).

Let $x_1, x_2 \in C_{1-\gamma}[J, X]$ and $t \in J$, then we have

$$\begin{aligned} |((Nx_1)(t) - (Nx_2)(t))\, t^{1-\gamma}| \leq{}& \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |K_{x_1}(s) - K_{x_2}(s)|\, ds \\ &+ \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_{x_1}(s) - K_{x_2}(s)|\, ds, \end{aligned} \tag{15}$$

And

$$\begin{aligned} |K_{x_1}(t) - K_{x_2}(t)| &= |f(t, x_1(t), x_1(\lambda t), K_{x_1}(t)) - f(t, x_2, x_2(\lambda t), K_{x_2}(t))| \\ &\leq K (|x_1(t) - x_2(t)| + |x_1(\lambda t) - x_2(\lambda t)|) + L |K_{x_1}(t) - K_{x_2}(t)| \\ &\leq \frac{2K}{(1-L)} |x_1(t) - x_2(t)|. \end{aligned} \tag{16}$$

By replacing (16) in the inequality (15), we get

$$\begin{aligned} |((Nx_1)(t) - (Nx_2)(t))\, t^{1-\gamma}| &\leq \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \left(\frac{2K}{(1-L)} B(\gamma, \alpha) (\tau)^{\alpha+\gamma-1} \|x_1 - x_2\|_{C_{1-\gamma}} \right) \\ &\quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)} t^{\alpha+\gamma-1} \left(\frac{2K}{1-L} \right) B(\gamma, \alpha) \|x_1 - x_2\|_{C_{1-\gamma}} \\ &\leq \left(\frac{2K}{1-L} \right) \frac{1}{\Gamma(\alpha)} B(\gamma, \alpha) \left(|Z| \sum_{i=1}^{m} c_i (\tau_i)^{\alpha+\gamma-1} + T^\alpha \right) \|x_1 - x_2\|_{C_{1-\gamma}} \end{aligned}$$

From (14), it follows that N has a unique fixed point which is solution of system (1).

Theorem 4.9. If the hypothesis (H3) and (14) are satisfied, then the system (1) is Ulam-Hyers stable

Proof. Let $\epsilon > 0$ and let $z \in C_{1-\gamma}^{\gamma}[J, X]$ be a function which satisfies the inequality:

$$\left| D_{0+}^{\alpha,\beta} z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha,\beta} z(\lambda t)) \right| \leq \epsilon \quad \text{for any } t \in J \tag{17}$$

and let $x \in C_{1-\gamma}^{\gamma}[J, X]$ be the unique solution of the following nonlinear neutral pantograph system

$$\begin{aligned} D_{0+}^{\alpha,\beta} x(t) &= f(t, x(t), x(\lambda t), D_{0+}^{\alpha,\beta} x(\lambda t)), \quad t \in J := [0, T], \\ I_{0+}^{1-\gamma} z(0) &= \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [0, T],\ \gamma = \alpha + \beta - \alpha\beta \end{aligned}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $0 < \lambda < 1$.

Using Lemma 2.14, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)ds,$$

where

$$A_x = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_x(s)ds.$$

On the other hand, if $x(\tau_i) = z(\tau_i)$, and $x(0) = z(0)$, then $A_x = A_z$. Indeed,

$$\begin{aligned} |A_x - A_z| &\leq \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} |K_x(s) - K_z(s))| ds \\ &\leq \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} \left(\frac{2K}{1-L}\right) |x(s) - z(s)| ds \\ &\leq \left(\frac{2K}{1-L}\right) \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^{m} c_i I_{0+}^{\alpha} |x(\tau_i) - z(\tau_i)| \\ &= 0. \end{aligned}$$

Thus,

$$Ax = Az.$$

Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)ds.$$

By integration of the inequality (17) and applying Lemma 4.6, we obtain

$$\left| z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)ds \right| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}. \tag{18}$$

We have for any $t \in J$

$$\begin{aligned} |z(t) - x(t)| &\leq \left| z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_z(s) - K_x(s)| ds \\ &\leq \left| z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)ds \right| \\ &\quad + \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| ds. \end{aligned}$$

By using (18), we have

$$|z(t) - x(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| ds$$

and to apply Lemma 2.13, we obtain

$$\begin{aligned} |z(t) - x(t)| &\leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left[1 + \frac{2\nu K}{(1-L)\Gamma(\alpha+1)} T^{\alpha}\right] \epsilon \\ &:= C_f \epsilon, \end{aligned}$$

Where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\varrho) = C_{f\in}$; $\psi(0) = 0$, then the system (1) is generalized Ulam-Hyers stable.

Theorem 4.10. Assume that (H3),(H4) and (14) are fulfilled, then the system (1) is Ulam-Hyers-Rassias stable.

Proof. Let $\in > 0$ and let $z \in C_{1-\gamma}^{\gamma} [J, X]$ be a function which satisfies the inequality:

$$\left| D_{0+}^{\alpha,\beta} z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\alpha,\beta} z(\lambda t)) \right| \leq \epsilon \varphi(t) \quad \text{for any } t \in J \tag{19}$$

and let $x \in C_{1-\gamma}^{\gamma} [J, X]$ be the unique solution of the following nonlinear neutral pantograph system

$$\begin{aligned} &D_{0+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\alpha,\beta} x(\lambda t)), \qquad t \in J := [0, T], \\ &I_{0+}^{1-\gamma} z(0) = \sum_{i=1}^{m} c_i x(\tau_i), \qquad \tau_i \in [0, T], \ \gamma = \alpha + \beta - \alpha\beta \end{aligned}$$

where $0 < \alpha < 1, 0 \leq \beta \leq 1$, and $0 < \lambda < 1$.

Using Lemma 2.14, we obtain

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) ds,$$

Where

$$A_z = \frac{|Z|}{\Gamma(\alpha)} t^{\gamma-1} \sum_{i=1}^m c_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} K_z(s) ds.$$

By integration of (19) and applying Lemma 4.6, we obtain

$$\begin{aligned} \left| z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) ds \right| &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \\ &\leq \epsilon \lambda_\varphi \varphi(t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |z(t) - x(t)| \leq & \left| z(t) - A_x - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s) ds \right| \\ & + \left(\frac{2K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, ds. \end{aligned}$$

By using (18), we have

$$|z(t) - x(t)| \leq \epsilon \lambda_\varphi \varphi(t) + \left(\frac{2K}{1-L} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - x(s)| \, ds$$

and to apply Lemma 2.13, we obtain

$$|z(t) - x(t)| \leq \left[\left(1 + \frac{2K\nu_1\lambda_\varphi}{1-L} \right) \lambda_\varphi \right] \epsilon \varphi(t),$$

where $\nu_1 = \nu_1(\alpha)$ is constant, then for any $t \in J$:

$$|z(t) - x(t)| \leq C_f \epsilon \varphi(t),$$

which completes the proof of the theorem.

5. Example

Example 5.1. Consider the following Hilfer type nonlinear neutral pantograph problem

$$\begin{aligned} & D_{1^+}^{\alpha,\beta} x(t) = \frac{1}{4} + \frac{1}{20} \left(x(t) + x\left(\frac{t}{2}\right) + D_{0^+}^{\alpha,\beta} x\left(\frac{t}{2}\right) \right), \quad t \in [1,2], \\ & I_{1^+}^{1-\gamma} x(1) = 2x\left(\frac{3}{2}\right), \quad \gamma = \alpha + \beta - \alpha\beta. \end{aligned} \tag{20}$$

Notice that this problem is a particular case of (1), where $0 < \lambda < 1$, $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{5}{6}$.

Set

$$f(t,u,v,w) = \frac{1}{4} + \frac{1}{20}u + \frac{1}{20}v + \frac{1}{20}w, \quad \text{for any } u,v,w \in X.$$

Clearly, the function f satisfies the conditions of Theorem 3.1.

For any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in X$ and $t \in [1,2]$.

$$|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| \leq \frac{1}{20} (|u - \overline{u}| + |v - \overline{v}|) + \frac{1}{20} |w - \overline{w}|.$$

Hence the condition (H3) is satisfied with $K = L = \frac{1}{20}$.

Thus condition from (14)

$$\frac{2K}{(1-L)} \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left(|Z| \sum_{i=1}^m c_i (\tau_i)^{\alpha+\gamma-1} + T^\alpha \right) = 0.2295 < 1.$$

where $|Z| = 0.8959$.

It follows from Theorem 4.8 that the problem (20) has a unique solution. Moreover, Theorem 4.10 implies that the problem (20) is Ulam-Hyers stable.

6. Conclusion

In this paper, we discussed existence, uniqueness and stability of Hilfer type neutral pantograph differential equations with nonlocal conditions. The main results is verified by a simulation example.

REFERENCES

[1] S. Andrs, J. J. Kolumbn, On the Ulam-Hyers stability of first order diff erential systems with nonlocal initial conditions, Nonlinear Analysis, 82,(2013),1-11.

[2] K. Balachandran, S. Kiruthika and J.J. Trujillo, Existence of solutions of Nonlinear fractional pantograph equations, Acta Mathematica Scientia, 33B, (2013),1-9.

[3] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence results for fractional order functional diff erential equations with infinite delay, Journal of Mathematical and Applications,338,(2008),1340-1350.

[4] Benchohra, F. Berhoun, Impulsive fractional diff erential equations with vari-able times, Computer and mathematics with applications 59 (2010) 1245-1252. doi:10.1016/j.camwa.2009.05.016

[5] K.M. Firati, M. D. Kassim, N.E. Tatar, Existence and uniqueness for a problem in-volving Hilfer fractional derivative, Computers and Mathematics with Applications, 64, (2012), 1616-1626.

[6] K.M. Firati, M. D. Kassim, N.E. Tatar, Non-existence of global solutions for a diff er-ential equation involving Hilfer fractional derivative, Electronic Journal of Diff erential Equations, 235, (2013), 1-10.

[7] R. W. Ibrahim, Generalized Ulam-Hyers stability for fractional diff erential equations, International Journal of mathematics,23,(2012),doi:10.1142/S0129167X12500565.

[8] S. M. Jung, Hyers-Ulam stability of linear diff erential equations of first order, Appl. Math. Lett., 17,(2004),1135-1140.

[9] P. Muniyappan, S. Rajan, Hyers-Ulam-Rassias stability of fractional diff erential equa-tion, International Journal of pure and Applied Mathematics, 102, (2015),631-642.

[10] I.A. Rus, Ulam stabilities of ordinary diff erential equations in a Banach space, Carpathian journal of Mathematics, 26, (2010),103-107

[11] H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, Applied Mathematics and Computation, (2014), doi:10.1016/j.amc.2014.10.083

[12] R.Hilfer, Application of fractional Calculus in Physics, World Scientific, Singapore, 1999.

[13] R.Hilfer, Y.Luchko, Z. Tomovski, Operational method for the solution of fractional dif-ferential equations with generalized Riemann-Lioville fractional derivative, Fractional calculus and Applications Analysis, 12, (2009), 289-318.

[14] A. Iserles, Exact and discretized stability of the pantograph equation, Applied Numer-ical Methods, 24, (1997),295-308.

[15] M.Z Liu, D. Li, Runge-Kutta methods for the multi-pantograph delay equation, Ap-plications and Mathematical Computations, 155, (2004), 853-871.

[16] I. Podlubny, Fractional diff erential equations, Academic Press, San Diego, 1999.

[17] D. Vivek, K. Kanagarajan, S. Sivasundaram, Dynamics and stability of pantograph equations via Hilfer fractional derivative, Nonlinear Studies, 23(4), (2016), 685-698.

[18] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional diff er-ential equations with Caputo derivative,Electronic Journal of Qualitative Theory of Diff erential Equations,63,(2011),110.

[19] J. Wang, Y. Zhou, New concepts and results in stability of fractional dif-ferential equations,Communications on Nonlinear Science and Numerical Simula-tions,17,(2012),2530-2538.

[20] J. Wang, Y. Zhang, nonlocal initial value problems for diff erential equations with Hilfer fractional derivative, Applied Mathematics and Computation, 266, (2015), 850-859.

Citation: *Elsayed, E.et al., (2018). Existence, Uniqueness and Stability of Hilfer Type Neutral Pantograph Differential Equations with Nonlocal Conditions. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 6(8), pp.42-53. http://dx.doi.org/10.20431/2347-3142.0608004*