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Approximate Solutions for Nonlinear Differential Equations Using PM with Dirichlet and Mixed Boundary Conditions

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Abstract: *In this paper, a perturbation method (PM) to solve a class of nonlinear problems is presented. As a case study, PM is employed to obtain approximate solutions of perturbed differential equations. Three examples are presented to more explanation about the application of proposed method.*

Keywords: *Dirichlet boundary conditions, Mixed boundary conditions, Nonlinear differential equation, Perturbation method, Approximate solutions*

1. Introduction

Perturbation methods are classical methods which have been used over a century to obtain approximate analytical solutions of various kinds of nonlinear problems. This procedure was originated by S.D. Poisson and extended by J.H. Poincare. Although, the method appeared in the early 19thcentury, the application of a perturbation procedure to solve nonlinear differential equations was performed later on that century. The most significant efforts were focused oncelestial mechanics, fluid mechanics, and aerodynamics [1,2].

It is possible to express a nonlinear differential equation in terms of one linear part and other nonlinear. The nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter) $\varepsilon \ll 1$.

Many differential perturbation techniques such as the method of multiple scales, the method of averaging, the renormalization method, the Lindstedt-Poincare method, the method of matched asymptotic, expansions, and their variants were developed [3, 4]. Also, the PM has been successfully applied to differential equations, integro-differential equation, and algebraic equations. In this paper, this method is applied to solve nonlinear differential equations.

This paper is organized as follows. Section 2 presents survey perturbation method. In Section 3, we provide an application of PM method, by solving three examples with applications in sciences and engineering. Finally, a brief conclusion is given in Section 4.

2. Survey Perturbation Method

Let the following nonlinear differential equation,

$$l(x) + \varepsilon N(x) = 0, \tag{1}$$

where x is an one-variable function as $x = x(t)$, $l(x)$ is a linear operator which, in general, contains derivatives in terms oft, and$N(x)$is a nonlinear operator, and ε is a small parameter.

Considering the nonlinear term in Eq. (1) to be a small perturbation and assuming that the solution for (1) can be written as a power series in terms of the small parameterε,

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots. \tag{2}$$

Substituting (2) into (1) and equating terms having identical powers of ε, we obtain a number of differential equations that can be integrated, recursively, to find the values for the functions, $x_0(t), x_1(t), x_2(t), \dots$.

3. APPROXIMATE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

3.1. Example

The following example emphasizes the use of PM for solving nonlinearODEs with Dirichlet boundary conditions. We will consider the following nonlinear differential equation.

$$\frac{d^2y(x)}{dx^2} + \varepsilon\left[y(x)\left(\frac{d^2y(x)}{dx^2}\right)^2 - \frac{dy(x)}{dx}\right] = 0, \qquad 0 \le x \le 1, \tag{3}$$

$$y(0) = 0, \quad y(1) = 0,$$

where ε is a positive parameter.

It is possible to find a handy solution for (3) by applying the PM method. Define terms:

$$l(y) = y''(x), \tag{4}$$

$$N(y) = y(x)y''^2(x) - y'(x), \tag{5}$$

We assume a solution for (3) in the following form as (2),

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \varepsilon^4 y_4(x) + \cdots. \tag{6}$$

On comparing the coefficients of like powers of ε,it can be solved for

$y_0(x), y_1(x), y_2(x), y_3(x), \cdots,$ and so on. Later it will be seen that, a very good handy result is obtained, by keeping up to fifth order approximation.

$$\varepsilon^0) \quad y_0'' = 0, \tag{7}$$

$$\varepsilon^1) \quad y_1'' - y_0' + 2y_0''y_1'' = 0 \tag{8}$$

$$\varepsilon^2) \quad y_2'' - y_1' + 2y_0''y_1'' = 0, \tag{9}$$

$$\varepsilon^3) \quad y_3'' - y_2' + 2y_0''y_2'' + y_1''^2 = 0, \tag{10}$$

$$\varepsilon^4) \quad y_4'' - y_3' + 2y_0''y_3'' + 2y_1''y_2'' = 0, \tag{11}$$

$$\varepsilon^5) \quad y_5'' - y_4' + 2y_0''y_4'' + 2y_1''y_3'' + y_2''^2 = 0, \tag{12}$$

The results obtained are,

$$y_0(x) = 1 - x, \tag{13}$$

$$y_1(x) = -\frac{1}{2}x^2 + \frac{1}{2}x, \tag{14}$$

$$y_2(x) = -\frac{1}{6}x^3 + \frac{1}{4}x^2 \quad -\frac{1}{12}x, \tag{15}$$

$$y_3(x) = -\frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{1}{6}x^2 + \frac{1}{24}x, \tag{16}$$

$$y_4(x) = -\frac{1}{40}x^5 + \frac{1}{16}x^4 - \frac{7}{18}x^3 + \frac{25}{48}x^2 + \frac{61}{360}x, \tag{17}$$

$$y_5(x) = -\frac{1}{240}x^6 + \frac{1}{80}x^5 - \frac{31}{72}x^4 + \frac{121}{48}x^3 + \frac{7}{90}x^2 - \frac{47}{144}x. \tag{18}$$

By substituting (13)-(18) into (6) we obtain a fifth order approximation for the solution of (3).

As a case study, we set$\varepsilon = 1$to obtain a handy approximation solution as follows,

$$y(x) = 1 - \frac{503}{720}x + \frac{131}{720}x^2 + \frac{319}{144}x^3 - \frac{71}{144}x^4 - \frac{1}{80}x^5 - \frac{1}{240}x^6. \tag{19}$$

In Figure 1, the plots of approximate solutions have been indicated for $\varepsilon = 1, 1/2$.

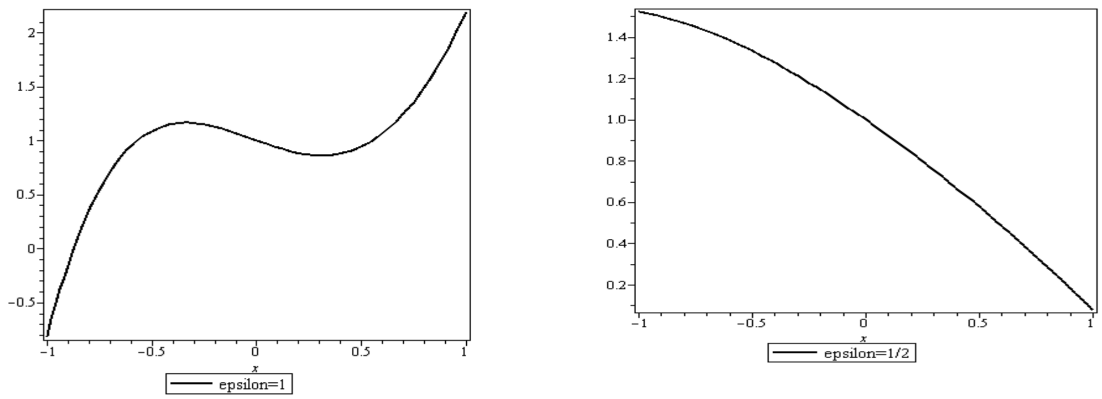

Fig1. *Approximate solutions of Example 3.1 for Left:* $\varepsilon = 1$, *Right:* $\varepsilon = 1/2$

3.2. Example

This example emphasizes the use of PM for solving nonlinear ODEs with mixed boundary conditions.

$$\frac{d^2y(x)}{dx^2} + \varepsilon\left[y(x)\frac{d^2y(x)}{dx^2} - \left(\frac{dy(x)}{dx}\right)^2\right] = 0, \qquad 0 \le x \le 1, \tag{20}$$

$$y(0) = 1, y(1) = 0,$$

where ε is a positive parameter.

To applyPM method, we define terms as follows,

$$l(y) = y''(x), \tag{21}$$

$$N(y) = y(x)y''(x) - y'^2(x). \tag{22}$$

Identifying ε with the PM parameter, we assume a solution for (20) in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \varepsilon^4 y_4(x) + \cdots. \tag{23}$$

Equating the terms with identical powers of εit can be solved for$y_0(x), y_1(x), y_2(x), \cdots$. We will see that a handy result is obtained, by keeping up to fifth order approximation.

$$\varepsilon^0)\quad y_0'' = 0, \tag{24}$$

$$\varepsilon^1)\quad y_1'' + y_0 y_1'' - y_0'^2 = 0, \tag{25}$$

$$\varepsilon^2)\quad y_2'' - 2y_0'y_1' + y_0 y_1'' + y_1 y_0'' = 0, \tag{26}$$

$$\varepsilon^3)\quad y_3'' - 2y_0'y_2' + y_0 y_2'' - y_1'^2 + y_1 y_1'' + y_2 y_0'' = 0, \tag{27}$$

$$\varepsilon^4)\quad y_4'' - 2y_0'y_3' - 2y_1'y_2' + y_0 y_3'' + y_1 y_2'' + y_2 y_1'' + y_3 y_0'' = 0, \tag{28}$$

$$\varepsilon^5)\quad y_5'' - 2y_0'y_4' - 2y_1'y_3' - y_2'^2 + y_0 y_4'' + y_1 y_3'' + y_2 y_2'' + y_3 y_1'' + y_4 y_0'' = 0. \tag{29}$$

After solving the above equation, one obtains,

$$y_0(x) = 1 - x, \tag{30}$$

$$y_1(x) = \frac{1}{2}x^2 - \frac{1}{2}x, \tag{31}$$

$$y_2(x) = -\frac{1}{6}x^3 + \frac{1}{6}x, \tag{32}$$

$$y_3(x) = \frac{1}{24}x^4 + \frac{1}{12}x^3 - \frac{1}{24}x^2 - \frac{1}{12}x, \tag{33}$$

$$y_4(x) = -\frac{1}{120}x^5 - \frac{1}{24}x^4 - \frac{1}{24}x^3 + \frac{1}{24}x^2 + \frac{1}{20}x, \tag{34}$$

$$y_5(x) = \frac{1}{48}x^6 - \frac{1}{16}x^5 - \frac{1}{36}x^4 + \frac{35}{144}x^3 + \frac{19}{144}x. \tag{35}$$

By substituting (30)-(35) into (23) we obtain a fifth order approximation for the solution of (20). Considering as a case study, the value of parameter $\varepsilon = 0.30$,we obtain a handy approximate solution as follows,

$$z(x) = 1 - \frac{889}{720}x + \frac{1}{2}x^2 + \frac{17}{144}x^3 - \frac{1}{36}x^4 - \frac{17}{240}x^5 - \frac{1}{48}x^6. \quad (36)$$

The plots of the approximate solutions are seen in Figure 2 for $\varepsilon = 1, 0.30$.

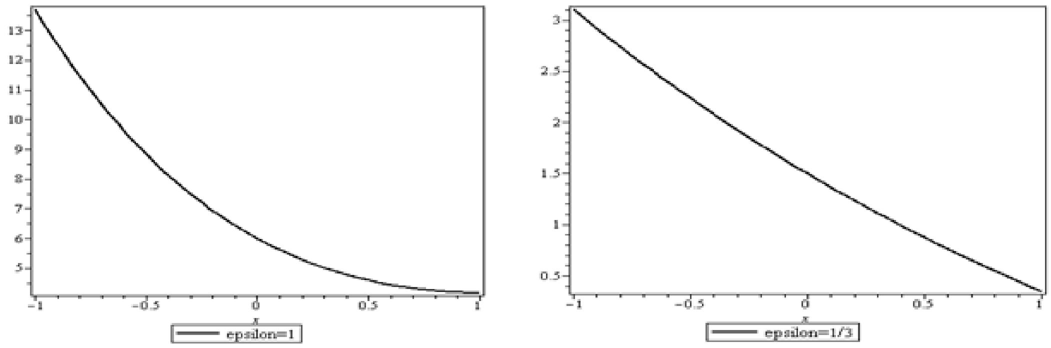

Fig2. *Approximate solutions of Example 3.2 for Left:*$\varepsilon = 1$, *Right:* $\varepsilon = 0.30$

3.3. Example

Consider the following linear differential equation.

$$\frac{d^2y(x)}{dx^2} + \varepsilon\frac{dy(x)}{dx} = 0, \qquad -1 \le x \le 0, \quad (37)$$

With boundary conditions,

$y(-1) = 1, y(0) = 1.$

In order to solve Eq. (37), define terms as follows,

$$l(y) = y''(x), \quad (38)$$

$$N(y) = y'(x). \quad (39)$$

We assume a solution for (37) in the following form,

$$z(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \varepsilon^4 y_4(x) + \cdots. \quad (40)$$

On comparing the coefficients of like powers ofε it can be solved for $y_0(x), y_1(x), y_2(x), y_3(x), \cdots$. A handy result is obtained, by keeping up to fifth order approximation,

$$\varepsilon^0) \quad y_0'' = 0, \quad (41)$$

$$\varepsilon^1) \quad y_1'' + y_0' = 0, \quad (42)$$

$$\varepsilon^2) \quad y_2'' + y_1' = 0, \quad (43)$$

$$\varepsilon^3) \quad y_3'' + y_2' = 0, \quad (44)$$

$$\varepsilon^4) \quad y_4'' + y_3' = 0, \quad (45)$$

$$\varepsilon^5) \quad y_5'' + y_4' = 0. \quad (46)$$

After solving the above equation, one obtains,

$$y_0(x) = 1 - x, \quad (47)$$

$$y_1(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1, \quad (48)$$

$$y_2(x) = -\frac{1}{6}x^3 + \frac{1}{4}x^2 - \frac{13}{12}x + 1, \quad (49)$$

$$y_3(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{13}{24}x^2 - \frac{3}{2}x + 1, \quad (50)$$

$$y_4(x) = -\frac{1}{120}x^5 + \frac{1}{48}x^4 - \frac{13}{72}x^3 + \frac{3}{4}x^2 - \frac{19}{12}x + 1, \quad (51)$$

$$y_5(x) = \frac{1}{720}x^6 - \frac{1}{240}x^5 + \frac{13}{288}x^4 - \frac{1}{4}x^3 + \frac{19}{24}x^2 + \frac{2281}{1440}x + 1. \quad (52)$$

By substituting (47)-(52) into (40) we obtain a fifth order approximation for the solution of (37). As a case study, we set $\varepsilon = 1$ to obtain a handy approximation solution as follows,

$$z(x) = 6 - \frac{5879}{1440}x + \frac{17}{6}x^2 - \frac{49}{72}x^3 + \frac{13}{288}x^4 - \frac{1}{80}x^5 + \frac{1}{720}x^6. \quad (53)$$

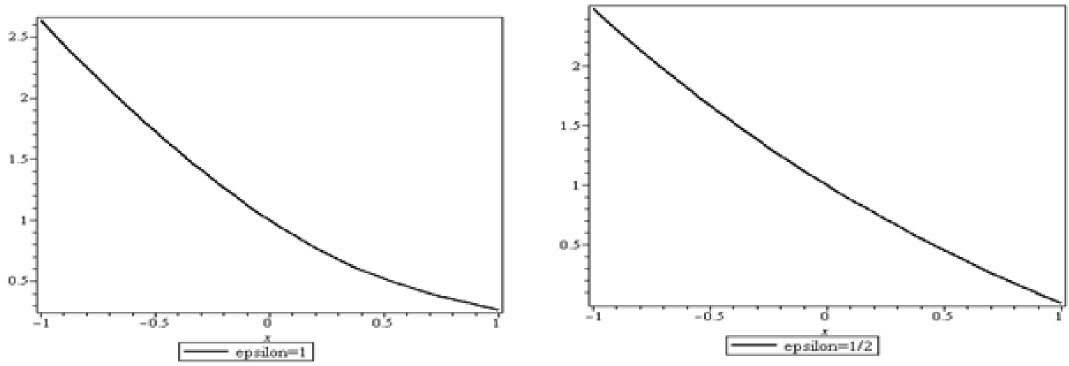

Fig3. *Approximate solutions of Example 3.3 for Left:* $\varepsilon = 1$*, Right:* $\varepsilon = 1/2$

4. Conclusions

In this study, PM was presented to construct analytical approximation solutions for nonlinear differential equations in the form of rapidly convergent series. We proposed three examples with Dirichlet and mixed boundary conditions, with good results. Although the solutions reported for other sophisticated methods have good accuracy, they are more complicated for applications than PM.

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