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The Spectral Norms of Geometric Circulant Matrices with Generalized Tribonacci Sequence

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Abstract: *In this paper, I study a geometric circulant matrix involving the generalized Tribonacci sequences and Chebyshev polynomials. Then I compute lower and upper bounds for the spectral norms of the matrix.*

Keywords: *geometric circulant matrix; generalized Tribonacci sequence; Cheby- shev polynomial; spectral norms; Euclidean norms*

1. Introduction

Circulant matrix have many special properties and have been one of the most important research areas in the field of the computation and pure mathematics. In particular, they have important position and application in solving coding theory, different types of partial and ordinary differential equations, numerical analysis and so on.

An n × n r−circulant matrix has the form:

$$C_r = \begin{pmatrix} c_0 & c_1 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

Obviously, C_r,the r-circulant matrix is determined by parameter $r$and its first row elements. When the parameter satisfies$r = 1$, we can get circulant matrix.

In rencent years, it's a brisk research topic that circulant matrices have been studied in many aspects. Particularly, manyscholars have learned comprehensively the norms of circulant matrices basing on the special properties. For example, Solak has computed the lower and upper bounds of spectral norms of circulant matrices involving the Fibonacci number and Lucas numbers in reference [1]. Later, Shen and Cen have has generalized the results of the reference [1] and obtained the bounds for the spectral norms of r-circulant matrices involving Fibonacci and Lucas numbers[2]. In 2014, coskun established norms, the eigenvalues and the determinant of circulant matrices with Cordonnier, Van Der laan numbers and Perrin by some properties of circulant matrix with third order linear recurrent sequence[3]. In reference [4], [5], they computed the lower and upper bounds for the spectral norms of Hankel matrices and Toeplitz matrices involving Pell number, Pell-Lucas numbers, respectively. In [6], they obtained the norms of semi circulant and circulant matrices with Horadam numbers. In [7], they studied eigenvalues, determinant and the spectral norms of circulant matrix involving the generalized k- Horadam numbers. In [8] authors have also studied the norms of many special matrices with generalized Tribonacci sequences and generalized Pell-Padovan. In 2016, Can and Naim have studied the bounds for the spectral norms of geometric circulant matrices involving the generalized Fibonacci number and Lucas numbers [9], and so on.

At this point, enlighted by above articles, in my present study, I obtained the lower and upper bounds of geometric circulant matrix involving the generalized Tribonacci sequences and Chebyshev polynomials.

2. PRELIMINARIES

The third order liner recurrence sequence has the form:

$Q_0 = a, Q_1 = b, Q_2 = c, Q_n = pQ_{n-1} + qQ_{n-2} + rQ_{n-3},$

where a,b,care positive integer.

Taking $p = q = r = 1$,then we can obtain the famous generalized Tribonacci sequence:

$R_n = R_{n-1} + R_{n-2} + R_{n-3}$,with initial conditions

$R_0 = a,\ R_1 = b,\ R_2 = c.$

The well-known Tribonacci sequence is defined by the following equations:

$T_0 = 0, T_1 = 1, T_2 = 1. T_n = T_{n-1} + T_{n-2} + T_{n-3}.$

For the sequenceQ_n,the characteristic equation $x^3 - px^2 - qx - r = 0$ has three distinct real roots, denoted by α, β, γ.Where

$\alpha + \beta + \gamma = p,$
$\alpha\beta\gamma = r,$
$\alpha\beta + \beta\gamma + \alpha\gamma = -q.$

The Binet's formula Q_n can be represented by

$Q_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}.$ Hence, clearly, for R_n, $\alpha + \beta + \gamma = 1$,$\alpha\beta\gamma = 1$,

$\alpha\beta + \beta\gamma + \alpha\gamma = -1.$

The binet's form is $R_n = \frac{\alpha^n}{X} + \frac{\beta^n}{Y} + \frac{\gamma^n}{W},$

$X = (\alpha - \beta)(\alpha - \gamma),\ Y = (\beta - \alpha)(\beta - \gamma)$,$W = (\gamma - \alpha)(\gamma - \beta).$

Definition1. The geometric circulant matrix C_{r^*} is defined by[9]:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

We denote it easily by $C_{r^*} = Circ_{r^*}(c_0, c_1, c_2, \cdots, c_{n-1})$.When the parameter $r = 1$, geometric circulant matrix turns into circulant matrix.

Definition2. Let us take any matrix $A = (a_{ij}) \in M_{m\times n}(C)$,the spectral norm and the Euclidean norm of matrix A are

$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)}$,$\|A\|_E = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$, respectively.

Where $\lambda_i(A^H A)$ is the eigenvalue of $A^H A$and A^H is the conjugate transpose of matrix$A$.

The following inequalities hold between the Euclidean norm and spectral norm[15]:

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E, \tag{1}$$

$$\|A\|_2 \leq \|A\|_E \leq \sqrt{n}\|A\|_2 \tag{2}$$

Definition3. Let $A=(a_{ij})$ and $B=(b_{ij})$ are $m \times n$ matrices, then the Hadamard product of A and B is the $m \times n$ matrix of element wise products, namely $A \circ B = (a_{ij}b_{ij})$.

Then we have the following inequalities[14]:

$\|A \circ B\|_2 \leq r_1(A)C_1(B)$,

Where

Lemma1.[8] For any $n \geq 1$.

$$\sum_{k=1}^{n} R_k^2 = \frac{4R_nR_{n+1} - 4R_0R_1 - \left(R_{n+1} - R_{n-1}\right)^2 + \left(R_{-2} + R_0\right)^2}{4}.$$ Hence,

$$\sum_{k=0}^{n-1} R_k^2 = \frac{4R_{n-1}R_n - 4ab - \left(R_n - R_{n-2}\right)^2 + \left(R_{-2} + a\right)^2 + 4a^2}{4}.$$

Lemma2. For the Chebyshev polynomial $T_n(x)$, $U_n(x)$,

$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

$$U_0(x) = 1, U_1(x) = 2x, U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

$$T_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n}{2} = \frac{\alpha^n+\beta^n}{2},$$

and $U_n(x) = \dfrac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}} = \dfrac{\alpha^{n+1}+\beta^{n+1}}{\alpha-\beta}$.

In particular,

$$\sum_{k=0}^{n-1} T_k^2 = \frac{1}{4}U_{2(n-1)}(x) + \frac{n}{2} + \frac{1}{4}.$$

3. Main Results

Theorem1. Let $R_{r^*} = Circ_{r^*}(R_0, R_1, R_2, \ldots, R_{n-1})$ be an $n \times n$ geometric circulant matrix.

(i) If $|r| > 1$, then

$$\sqrt{\sum_{k=0}^{n-1} R_k^2} \leq \|R_{r^*}\|_2 \leq \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\sum_{k=0}^{n-1} R_k^2},$$

(ii) If $|r| < 1$, then

$$|r|\sqrt{\frac{|r|^{2n+4}U_0 - |r|^{2n+2}K_0 + |r|^{2n}U_{-2} + |r|^4U_{2n} + |r|^2K_1 - U_{2n-2}}{|r|^6 + 3|r|^4 + (9-\Delta_4)|r|^2 - 1}} \leq$$

$$\|R_{r^*}\|_2 \leq \sqrt{n\sum_{k=0}^{n-1} R_k^2}.$$

Proof.

$$R_{r^*} = \begin{pmatrix} R_0 & R_1 & R_1 & \cdots & R_{n-2} & R_{n-1} \\ rR_{n-1} & R_0 & R_1 & \cdots & R_{n-3} & R_{n-2} \\ r^2R_{n-2} & rR_{n-1} & R_0 & \cdots & R_{n-4} & R_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}R_1 & r^{n-2}R_2 & r^{n-3}R_3 & \cdots & rR_{n-1} & R_0 \end{pmatrix}_{n\times n}.$$

(i) From $|r| > 1$ and by using the definition of Euclidean norm, we have

$$\begin{aligned}\|R_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)R_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2R_k^2\\ &\geq \sum_{k=0}^{n-1}(n-k)R_k^2+\sum_{k=1}^{n-1}kR_k^2\\ &= n\sum_{k=0}^{n-1}R_k^2,\end{aligned}$$

that is, $\frac{1}{\sqrt{n}}\|R_{r^*}\|_E \geq \sqrt{\sum_{k=0}^{n-1}R_k^2},$ from (1), I have $\sqrt{\sum_{k=0}^{n-1}R_k^2} \leq \|R_{r^*}\|_2.$

For another, let the matrices A and B be presented by

$$A=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1\\ r & 1 & 1 & \cdots & 1 & 1\\ r^2 & r & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1\end{pmatrix}_{n\times n}$$

$$B=\begin{pmatrix} R_0 & R_1 & R_1 & \cdots & R_{n-2} & R_{n-1}\\ R_{n-1} & R_0 & R_1 & \cdots & R_{n-3} & R_{n-2}\\ R_{n-2} & R_{n-1} & R_0 & \cdots & R_{n-4} & R_{n-3}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ R_1 & R_2 & R_3 & \cdots & R_{n-1} & R_0\end{pmatrix}_{n\times n}$$

then $R_{r^*}=A\circ B.$ So,

$\|R_{r^*}\|_2=\|A\circ B\|_2\leq r_1(A)C_1(B),$

$$\begin{aligned} r_1(A) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|a_{nj}|^2}\\ &= \sqrt{1+|r|^2+\cdots+|r^{n-1}|^2}=\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}},\end{aligned}$$

$$c_1(B)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|b_{in}|^2}=\sqrt{\sum_{k=0}^{n-1}R_k^2}$$

So we have

$$\|R_{r^*}\|_2\leq r_1(A)c_1(B)=\sqrt{\tfrac{1-|r|^{2n}}{1-|r|^2}\sum_{k=0}^{n-1}R_k^2}.$$

Thus, we can obtain the inequality:

$\sqrt{\sum_{k=0}^{n-1}R_k^2}\leq\|R_{r^*}\|_2\leq\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\sum_{k=0}^{n-1}R_k^2}.$ (ii)From $|r|<1,$

$$\begin{aligned}\|R_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)R_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2R_k^2\\ &\geq \sum_{k=0}^{n-1}(n-k)|r^{n-k}|^2R_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2R_k^2\\ &= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{R_k}{|r|^k}\right)^2\\ &= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{\frac{\alpha^k}{X}+\frac{\beta^k}{Y}+\frac{\gamma^k}{W}}{|r|^k}\right)^2,\end{aligned}$$

By taking

$U_n = \frac{\alpha^n}{X^2} + \frac{\beta^n}{X^2} + \frac{\gamma^n}{W^2}$, $\Delta_n = \alpha^n + \beta^n + \gamma^n$ and the relation of

$\Delta_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma) = 3,$

we have

$$n|r|^{2n} \sum_{k=0}^{n-1} \left(\frac{\frac{\alpha^k}{X} + \frac{\beta^k}{Y} + \frac{\gamma^k}{W}}{|r|^k}\right)^2$$
$$= n|r|^2 \frac{|r|^{2n+4}U_0 - |r|^{2n+2}K_0 + |r|^{2n}U_{-2} + |r|^4 U_{2n} + |r|^2 K_1 - U_{2n-2}}{|r|^6 + 3|r|^4 + (9 - \Delta_4)|r|^2 - 1}$$

where $K_0 = 3U_0 - U_2,\ K_1 = 3U_{2n} - U_{2n+2}$.

For another, the matrices $A$and B as mentioned above:

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ r^2 & r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{pmatrix}_{n\times n}$$

$$B = \begin{pmatrix} R_0 & R_1 & R_1 & \cdots & R_{n-2} & R_{n-1} \\ R_{n-1} & R_0 & R_1 & \cdots & R_{n-3} & R_{n-2} \\ R_{n-2} & R_{n-1} & R_0 & \cdots & R_{n-4} & R_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_1 & R_2 & R_3 & \cdots & R_{n-1} & R_0 \end{pmatrix}_{n\times n}$$

In this case, $R_{r^*} = A \circ B$.

So,

$\|R_{r^*}\|_2 = \|A \circ B\|_2 \le r_1(A)C_1(B),$

$$r_1(A) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{n},$$

$$c_1(B) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{in}|^2} = \sqrt{\sum_{k=0}^{n-1} R_k^2}.$$

So,

$$\|R_{r^*}\|_2 \le \sqrt{n \sum_{k=0}^{n-1} R_k^2},$$

Therefore, we have

$$|r| \sqrt{\frac{|r|^{2n+4}U_0 - |r|^{2n+2}K_0 + |r|^{2n}U_{-2} + |r|^4 U_{2n} + |r|^2 K_1 - U_{2n-2}}{|r|^6 + 3|r|^4 + (9 - \Delta_4)|r|^2 - 1}} \le \|R_{r^*}\|_2 \le \sqrt{n \sum_{k=0}^{n-1} R_k^2}.$$

Theorem2. Let $T_{r^*} = Circ_{r^*}(T_0, T_1, T_2, \dots, T_{n-1})$ be an $n \times n$ geometric circulant matrix.

(i) If $|r| > 1$, then

$$\sqrt{\sum_{k=0}^{n-1} T_k^2} \le \|R_{r^*}\|_2 \le \sqrt{\frac{1-|r|^{2n}}{1-|r|^2} \sum_{k=0}^{n-1} T_k^2},$$

(ii) If $|r| < 1$, then

$$\frac{|r|}{2} \sqrt{\frac{2|r|^{2n+2} - 2|r|^{2n}T_2 - 2|r|^2 T_{2n} + 2T_{2n-2}}{|r|^4 - 2|r|^2 T_2 + 1} + \frac{2|r|^{2n} - 2}{|r|^2 - 1}} \le \|R_{r^*}\|_2 \le \sqrt{n \sum_{k=0}^{n-1} T_k^2}.$$

Proof.

$$T_{r^*}=\begin{pmatrix} T_0 & T_1 & T_1 & \cdots & T_{n-2} & T_{n-1} \\ rT_{n-1} & T_0 & T_1 & \cdots & T_{n-3} & T_{n-2} \\ r^2T_{n-2} & rT_{n-1} & T_0 & \cdots & T_{n-4} & T_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}T_1 & r^{n-2}T_2 & r^{n-3}T_3 & \cdots & rT_{n-1} & T_0 \end{pmatrix}_{n\times n}.$$

(i) For $|r|>1$, from the definition of Euclidean norm, we have

$$\begin{aligned} \|T_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)T_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2T_k^2 \\ &\geq \sum_{k=0}^{n-1}(n-k)R_k^2+\sum_{k=1}^{n-1}kR_k^2 \\ &= n\sum_{k=0}^{n-1}T_k^2, \end{aligned}$$

In this case, $\frac{1}{\sqrt{n}}\|T_{r^*}\|_E \geq \sqrt{\sum_{k=0}^{n-1}T_k^2},$ then by using (1),

we have $\sqrt{\sum_{k=0}^{n-1}T_k^2} \leq \|T_{r^*}\|_2.$ For another, let the matrices C and D be defined by the form:

$$C=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ r^2 & r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{pmatrix}_{n\times n}$$

$$D=\begin{pmatrix} T_0 & T_1 & T_1 & \cdots & T_{n-2} & T_{n-1} \\ T_{n-1} & T_0 & T_1 & \cdots & T_{n-3} & T_{n-2} \\ T_{n-2} & T_{n-1} & T_0 & \cdots & T_{n-4} & T_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_1 & T_2 & T_3 & \cdots & T_{n-1} & T_0 \end{pmatrix}_{n\times n}$$

then $T_{r^*}=C\circ D$. So,

$$\|T_{r^*}\|_2=\|C\circ D\|_2\leq r_1(C)C_1(D),$$

$$\begin{aligned} r_1(C) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|c_{nj}|^2} \\ &= \sqrt{1+|r|^2+\cdots+|r^{n-1}|^2}=\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}}, c_1(D)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|d_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|d_{in}|^2}=\sqrt{\sum_{k=0}^{n-1}T_k^2} \end{aligned}$$

So we have $\|T_{r^*}\|_2\leq r_1(C)c_1(D)=\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\sum_{k=0}^{n-1}T_k^2}.$

Thus, we can obtain

$$\sqrt{\sum_{k=0}^{n-1}T_k^2}\leq\|T_{r^*}\|_2\leq\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\sum_{k=0}^{n-1}T_k^2}.$$

(iii) From $|r|<1$,

$$\begin{aligned}
\|T_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)T_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2T_k^2 \\
&\geq \sum_{k=0}^{n-1}(n-k)|r^{n-k}|^2T_k^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2T_k^2 \\
&= n|r|^{2n}\sum_{k=0}^{n-1}(\frac{T_k}{|r|^k})^2 \\
&= \frac{n|r|^{2n}}{4}\left(\sum_{k=0}^{n-1}(\frac{\alpha^2}{|r|^2})^k+\sum_{k=0}^{n-1}(\frac{\beta^2}{|r|^2})^k+2\sum_{k=0}^{n-1}(\frac{\alpha\beta}{|r|^2})^k\right) \\
&= \frac{n|r|^2}{4}\left(\frac{2|r|^{2n+2}-2|r|^{2n}T_2-2|r|^2T_{2n}+2T_{2n-2}}{|r|^4-2|r|^2T_2+1}+\frac{2|r|^{2n}-2}{|r|^2-1}\right).
\end{aligned}$$

Hence, $\frac{|r|}{2}\sqrt{\frac{2|r|^{2n+2}-2|r|^{2n}T_2-2|r|^2T_{2n}+2T_{2n-2}}{|r|^4-2|r|^2T_2+1}+\frac{2|r|^{2n}-2}{|r|^2-1}}\leq \|R_{r^*}\|_2$.For another, let the matrices C and D be defined as the same as mentioned above:

$$C=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ r^2 & r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{pmatrix}_{n\times n}$$

$$\begin{aligned}
r_1(C) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|c_{nj}|^2} \\
&= \sqrt{n},
\end{aligned}$$

$$c_1(D)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|d_{ij}|^2}=\sqrt{\sum_{j=1}^{n}|d_{in}|^2}=\sqrt{\sum_{k=0}^{n-1}T_k^2}$$

Hence, we have

$$\|R_{r^*}\|_2\leq\sqrt{n\sum_{k=0}^{n-1}T_k^2}.$$

4. Conclusion

In this paper, we approximated lower and upper bounds of the spectral norms of geometric circulant matrices with the generalized Tribonacci sequence and the Chebyshev polynomial.

In particular cases:

- Taking $a=0, b=c=1$, then we get the inequality of the spectral norms with the classical Tribonacci number.
- Taking $r=1$, we can get the same conclusion as the circulant matrix.

Author' Contributions

I contributed to each part of this work seriously and read and approved the final version of the manuscript.

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