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An Inequality for Closed Manifolds with Timelike Immersion and Negative Gauss Curvature

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Abstract: *In this paper an inequality for closed manifolds with timelike immersion and negative Gauss curvature is derived. It is computed by means of the mean curvature H and Gauss curvature G of timelike immersed manifold.*

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- $\int_V H^2 dV \geq \int_V \sqrt{|-3G^2-2|}dV$.

Keywords: *Total Absolute Curvatue, Lorentz Space, Timelike Immersion, Mean Curvature, Gauss Curvature*

1. Introduction

Let M^2 be an oriented closed surface with a timelike immersion : $M^2 \to L^4$. Let $F(M^2)$ and $F(L^4)$ be the bundles of orthonormal frames $M^2$and L^4 respectively. Let B be the set of elements $b = (p, l_1, l_2, l_3, l_4)$ such that $(p, l_1, l_2) \in F(M^2)$ and $b = (x(p), l_1, l_2, l_3, l_4) \in F(L^4)$ whose orientation is coherent with the one of L^4, identifying l_i with $dx(l_i)$, $i = 1,2$ where l_i are unit vectors and l_2 is a timelike vector.

Define $\tilde{x} : B \to F(L^4)$ naturally by $b \to (x(p), l_1, l_2, l_3, l_4)$. The structure equations of $L^4$are given by

$$dx = \sum \widetilde{w}_A l_A \qquad dl_A = \sum \widetilde{w}_{AB} l_B \qquad \widetilde{w}_{AB} + \widetilde{w}_{BA} = 0$$

$$d\widetilde{w}_A = \sum \widetilde{w}_B \Lambda \, \widetilde{w}_{BA} \qquad d\widetilde{w}_{AB} = \sum \widetilde{w}_{AC} \, \Lambda \widetilde{w}_{CB} \quad A, B, C = 1,2,3,4$$

where $\widetilde{w}_A$, $\widetilde{w}_{AB}$ are differential 1- forms on $F(L^4)$.

Let w_A ,w_{AB} be induced 1- forms on B from $\widetilde{w}_A$, $\widetilde{w}_{AB}$ by the mapping $\tilde{x}$. Then we have

$$w_3 = w_4 = 0$$

$$w_{i3} = A_{3i1} w_1 + A_{3i2} w_2$$

$$w_{i4} = A_{4j1} w_1 + A_{4j2} w_2 \quad ; \quad i, j = 1,2$$

Let $(p, l_1, l_2, \tilde{l}_3, \tilde{l}_4)$ be a local cross-section of $B \to F(M^2)$. The restriction of A_{rij} onto the image of local cross-section is denoted by $\bar{A}_{rij}$ where $= 3, 4$.

We can compute second fundamental form as

$$II(dp,dp) =< S(dp),dp >$$

where S is the shape operator of the immersion

$$II(dp, dp) = w_1^2 < S(l_1), l_1 > +2w_1 w_2 < S(l_1), l_2 > +w_2^2 < S(l_2), l_2 >$$

$$S(l_1) = D_{11} l_4 = A_{411} l_1 - A_{421} l_2$$

$$S(l_2) = D_{12} l_4 = A_{412} l_1 - A_{422} l_2$$

$$S = \begin{pmatrix} A_{411} & -A_{421} \\ A_{412} & -A_{422} \end{pmatrix}$$

$$II(dp, dp) = A_{411} w_1^2 + 2A_{412} w_1 w_2 + A_{422} w_2^2$$

is the second fundamental form.

Theorem 1: Let M^2 be a 2-dimensional oriented closed manifold with a timelike immersion : $M^2 \rightarrow L^4$. If (A_{4ij}) is the shape operator of the timelike immersion then Lipschitz- Killing curvature $K(p,l)$ is given by

$$K(p,l) = -\lambda_1(p) cos^2\theta - \lambda_2(p) sin^2\theta$$

where l is the unit normal vector and

$$\lambda_1(p) = \det(\bar{A}_{3ij})$$

$$\lambda_2(p) = \det(\bar{A}_{4ij})$$

Proof: Choose l as

$$l = l_4 = \cos\theta\, \tilde{l}_3 + sin\theta \tilde{l}_4$$

$$A_{4ij} = cos\theta \bar{A}_{3ij} + sin\theta \bar{A}_{4ij}\ ; i, j = 1,2$$

The Lipschitz- Killing curvature $K(p,l)$ is determined by

$$K(p,l) \equiv \det A_{4ij} = det \begin{pmatrix} cos\theta \bar{A}_{311} + sin\theta \bar{A}_{411} & -cos\theta \bar{A}_{312} - sin\theta \bar{A}_{412} \\ cos\theta \bar{A}_{312} + sin\theta \bar{A}_{412} & -cos\theta \bar{A}_{322} - sin\theta \bar{A}_{422} \end{pmatrix}$$

The determinant is a quadratic form of $cos\theta$ and $sin\theta$. It will be derived as

$$K(p,l) = -\lambda_1(p) cos^2\theta - \lambda_2(p) sin^2\theta$$

By using an orthonormal frame where

$$\lambda_1(p) = \det(\bar{A}_{3ij})$$

$$and$$

$$\lambda_2(p) = \det(\bar{A}_{4ij})$$

$\lambda_1(p)$, $\lambda_2(p)$ are continuous on M^2.The Gauss curvature $G(p)$ is given by

$$G(p) = \lambda_1(p) + \lambda_2(p)$$

as in [1].

Theorem 2: Let M^2 be a 2-dimensional oriented closed manifold with a timelike immersion : $M^2 \rightarrow L^4$. If $G(p) = \lambda_1(p) + \lambda_2(p)$ is negative Gauss curvature of M^2 then the total absolute curvature $K^*(p)$ at point p is

$$K^*(p) = -\pi G(p)$$

on V and

$$K^*(p) = (2\alpha - \pi) G(p) + 4\sqrt{-\lambda_1 \lambda_2}$$

on U where

$$U = \{p \in M^2, \lambda_1(p) > 0\}$$

$$V = \{p \in M^2, \lambda_1(p) \leq 0\}$$

Proof: Since λ_1 and λ_2 are both negative on V we have

$K^*(p) = \int_0^{2\pi} |K(p,l)| d\theta$ where $K(p,l)$ is the Lipschitz- Killing curvature

$$K^*(p) = \int_0^{2\pi} |K(p,l)| d\theta$$
$$= \int_0^{2\pi} |-\lambda_1(p)cos^2\theta - \lambda_2(p)sin^2\theta|\, d\theta$$
$$= \int_0^{2\pi} |-1||\lambda_1(p)cos^2\theta + \lambda_2(p)sin^2\theta|\, d\theta$$
$$= \int_0^{2\pi} -(\lambda_1(p)cos^2\theta + \lambda_2(p)sin^2\theta) d\theta$$
$$= -\pi(\lambda_1(p) + \lambda_2(p))$$
$$= -\pi G(p)$$

Since λ_1 is positive on U and $G(p) \leq 0$ we have a negative λ_2 such that $|\lambda_2| \geq |\lambda_1|$.

Total absolute curvature on U is

$$K^*(p) = \int_0^{2\pi} |K(p,l)| d\theta$$
$$= \int_0^{2\pi} |-\lambda_1(p)cos^2\theta - \lambda_2(p)sin^2\theta|\, d\theta$$
$$= \int_0^{2\pi} |-1||\lambda_1(p)cos^2\theta + \lambda_2(p)sin^2\theta|\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} |(\lambda_1 + \lambda_2) + (\lambda_1 - \lambda_2)cos2\theta|\, d\theta$$
$$= \frac{1}{2}(\lambda_1 - \lambda_2)\int_0^{2\pi} \left|\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} + cos2\theta\right| d\theta$$

Define an angle α such that

$$cos\alpha = -\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\,; \quad 0 < \alpha \leq \frac{\pi}{2} \quad \text{so} \quad sin\alpha = \frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1-\lambda_2}$$

$$K^*(p) = \frac{1}{2}(\lambda_1 - \lambda_2)\int_0^{2\pi} |cos2\theta - cos\alpha| d\theta$$
$$= \frac{1}{4}(\lambda_1 - \lambda_2)\int_0^{4\pi} |cost - cos\alpha| dt$$
$$= (2\alpha - \pi)G(p) + 4\sqrt{-\lambda_1\lambda_2}$$

Theorem 3: Let M^2 be a 2-dimensional oriented closed manifold with a timelike immersion $: M^2 \to L^4$. If $G(p) = \lambda_1(p) + \lambda_2(p)$ is negative Gauss curvature of M^2 then for the mean curvature H of M^2 in L^4 we have

$$\int_V H^2\, dV \geq \int_V \sqrt{|-3G^2 - 2|}$$

Proof: Let for the frame (p, l_1, l_2, l_3, l_4) ; l_1 *and* l_2 be the principal directions with respect to l_4.

Choose $\bar{A}_{rij}$ as follows

$$\bar{A}_{311} = a\,;\ \bar{A}_{312} = \bar{A}_{321}\ - c\,;\ \bar{A}_{322} = -b$$
$$\bar{A}_{411} = d\,;\ \bar{A}_{422} = -e\,;\ \bar{A}_{412} = \bar{A}_{421} = 0$$

where a,b,c,d,e are all positive. $\bar{A}_{3ij} = \begin{pmatrix} a & -c \\ -c & -b \end{pmatrix}$ *and* $\bar{A}_{4ij} = \begin{pmatrix} d & 0 \\ 0 & -e \end{pmatrix}$ then

$$\lambda_1(p) = \det(\bar{A}_{3ij}) = -ab - c^2$$
$$\lambda_2(p) = \det(\bar{A}_{4ij}) = -de$$

where $\lambda_2 \leq \lambda_1 \leq 0$.

Shape operator is given by $S = \begin{pmatrix} a & -c & 0 & 0 \\ -c & -b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & -e \end{pmatrix}$. Mean curvature is then

$$H = \frac{a-b+d-e}{2}$$

$$H^2 = \frac{(a-b+d-e)^2}{4}$$

$$4H^2 = (a-b)^2 + (d-c)^2 + 2(a-b)(d-e)$$

$$= (a-b)^2 + (d-c)^2 + 2(ad - ae - bd + be)$$

Since

$$\bar{A}_{311}\bar{A}_{411} + \bar{A}_{322}\,\bar{A}_{422} = \bar{A}_{311}\bar{A}_{422} + \bar{A}_{322}\bar{A}_{411} = \bar{A}_{312}\bar{A}_{412} + \bar{A}_{421}\bar{A}_{312}$$

We have $\quad ad - ae - bd + be = 0$

$$4H^2 = (a-b)^2 + (d-e)^2$$

$$4H^2 \geq 4|ab| + 4|de|$$

$$4H^2 \geq 8\sqrt{|abde|}$$

$$\lambda_1\lambda_2 = (-ab - c^2)(-de)$$

$$= abde + dec^2$$

$$abde = \lambda_1\lambda_2 - dec^2$$

$$abde = \lambda_1\lambda_2 + \lambda_2 c^2$$

Let $c = 1$ we have $abde = \lambda_1\lambda_2 + \lambda_2$

$$4H^2 \geq 8\sqrt{|\lambda_1\lambda_2 + \lambda_2|}$$

$$H^2 \geq 2\sqrt{|\lambda_1\lambda_2 + \lambda_2|}$$

$$H^4 \geq 4|\lambda_1\lambda_2 + \lambda_2|$$

For $V = \{p \in M^2, \lambda_1(p) \leq 0\}$ we get

$$\int_V H^4\, dV \geq \int_V 4|\lambda_1\lambda_2 + \lambda_2|\, dV$$

If we substitute $G(p) = \lambda_1(p) + \lambda_2(p)$ in $|\lambda_1(p)\lambda_2(p) + \lambda_2(p)| = |\lambda_2(p)\lambda_1(p) + 1|$ then we get

$$|\,\lambda_1(p)\lambda_2(p) + \lambda_2(p)| = |\,\lambda_2(p)[(G(p) - \lambda_2(p)) + 1]\,|$$

$$= \left|\,\lambda_2(p)G(p) - \lambda_2^2(p) + \lambda_2(p)\,\right|$$

and since $\quad \int_V -\lambda_2 dV \geq -\frac{1}{2}\int_V GdV \quad$ in [1]

$$\int_V H^4\, dV \geq \int_V 4\left|\lambda_2 G - \lambda_2^2 + \lambda_2\right| dV$$

$$\geq \int_V 4\left|\lambda_2(G+1) - \lambda_2^2\right| dV$$

since $\lambda_2(p)(G(p)+1) - \lambda_2^2(p) \leq 0$ on V

We get the inequality

$$\int_V H^4\, dV \geq \int_V (-4\lambda_2(G+1) - \lambda_2^2)\, dV$$

Since $-\frac{1}{2}G \leq -\lambda_2$

$$\int_V H^4\, dV \geq \int_V [-\frac{1}{2}G(G+1) - \left(-\frac{1}{2}G\right)^2]\, dV = \int_V (-3G^2 - 2)dV$$

$$H^4 \geq -3G^2 - 2 \Rightarrow H^2 \geq \sqrt{|-3G^2 - 2|}$$

Finally we get the inequality

$$\int_V H^2\, dV \geq \int_V \sqrt{|-3G^2 - 2|}\, dV$$

REFERENCES

[1] Bang-Yen Chen , On an inequality of T.J.Willmore, Proceedings of the American Mathematical Society, 1970

[2] B.O'Neill, Semi-Riemannian Geometry, Academic Press, NewYork, 1983

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