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Distribution Patterns of the Terms of n^{th} Order Determinant/ Cycles of S_n "A Series Delineation"

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Abstract: *The methods of evaluating a determinant described so far, focus primarily on the associated value. A focus over the permutation method, engenders a series representing the distribution patterns of the terms of the determinant as well as the permutation cycles of a symmetric group of the same order. Our paper aims to analyze the basic ideas therein with logical corroboration.*

Keywords: *Permutation, determinant, distribution pattern, series representation*

1. Introduction

Multiple methods of evaluating a determinant have been described in the literature. Notable amongst them include Laplace method [1] and permutation method [2]. Each of the methods offers its own sets of merits and drawbacks. A determinant of n^{th} order consists of $n!$ terms with each term as a product of n elements, one from each row and one from each column. The permutation method of evaluating a n^{th} order determinant involves the permutation of the symbols (suffices) representing either the n rows only or the n columns only, when the variation of the other being rendered in the natural order of occurrence of the elements in the product comprising the term [3]. Therefore, when suffices representing rows are subjected to change in the natural order of occurrence of the elements, the permutations of the symbols representing the column solely determine the terms of the determinant. The positive or negative sign is assigned to each term according as the corresponding permutation is even or odd respectively.

2. Results and Discussion

In a determinant of a square matrix, $A=(a_{ij})$ of order n, there are n different symbols, 1,2,3,4,……..,n representing the columns which can be arranged among themselves in $n!$ ways, thus giving rise to $n!$ terms, each term being the product of n elements. To write it in the form of n columns each containing $(n-1)!$ terms we fix 1 in the first column written $(n-1)!$ times followed by remaining n-1 digits 2,3,4,…,n each written $(n-2)!$ times in the respective order of their occurrence. The third will then be the one among the remaining n-2 digits, each written $(n-3)!$ times in the order of occurrence and so forth. The last digit as a result ends up being the last remaining one. The other columns, 2^{nd},3^{rd}, ..., n^{th} can then be obtained by adding 1,2,3, …, n-1 to the respective symbols representing the first column [4]. The first column, indeed, represents the value of the determinant of order n-1. The reason, why just the first column represents the determinant of order n-1, despite of the involvement of n elements (symbols) can be better understood on the basis of the permutation of the elements (symbols) determining the terms of the determinant.

In the first column the first element in each of the product representing the term is a_{11} (represented by 1). This means that the first digit always occupies the first position, or is not permuted throughout the column. The actual permutation therefore takes place of the remaining n-1 digits 2,3,4, …, n. This happens in none of the other columns. In the other columns, none of the elements(digits) are found to occur throughout in the natural specified order as though if one does in a term, the repetition does not last throughout the column. Hence,the actual permutation involving n digits takes place in those columns. Therefore in each digit group representing the term of the first column, if we delete 1 from the first position and replace each of the remaining digits representing the term by the digit less by 1 than itself i.e. writing 1 2 3 4…….(n-1) in place of 1 2 3 ……. (n-1) n by means of the replacement as shown:

~~1~~ 2 3 4 ………… (n-2) (n-1) n, this column can now be made to represent a determinant of order n-1. Extrapolating this further, its first column being a determinant of order n-1, can be arranged in the form of n-1 columns each containing *(n-2)*! terms. Arguing similarly its 1^{st} column can in turn be made to represent a determinant of order n-2. A successive execution of the idea eventually leaves us with just one digit (symbol) 1, which represents a_{11} or the value of the determinant of order 1. Regardless of the associated value, if we apply this concept to determinant of order 4 gives a series representation of its number of terms. Number of terms of a determinant of order 4 = (4-1)(4-1)! + (4-2)(4-2)! + (4-3)(4-3)! + 1= 3.3! + 2.2! + 1.1! + 1 = 24 = 4!

Based on the assertion made above, we posit that the number of terms of every determinant, except of order 1, when expressed in the form of series, contains in it the series representing the number of terms of the determinant of order less by one than itself. The application of this idea leads us to the generation of series specifying the number of terms of a determinant. The *n*! terms of the n^{th} order determinant can be split into n columns each containing *(n-1)*! terms. A successive splitting of the first column, that represents a determinant of order less by one, ultimately leaves us with the number of terms of a determinant of order 1, which is 1!=1. The *n*! terms of a n^{th} order determinant can therefore be represented in the form of series:n! $=\sum_{k=2}^{n-1}(k-1)(k-1)!$ +1. This fact can be proved mathematically as shown in the following proposition.

2.1 Proposition *For any positive integer n, we have:* $n! = \sum_{k=2}^{n-1}(k-1)(k-1)! + 1$.

Proof: We have, $t_1 = 1$. Also, for n>1, k^{th} term, $t_k=(k-1)(k-1)! = k! - (k-1)!$

$\therefore\ t_2 = 2! - 1!$

$t_3 = 3! - 2!$

. . .

. . .

. . .

$t_n = n! - (n-1)!$

Adding vertically yields, $t_1 + t_2 + t_3 + \dots + t_n = n!$

i.e. $1 + \sum_{k=2}^{n-1}(k-1)(k-1)! = n!$.

Next we discuss the distribution pattern of the cycles of a symmetric group S_n .The permutation method of evaluating a determinant offers several advantages besides determining the associated value. The procedure adopted to analyze one of such advantages itself reveals the pragmatic the justification of its application to another, as the analogy of the title demands. One amongst such benefits is that it enables to clarify the generation of the cycles of a permutation group of order *n* as well as their distribution patterns. The permutation of the digits 1, 2, 3, . . .,*n* serves as the basis for explanation [4]. For this we begin with the definition of symmetric group.

Definition: Let S be a non empty set and A(S), the set of all bijections from S to S. Under the operation of composition of functions, A(S) is a group called the group of permutations on S. If S= {1, 2, 3, …, n}, then A(S) is called the symmetric group on n letters and is denoted by S_n. [4]

Let us consider a one to one mapping '*f*' from the ordered set (1,2,3, ………. ,n) onto each of the

n! ordered set of digits representing the *n!* terms of a determinant, such that the k^{th} element of the first set is mapped to the k^{th} element of the second set. In these mappings, if we take in account only those digits that are mapped by '*f*' to something different and rearrange them in order such that each digit follows its image (the image of the last digit being the first one), this arrangement constitutes the $n!$ cycles of the symmetric group S_n in the form of n column each containing $(n-1)!$cycles [4].

If we choose any one of the $(n-1)!$ rows of the representation and move along, it is observed clearly that each digit is found to mapped to itself only once in the row. Hence, the average length of the n cycles in each row is n-1 (considering the sum of lengths in the case of the product of disjoint cycles). As a consequence, the total length of n cycles in each row is n(n-1) and hence the total length of all the $n!$ cycles of the $(n-1)!$ rows is n(n-1) $(n-1)!$ $=(n)!$ $(n-1)$. It is interesting to note that the distribution of the lengths across the columns is however, not uniform as in the case of rows. The last $(n-1)$ columns which inherit the permutation of all the n digits share the lengths equally whilst the first column, where at most the permutation of $(n-1)$ digits can occur, contains the sum of lengths of the cycles of the symmetric group S_{n-1} and hence the total sum of lengths of the cycles in it is $(n-1)!$ $(n-2)!$. The total length of cycles to be distributed among the last $(n-1)$ columns is therefore $n!(n-1)$ - $(n-1)!$ $(n-2)$ and thence [$n(n-1)!$ - $(n-1)!$ $(n-2)$] /$n-1$ or n - $(n-2)!$ $(n-2)$ in each. Arguing in an analogous fashion, the first column containing the cycles of a symmetric group of order less by one than the pre-existing one i.e.

S_{n-2}, the distribution of cycles among $n-1$ column each containing $(n-2)!$cycles can be similarly made. Application of this idea thereof in succession forces us to reach to S_1 ultimately that has just one cycle representing the identity permutation. The distribution pattern of the lengths of the cycles of S_n can therefore be represented by a series as in the following Proposition:

2.2 Proposition: For any positive integer n,

$$n!(n-1) = \sum_{r=1}^{n}[r!\,(r-1)-(r-1)!\,(r-2)] \quad (1)$$

$$= \sum_{r=1}^{n}(r-1)[((r-1)^2+1)(r-2)!] \quad (2)$$

Proof: The proof follows readily, as for each value of r the right side of the expression (1) is the difference of two parts with one positive and the other negative. For r=1, the value is zero, and for r=2, the negative part is zero. Moreover, for the rest of the values of r, the positive part for one value of r cancels the negative part of the succeeding value of r. Consequently, in the sum for the n values of r, we will finally be left with the positive part of the n^{th} value of r, which is $n!(n-1)$.

As an illustration, we apply the above notion to find out the distribution of the terms of a determinant of order 6 as well as the distribution of the cycles (including lengths of cycles) in S_6.

In a determinant of order 6, the number of terms = 6! = 720.

Now, $6! = 1+ 1\times1! + 2\times2! + 3\times3! + 4\times4! + 5\times5!$

$= 1+ 1\times1 + 2\times4 + 3\times6 + 4\times24 + 5\times120$

The sum of the first 'r' terms denote the number of terms in first column (and hence the other columns) of the determinant of order 'r+1'. The distribution of the number of cycles takes place accordingly. Also,

total length (sum of lengths) of all the cycles in S_6 = $n!(n-1) = 6!(6-1) = 720\times5 = 3600$.

Using, $n!(n-1) = \sum_{r=1}^{n}(r-1)[((r-1)^2+1)(r-2)!]$, we get

$6!\times5 = 0 + 1\times2 + 2\times5 + 3\times20 + 4\times102 + 5\times624$.

The sum of the first 'r' terms denote the total length of all the cycles of a symmetric group S_r and hence the sum of lengths of all the cycles in the first column of the symmetric group S_{r+1}. Hence, it is clear that the distribution of the length of the cycles across the column is not uniform.

3. Conclusion

The mathematical modulation of the distribution of the terms of an^{th} order determinant and the cycles of a symmetric group S_n in the form of series and the corresponding distribution as directed by the series above serves as a tool for the proposed work. Moreover, it enables to understand how the distribution pattern of two determinants or two symmetric groups of different orders are correlated.

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