# Efficiency and Vectorial Optimization

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**Abstract:** *This scientific research report was presented, in the quality of session chair and as an expert speech, at the 5th International Multi – Disciplinary Conference organized by Institute of Research Engineers and Doctors - IRED, USA September 02 – 03, 2017, Zurich, Switzerland. It is devoted to the general Efficiency presented inside the best appropriate environment of the infinite dimensional Ordered Vector Spaces, with its projections in Vectorial Optimization, that is, in Euclidean spaces $R^n, n \in N^*$ which can be extended, mutatis mutandis, to $R^I$ for every non –empty set $I$. Following our refined recent results, we especially use the largest class of the Convex Cones discovered till now in separated Locally Convex Spaces, named by us "Isac's Cones", and ensuring the existence together with important properties for the efficient points under completeness instead of compactness, with pertinent applications.The couched Open problems are generated by the new links between the General Efficiency, the Vector (Strong) Optimization and the Choquet Boundaries, since the Efficiency is strongly related to the General Optimization, the Potential Theory and conversely, with projections in the new recent and future fields of Scientific Research: Theory and Applications of the Generalized Dynamical Systems, Fixed points Theory, the Best Approximation Theory, the study of the Conically Bounded Sets, the study of the Nuclearity for the Topological Vector Spaces, Vector Optimization for multivalued functions and so on.*

**Keywords:** *Ordered Vector Space, General Efficiency, Multifunction, Pareto Optimality, Locally Convex Space, Isac's (nuclear, supernormal) Cone (set), Choquet Boundary, Best Approximation, Spline function, Multi – Objective Optimization, Strong and Vectorial Optimal Solution.*

## 1. INTRODUCTION

Until now, the basis on which was developed our general concept of the Efficiency in the Ordered Linear Spaces and its Applications, supplied by the Ordered Locally Convex Spaces, is represented about the vastness class of the Convex Cones introduced by Professor Isac in 1981, published in 1983, called by us and, officially recognized, as *"Isac's Cone"* in 2009, after the acceptance of this last agreed denomination by Professor Isac. To Professor Isac is devoted this scientific research work.We think that it is not possible to obtain more than these results concerning the General Efficiency in the world of the separated Locally Convex Spaces.They represent the most development background. All the elements concerning the ordered topological vector spaces used here are in acordance with *Nachbin, L., 1965* and *Peressini, A., L., 1967*. The bibliography of this research work represents a synthesis of all the examined references.

## 2. GENERAL EFFICIENCY IN THE ORDERED VECTOR SPACES

Let $E$ be any real or complex vector space with the origin element $\theta$, ordered by a convex cone $K$, that is, $K + K \subseteq K$ and $\chi K \subseteq K$ for all $\chi \in R_+$, $K_1$ a non - void subset of $K$ and $A$ a non - empty subset of $E$. The following definition introduces a new concept of *the approximate efficiency*, which expresses *the general efficiency*, generalizes the well known notion of Pareto type efficiency in every Euclidean space and not only.

**Definition 1.** *(Postolică,V., 2002, 2008). We say that $a_0 \in A$ is a $K_1 - minimal\ efficient\ point\ of\ A$, in notation, $a_0 \in MIN(A, K, K_1)$ if it satisfies one of the following equivalent conditions:*

*(i)* $\quad A \bigcap (a_0 - K - K_1) \subseteq a_0 + K + K_1;$

*(ii)* $\quad (K + K_1) \bigcap (a_0 - A) \subseteq -K - K_1;$

In a similar manner one defines the $K_1 - maximal\ efficient\ points$ by replacing $K + K_1$ with $-(K + K_1)$. The set of such as these points will be denoted by $MAX(A, K, K_1)$. Whenever $K_1 = \{\theta\}$ we consider $MIN(A, K, K_1) = MIN(A, K)$ and $MAX(A, K, K_1) = MAX(A, K)$, respectively. *All the efficient points of the set A with respect to the convex cone $K$ following $K_1$ are represented by the set* $eff(A, K, K_1) = MIN(A, K, K_1) \cup MAX(A, K, K_1)$. Consequently, $eff(A, K) = MIN(A, K) \cup MAX(A, K)$. It is obvious that

$$A \bigcap (a_0 \mp K) \subseteq a_0 \pm K_1 \Rightarrow A \bigcap (a_0 \pm K \pm K_1) \subseteq a_0 \mp K \mp K_1 \Rightarrow A \bigcap (a_0 \mp K_1) \subseteq a_0 \pm K,$$

which suggests other concepts for the *efficiency and its approximate term* in *the general ordered linear spaces.*

**Remark 1.** We notice that $a_0 \in eff(A, K, K_1)$ iff it is a fixed point for at least one of the multifunctions $F_\mp : A \to 2^A$ defined by $F_\mp(t) = \{a \in A : A \bigcap (a \mp K \mp K_1) \subseteq t \pm K \pm K_1\}$, that is, $a_0 \in F_\mp(a_0)$, with the corresponding signs. Consequently, for the existence of these general efficient points we can also apply appropriate fixed points theorems concerning the multifunctions (see, for instance, *Cardinali, T., Papalini, F., 1994, Park, S.,1992, Patriche, M.,2013, Zhang Cong – Jun, 2005 and any other proper scientific papers*).

**Remark 2.** *Németh, A.B., 1989*, proved that, whenever $K_1 \subseteq K \setminus \{\theta\}$, the existence of this new type of the efficient points for the lower bounded sets characterizes the semi - Archimedian Ordered Vector Spaces and the Regular Ordered Locally Convex Spaces.

**Remark 3.** Whenever $K$ is pointed, that is, $K \bigcap (-K) = \{\theta\}$, then

$a_0 \in MIN(A, K, K_1)(a_0 \in MAX(A, K, K_1))$ means that $A \bigcap (a_0 \mp K \mp K_1) = \varnothing$ or, equivalently, $(\pm K + K_1) \bigcap (a_0 - A) = \varnothing$ for $\theta \notin K_1$ and $A \bigcap (a_0 \mp K \mp K_1) = \{a_0\}$, respectively, if $\theta \in K_1$. Clearly, $K \subseteq K + K_1$ if $\theta \in K_1$ and, if the convex cone $K$ is pointed, then $K \subseteq K + K_1 \Leftrightarrow \theta \in K_1$. This inclusion is not always valid as we can see taking $E = R^2$, $K = \{(x; y) \in R^2 : x \in R, y \geq 0\}$ and $K_1 = \{(1,1)\}$, but, in our considered case, it remains true for $K_1 = \{(0, -1)\}$. Whenever $K_1 = \{\theta\}$, from **Definition 1** one obtains the usual concept of the efficient (*Pareto minimal, optimal or admissible*) point : $a_0 \in MIN(A, K)(MAX(A, K))$ if it fulfils *(i)*, *(ii)* or any of the next equivalent properties:

*(iii)* $\quad (A \pm K) \bigcap (a_0 \mp K) \subseteq a_0 \pm K$ ;

*(iv)* $\quad K \bigcap (a_0 - A \mp K) \subseteq \mp K$

This shows that $a_0$ is a fixed point for at least one of the following multifunctions:

$$F_1 : A \to A, F_1(t) = \{a \in A : A \bigcap (a \mp K) \subseteq t \pm K\},$$

$$F_2 : A \to A, F_2(t) = \{a \in A : A \bigcap (t \mp K) \subseteq a \pm K\},$$

$$F_3 : A \to A, F_3(t) = \{a \in A : (A \pm K) \bigcap (a \mp K) \subseteq t \pm K\},$$

$$F_4 : A \to A, F_4(t) = \{a \in A : (A \pm K) \bigcap (t \mp K) \subseteq a \pm K\},$$

that is, $a_0 \in F_i(a_0)$ for some $i = \overline{1,4}.$ If, in addition, $K$ is pointed, then $a_0 \in A$ is an efficient point of $A$ with respect to $K$ if and only if one of the following equivalent relations holds :

(v)  $A \bigcap (a_0 \mp K) = \{a_0\}$;

(vi)  $A \bigcap (a_0 \mp K \setminus \{\theta\}) = \varnothing$;

(vii)  $(\pm K) \bigcap (a_0 - A) = \{\theta\}$;

(viii)  $(\pm K \setminus \{\theta\}) \bigcap (a_0 - A) = \varnothing$;

(ix)  $(A \pm K) \bigcap [a_0 \mp (K \setminus \{\theta\})] = \varnothing$.

We also notice that

$$MIN(A,K)(MAX(A,K)) = \bigcap_{\{0\} \neq K_2 \subseteq K} MIN(A,K,K_2)(\bigcap_{\{0\} \neq K_2 \subseteq -K} MAX(A,K,K_2)).$$ Moreover,

$a_0 \in eff(A,K)$ if only if it is *a critical (ideal or balance) point ( see, for example, Kim, W. K., Tan, K.K., 2001; Isac,G. 1985; Isac, G., Bulavsky, V. A., Kalashnikov, V., V. 2002; Isac,G., Postolică, V.,1993; Postolică, V., Scarelli, A., Venzi,L., 2001; Postolică, V., 2004 and the corresponding references)* for *the generalized dynamical systems* $\Gamma_\mp : A \to 2^A$ defined by $\Gamma_\mp(a) = A \bigcap (a \mp K), a \in A.$ In this way , $eff(A,K)$ describes *the balance extremum moments* for $\Gamma_\mp$, which in *the abstract market context*, expresses *the competitive* equilibrium/non-*equilibrium consisting of the general relation price/consumption.* Considering $K_1 = \{\varepsilon\} (\varepsilon \in K \setminus \{\theta\})$, one obtains that $a_0 \in eff(A,K,K_1)$ iff $A \bigcap (a_0 - \varepsilon \mp K) = \varnothing$. In all these cases, for the set $MIN(A,K,K_1)$ we used the notation $\varepsilon - MIN(A,K)$. It is obvious that $MIN(A,K) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - MIN(A,K)]$, with the natural projections for $MAX(A,K)$ obtained by replacing the convex cone $K$ with $-K$. When referring to the existence of the efficient points, the significant properties of the sets containing these points and the applications in the usual or extended contexts we mention, as a comprehensive bibliography, *(Aubin, J.P., 1993; Benson, H. P., 1077, 1979,1983;Borwein, J. M., 1977, 1983;Dauer, J. P., Gallagher, R. J., 1990; Dominiak, Cezary, 2006; Ehrgott, Matthias, 2005;Fishburn, P. C.,1984; Gass, Saul.,I., Harris, Carl., M., 2001; Geoffrion, A. M;, 1968; Hartley, R., 1978; Henig, M. I., 1982; Heyne, Paul, 2000; Hyers, D., H., Isac, G., Rassias,T.M.,1997; Isac, G., 1981, 1983, 1985, 1994, 1998,2003, Isac, G., Bahya, A. O., 2002; Isac, G., Postolică, V.,1989, 1993,1995, 1997, 1999, 2001, 2002, 2009, 2014, 2015, 2016; Isac, G., Tammer,Chr.,2003; Jablonsky, J.,2006; Loridan, P., 1984; Luc, D. T., 1989; Luptáčik, M., Bőhm, B., 2006; Németh, A., B., 1989; Ng., K., F., Zheng, X., Y., 2002; Postolică, V.,1989, 1993, 1994, 1997, 1999, 2001, 2002, 2004, 2008, 2009; Songsak Sriboonchitta, Wing – Keung Wong, Sompong Dhopongsa, Hung, T., Nguyen, 2009; Sontag, Z., Zălinescu, C., 2000; Stewart, D., T., 2004; Truong, X. D. H., 1994; Zimmermann, H., 2000),* with widely established results.

**Open Problem 1.** *Find general criteria to obtain $eff(A,K,K_1) \subseteq eff(A,\widetilde{K},K_1)$ from $K \subseteq \widetilde{K}$ when $K, \widetilde{K}$ are convex cones in $E$ and $eff(A,K,K_1) \subseteq eff(A,K,K_2)$ if $K_1 \subseteq K_2 \subseteq K$, respectively.*

In order to offer a first solution, we remark that if $K \subseteq \widetilde{K} \subseteq K + K_1$, then

$eff(A, K, K_1) = eff(A, \widetilde{K}, K_1)$: it is sufficient to prove that $MIN(A, K, K_1) = MIN(A, \widetilde{K}, K_1)$. Indeed, if $a_0 \in MIN(A, K, K_1)$, then $A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1$. On the other hand, $A \cap (a_0 - \widetilde{K} - K_1) \subseteq A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1 \subseteq a_0 + \widetilde{K} + K_1$, that is,

$a_0 \in MIN(A, \widetilde{K}, K_1)$. Let now $A \cap (a_0 - \widetilde{K} - K_1) \subseteq a_0 + \widetilde{K} + K_1$. Then,

$A \cap (a_0 - K - K_1) \subseteq A \cap (a_0 - \widetilde{K} - K_1) \subseteq a_0 + \widetilde{K} + K_1 \subseteq a_0 + K + K_1$. So,

$a_0 \in MIN(A, K, K_1)$. Consequently, $MIN(A, K, K_1) = MIN(A, \widetilde{K}, K_1)$. This completes the proof. In a similar manner one proves that $MAX(A, K, K_1) = MAX(A, \widetilde{K}, K_1)$ under the same hypothesis and we conclude that $eff(A, K, K_1) = eff(A, \widetilde{K}, K_1)$. Let now $K_1 \subseteq K_2 \subseteq K_1 + K$ and $a_0 \in MIN(A, K, K_1)$, that is, $A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1$. Then,

$A \cap (a_0 - K - K_2) \subseteq A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1 \subseteq a_0 + K + K_2$. So,

$a_0 \in MIN(A, K, K_2)$. At the same time, if $A \cap (a_0 - K - K_2) \subseteq a_0 + K + K_2$, then $A \cap (a_0 - K - K_1) \subseteq A \cap (a_0 - K - K_2) \subseteq a_0 + K + K_2 \subseteq a_0 + K + K_1$, that is,

$a_0 \in MIN(A, K, K_1)$. Therefore, $MIN(A, K, K_1) = MIN(A, K, K_2)$. The above assumption on $K, K_1$ and $K_2$ ensures, in a similar manner, that $MAX(A, K, K_1) = MAX(A, K, K_2)$ and thus $eff(A, K, K_1) = eff(A, K, K_2)$. We also have $eff(A, \widetilde{K}) \subseteq eff(A, K)$ if $K$ is pointed and $K \subseteq \widetilde{K}$. The hypotheses $K \subseteq \widetilde{K} \subseteq K + K_1$ and $K_1 \subseteq K_2 \subseteq K_1 + K$ seem to be too strong to solve the formulated open problem.

**Open Problem 2.** *Let us denote by* $2^E$, $2^K$, $2^{K_1}$ *the families of all subsets for* $E$, *all the convex cones in* $E$, *all the non – empty subsets of these cones, respectively, and the multifunction* $eff : 2^E \times 2^K \times 2^{K_1} \rightarrow 2^E$ *defined by* $(A, K, K_1) \in (2^E \times 2^K \times 2^{K_1}) \rightarrow eff(A, K, K_1) \in 2^E$. *The study of the properties for these multifunctions, starting with their restrictions.*

Thus, for example, *if* $E$ *is any linear topological vector space, the lower (upper) semicontinuity* and *the continuity* of this map are important for *the stability investigation* in *the optimization problems* concerning *the efficiency*.

Let us consider the set $SMIN(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ of all *strong minimum points* of $A$ with respect to $K$ by $K_1$ and $SMAX(A, K, K_1) = \{a_2 \in A : A \subseteq a_2 - K - K_1\}$ the set of all *strong maximum points* of $A$ with respect to $K$ by $K_1$, respectively, and

$S(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\} \cup \{a_2 \in A : A \subseteq a_2 - K - K_1\}$

The following theorem shows the immediate connection between *the approximate efficiency* and *the strong optimization* through the agency of *the cone minimal (cone maximal) efficient points.*

**Theorem 1.** *If* $S(A, K, K_1) \neq \varnothing$, *then* $S(A, K, K_1) = eff(A, K, K_1)$.

**Proof.** In fact, we prove that $SMIN(A, K, K_1) \neq \varnothing \Rightarrow MIN(A, K, K_1) = SMIN(A, K, K_1)$ and $SMAX(A, K, K_1) \neq \varnothing \Rightarrow MAX(A, K, K_1) = SMAX(A, K, K_1)$, so it is sufficient to show the first implication. Clearly, $SMIN(A, K, K_1) \subseteq MIN(A, K, K_1)$. Indeed, if $a_0 \in SMIN(A, K, K_1)$ and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then $a \in a_0 + K + K_1$, that is, $a_0 \in MIN(A, K, K_1)$ by virtue of the point (i) contained in Definition 1. Suppose now that $\bar{a} \in SMIN(A, K, K_1) \neq \varnothing$ and

there exists $a_0 \in MIN(A, K, K_1) \setminus SMIN(A, K, K_1)$. Out of the fact $\bar{a} \in SMIN(A, K, K_1)$, it follows that $a_0 \in \bar{a} + K + K_1$, that is, $\bar{a} \in a_0 - K - K_1$, from which, since $\bar{a} \in A$ and $a_0 \in MIN(A, K, K_1)$, we conclude that $\bar{a} \in a_0 + K + K_1$. Therefore,

$A \subseteq \bar{a} + K + K_1 \subseteq a_0 + K + K_1$, in contradiction with $a_0 \notin SMIN(A, K, K_1)$, as claimed.

**Remark 4.** It is obvious that we have always $S(A, K, K_1) \subseteq eff(A, K, K_1)$. The previous theorem shows that, every time when there exists at least one strong cone minimum or cone maximum point, the set of all efficient points coincides with the set of all these points, respectively. This is important also for the numerical methods, because if there exist *"strong efficient"* points, then they are the only *"vectorial efficient"* points. Using this conclusion and the abstract construction given by us in *(Postolică, V. 1991, 1988)* for the splines in *the H - locally convex* spaces introduced in *(Precupanu, T., 1969)* as separated locally convex spaces with any semi - norm satisfying the parallelogram law, linear topological spaces also studied with strictness in *(Kramar, E.,1981)*, it follows that the only best simultaneous and vectorial approximation for each element in the direct sum of any (closed) linear subspace and its orthogonal, with respect to a linear (continuous) operator between two arbitrary *H - locally convex spaces*, is its *spline function.*We also note that it is possible to have $S(A, K, K_1) = \varnothing$ and $eff(A, K, K_1) = A.$ Thus, for example, if one considers $X = R^n (n \in N, n \geq 2)$, $K = R_+^n$, $K_1 = \{(0,...,0)\}$ and for each real number $c$ one defines $A_{0c} = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i = c \right\}$,

$A_{1c} = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i \geq c \right\}$, $A_{2c} = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i \leq c \right\}$ then it is clear that

$S(A_{0c}, K, K_1) = S(A_{1c}, K, K_1) = S(A_{2c}, K, K_1) = \varnothing$,
$eff(A_{0c}, K, K_1) = MIN(A_{0c}, K, K_1) = MAX(A_{0c}, K, K_1) = A_{0c}$,
$MIN(A_{1c}, K, K_1) = MAX(A_{2c}, K, K_1) = A_{0c}$ so, by extrapolation,

$eff(A_{1c}, K, K_1) = A_{0c} \cup \{(+\infty, +\infty, ..., +\infty)\}$ and $eff(A_{2c}, K, K_1) = A_{0c} \cup \{(-\infty, -\infty, ..., -\infty)\}$. At the same time, in the usual real linear space of all sequences composed by real numbers, ordered by the convex cone $K = \{(x_n) : n \in N^*, n \geq 2, x_n \geq 0, \forall n \geq 2\}$, for $A = \{(x_{n\alpha}) : n \in N^*, n \geq 2, \alpha > 0\}$ with

$x_{n\alpha} = (n-1)^{-\alpha} - n^{-\alpha}, n \in N^*, n \geq 2, \alpha > 0$ and $K_1 = \{(0, 0, ....)\}$, we have

$eff(A, K, K_1) = S(A, K, K_1) = \varnothing$. If one considers $E = R^2$, $K = R_+^2 = \{(x, y) \in R^2 : x, y \geq 0\}$, $K_1 = \{(0, 0)\}$ and $A = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ then, $S(A, K) = \varnothing$,

$MIN(A, K) = MAX(A, K) = \widehat{A_1 B_1} \cup \widehat{C_1 D_1}$, so $eff(A, K) = \widehat{A_1 B_1} \cup \widehat{C_1 D_1}$, where $A_1(1, 0)$, $B_1(0, 1)$, $C_1(-1, 0)$, $D_1(0, -1)$, with the corresponding extension in any Euclidean space $R^n, n \in N^*$. Some of the above examples show that, in general, the implication $S(A, K, K_1) = \varnothing \Rightarrow eff(A, K, K_1) = A$ is not true. However, at least from the numerical methods point of view, we are interested for

   **Open Problem 3.** *Characterizations of the equivalence*
$$S(A, K, K_1) = \varnothing \Leftrightarrow eff(A, K, K_1) = A .$$

## 3. ISAC'S CONES AND THE EFFICIENCY

Initially, the next concept of convex cone named by its author *nuclear, supernormal (Isac, G., 1980, 1981, 1983, 1985)* and, by us, *Isac's cone (Postolică, V., 2009)* was introduced in the following context, being the most appropriate for the general efficiency by optimization and conversely *(see,*

*for instance, Isac,G., 1980,1981, 1983, 1985, 1994, 1998, 2003; Isac, G., Bahya, A.O., 2002; Isac, G., Postolică, V., 1993, 2005; Isac, G., Tammer, Chr., 2003).* As we see, to ensure the existence of the efficient points for a

non - empty set in Hausdorff separated locally convex space, the main reason will be to replace *the compactness* with *the completeness* hypothesis imposed on the set and the ordering cone to be an Isac's cone.

**Definition 2.** *(Treves, F., 1967). A locally convex space is any couple* $(X, Spec(X))$ *which is composed of a real or complex linear space* $X$ *and a family* $Spec(X)$ *of seminorms on* $X$ *such that:*

*(i)* $\quad \chi p \in Spec(X), \forall \chi \in R_+, p \in Spec(X);$

*(ii)* $\quad$ *if* $p \in Spec(X)$ *and* $q$ *is an arbitrary seminorm on* $X$ *such that* $q \leq p$, *then* $q \in Spec(X);$

*(iii)* $\quad \sup(p_1, p_2) \in Spec(X), \forall p_1, p_2 \in Spec(X)$ *where*
$\sup(p_1, p_2)(x) = \sup(p_1(x), p_2(x)), \forall x \in X$.

It is well known that whenever such a family as this $Spec(X)$ is given on a real vector space $X$, there exists a locally convex topology $\tau$ on $X$ such that $(X, \tau)$ is a topological linear space and a seminorm $p$ on $X$ is

$\tau$ - continous iff $p \in Spec(X)$. A non-empty subset $\mathrm{B}$ of $Spec(X)$ is a *base* for it if for every $p \in Spec(X)$ there exist $\chi > 0$ and $q \in \mathrm{B}$ such that $p \leq \chi q$ and $(X, \tau)$ is a Hausdorff locally convex space iff $Spec(X)$ has a base $\mathrm{B}$, named a *Hausdorff base*, with the property that

$\{x \in X : p(x) = 0, \forall p \in \mathrm{B}\} = \{\theta\}$ where $\theta$ is the null vector in $X$. In this paper we will suppose that the space $(X, \tau)$, sometimes denoted by $X$, is a Hausdorff locally convex space. We recall that every non - empty subset $K$ of $X$ satisfying the following properties: $K + K \subseteq K$ and $\chi K \subseteq K, \forall \chi \in R_+$ is named *convex cone*. If, in addition, $K \bigcap K = \{\theta\}$, then $K$ is called *pointed*. Clearly, any pointed convex cone $K$ in $X$ generates an ordering on $X$ defined by $x \leq y(x, y \in X)$ iff $y - x \in K$. If $X^*$ is the dual of $X$, then the *dual cone* of $K$ is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$ and its corresponding *polar* is $K^0 = -K^*$. It is well known that a pointed convex cone $K \subset (X, Spec(X))$ is *normal* with respect to the topology defined by $Spec(X)$ if it fulfils one of the next equivalent assertions:

*(i) there exists a base* $\Omega$ *of neighborhoods for the origin* $\theta$ *in* $X$ *such that*
$$V = (V + K) \cap (V - K), \forall V \in \Omega;$$
*(ii) there exists a base* $\mathrm{B}$ *of* $Spec(X)$ *with* $p(x) \leq p(y), \forall x, y \in K, x \leq y, \forall p \in B;$

*(iii) for any two nets* $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subset K$ *with* $\theta \leq x_i \leq y_i, \forall i \in I$ *and* $\lim y_i = \theta$

*it follows that* $\lim x_i = \theta$. *In particular, a convex cone* $K$ *is normal in a normed linear space* $(E, \| . \|)$ *iff there exists* $t \in (0, \infty)$ *such that* $x, y \in E$ *and* $y - x \in K$ *implies that* $\| x \| \leq t \| y \|$. It is well known (*Treves,F., 1967*) that, the concept of *normal cone* together with its natural generalizations, represents the most important notion in *the Theory and the Applications of the Convex Cones in the general Topological Ordered Vector Spaces*. Thus, for example, *for every separated locally convex space* $(X, Spec(X))$ *and any closed normal cone* $K \subset (X, Spec(X))$ *we have* $X^* = K^* - K^*$ *(for more details see Hyers, D., H., Isac, G., Rassias,T.M., 1997; Isac, G., 1980).* Each pointed convex cone $K \subset (X, Spec(X))$ for which there exists a non - empty, convex

bounded set $B \subset X$ such that $\theta \notin \bar{B}$ and $K = \bigcup\limits_{\chi \geq 0} \chi B$ is called *well-based*. A cone $K \subset (X, Spec(X))$ is *well - based iff there exists a base* $B = \{p_i\}_{i \in I}$ *of* $Spec(X)$ *and a linear continuous functional* $f \in K^*$ *such that for every* $p_i \in B$ *there exists* $c_i > 0$ *with* $c_i p_i(x) \leq f(x), \forall x \in K$ *(Hyers, D., H., Isac, G., Rassias,T.M., 1997; Isac, G., 1980)*. Evidently, *every well - based* cone is *a normal cone*, but, in general, the converse is not true, as we can see in the examples below, starting from the next basic notion.

**Definition 3.** *(Isac, G., 1981,1983). In a Hausdorff locally convex space* $(X, Spec(X))$, *a pointed convex cone* $K \subset X$ *is nuclear (supernormal) with respect to the topolgy induced by* $Spec(X)$ *if there exists a base* $B = \{p_i\}_{i \in I}$ *of* $Spec(X)$ *such that for every* $p_i \in B$ *there exists* $f_i \in X^*$ *with*

$$p_i(x) \leq f_i(x), \forall x \in K .$$

**Remark 5.** Initially, the regretted Professor George Isac considered that Treves'definition was *more scientifically productive* for its *cones*. Afterwards, he accepted our proposal. Thus, for the first time, we called any such as this cone *"Isac's cone"* published and officially recognized in *(Postolică,V., 2009)*. For this reason, we will call these cones *"supernormal"* or *"Isac's cones"* as folows, taking into account that the above definition of any locally convex space is equivalent with the following: let $X$ be a real or complex linear space and let $P = \{p_\alpha : \alpha \in A\}$ be a family of seminorms defined on $X$. For every $x \in X, \varepsilon > 0$ and $n \in N^*$ let

$$V(x; p_1, p_2, ..., p_n; \varepsilon) = \{y \in X : p_\alpha(y - x) < \varepsilon, \forall \alpha = \overline{1, n}\}$$ be the family

$$\triangle_0(x) = \left\{V(x; p_1, p_2, ..., p_n; \varepsilon) : n \in N^*, p_\alpha \in P, \alpha = \overline{1, n}, \varepsilon > 0\}\right\},$$ with the following

properties :

 $(V_1)$ $x \in V, \forall V \in \triangle_0(x)$ ;

 $(V_2)$ $\forall V_1, V_2 \in \triangle_0(x), \exists V_3 \in \triangle_0(x) : V_3 \subseteq V_1 \cap V_2$ ;

 $(V_3)$ $\forall V \in \triangle_0(x), \exists U \in \triangle_0(x), U \subseteq V$ such that $\forall y \in U, \exists W \in \triangle_0(y)$ with $W \subseteq V$ .

Therefore, $\triangle_0(x)$ is *a base of neighborhoods* for $x$ and taking $\triangle(x) = \{V \subseteq X : \exists U \in \triangle_0(x)$ with $U \subseteq V\}$, the set

$$\tau = \{D \subseteq X : D \in \triangle(x), \forall x \in D\} \cup \{\varnothing\}$$ is the locally convex topology generated by the family $P$. Obviously, the usual operations which induce the structure of linear space on $X$ are continuous with respect to this topology. The corresponding topological space $(X, \tau)$ is a Hausdorff locally convex space iff the family $P$ is sufficient, that is, $\forall x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$ with $p_\alpha(x_0) \neq 0$. In this context, a convex cone $K \subset X$ is an Isac's cone iff

$$\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \leq f_\alpha(x), \forall x \in K .$$

This is an *equivalent definition* of *Isac's cones* in separated locally convex spaces. The *best special, refined* and *non - trivial Isac's cones* class associated to *normal cones* in *Hausdorff locally convex spaces* was introduced and studied in *(Isac, G., Bahya,A., O., 2002)* as *the full nuclear cones* family defined as follows: if $(X, Spec(X))$ is an arbitrary locally convex space, $B \subset Spec(X)$ is a Hausdorff base of $Spec(X)$ and $K \subset X$ is a normal cone, then for any mapping $\varphi : B \to K^* \setminus \{\theta^*\}$ one says that the set

$K_\varphi = \{x \in X : p(x) \le \varphi(p)(x), \forall p \in B\}$ is *the full nuclear cone* associated to $K$ whenever $K_\varphi \ne \{\theta\}$. In fact, $K$ is *supernormal* iff there exists

$\varphi : P \to K^* \smallsetminus \{0\}$ such that $K \subseteq K_\varphi$ (**Poposition 3.1** given by *(Isac, G. and Postolică, V., 2005)*. In the above equivalent context, if $\varphi : P \to K^*$ is a function, then the convex cone $K_\varphi = \{x \in X : p_\alpha(x) \le \varphi(p_\alpha)(x), \forall p_\alpha \in P\}$ is *the full nuclear cone* associated to K, $P$ and $\varphi$. In a real normed linear space $(E, \|.\|)$, a non - empty set $T \subset E$ is called *a Bishop - Phelps cone* if there exists $y^*$ in the usual dual space $E^*$ of $E$ and $\alpha \in (0,1]$ such that

$T = \{y \in E : \alpha \|y\| \le y^*(y)\}$. Following the applications of such as these cones in *Nonlinear Analysis*, including *the Optimization Programs* with

*Vector - Valued Objective Mappings*, one concludes that, for Haudorff locally convex spaces, *the full nuclear cones are natural generalizations of Bishop - Phelps Cones in Normed Vector Spaces.*

The beginning and the considerations in **Section 4** of *(Isac, G., Bahya, A. O., 2002)* suggested us to consider for each function $\varphi : P \to K^* \backslash \{0\}$ the full nuclear cone

$K_\varphi = \{x \in X : p_\alpha(x) \le \varphi(p_\alpha)(x), \forall p_\alpha \in P\}$ in order to give the next generalization of **Theorem 7** in *(Isac, G., Bahya, A.O., 2002)*. It represents an extension of the main result indicated by *(Isac, G., Postolică, V., 2005)*, an important link between *the efficiency*, *the full nuclear cones*, *strong optimization* and a significant result concerning *the multiple scalarization* for this general concept of the efficiency.

**Theorem 2.** *If $K$ is any Isac's cone in every Hausdorff locally convex space $(X, P = \{p_\alpha : \alpha \in A\})$, then*

(i) $MIN(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \backslash \{0\}}} SMIN(A \cap (a - K - K_1), K_\varphi);$

(ii) $MAX(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \backslash \{0\}}} SMAX(A \cap (a + K + K_1), K_{-\varphi});$

(iii)
$eff(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \backslash \{0\}}} [SMIN(A \cap (a - K - K_1), K_\varphi) \cup SMAX(A \cap (a + K + K_1), K_{-\varphi})].$

*for any non - empty subset $K_1$ of K. eff $(A, K, K_1) = eff(A, K)$*

**Proof.** It follows, mutatis mutandis, the same line as for **Theorem 4.3** given by *Isac, G.* and *Postolică, V.* in 2005.

**Remark 6.** To obtain (ii) from (i) it is not about of a simple formality by replacing $K$ with $(-K)$, since $(-K)_\varphi = -K_{-\varphi}$. If $\theta \notin K_1$, then $a_0 \in eff(A, K, K_1)$ implies that $A \cap (a_0 - K - K_1) = \varnothing$. Therefore, it is not possible to have $a_0 \in S(\varnothing, K_\varphi)$. In the case of $\theta \in K_1$, then $eff(A, K, K_1) = eff(A, K)$ and

$a_0 \in eff(A, K)$ iff $A \cap (a_0 - K) = \{a_0\}$, so in the right member of the first proved inclusion it can be selected any convex cone, not necessary $K_\varphi$. The hypothesis $K \subseteq K_\varphi$ imposed upon the convex cone $K$ and automatically satisfied since $K$ is an Isac's cone was used only to prove the next inclusion, with its natural projection for $MAX(A, K, K_1)$:

$$MIN(A, K, K_1) \subseteq \bigcup_{\substack{a \in A \\ \varphi:P \to K^* \setminus \{0\}}} SMIN(A \cap (a - K - K_1), K_\varphi).$$

**Corollary 2.1.** *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary Isac's cone K with its dual cone $K^*$ we have*

(i) $MIN(A, K) = \bigcup_{\substack{a \in A \\ \varphi:P \to K^* \setminus \{0\}}} SMIN(A \cap (a - K), K_\varphi);$

(ii) $MAX(A, K) = \bigcup_{\substack{a \in A \\ \varphi:P \to K^* \setminus \{0\}}} SMAX(A \cap (a + K), K_{-\varphi});$

(iii)

$$eff(A, K) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \setminus \{0\}}} [SMIN(A \cap (a - K), K_\varphi) \cup SMAX(A \cap (a + K), K_{-\varphi})].$$

**Remark 7.** The hypothesis of the above theorem involves $K$ to be pointed. Consequently, $\theta \in K_1$ iff $\theta \in K + K_1$. If $a_0 \in S(A \cap (a - K - K_1), K_\varphi)$ for some $\varphi : P \to K^*$ and $a \in A$ with $a_0 = a - k - k_1$, $k \in K$, $k_1 \in K_1$, then $K \cap (a_0 - A) = \{\theta\}$ because $A \cap (a - K - K_1) \subseteq a_0 + K_\varphi$ in any such a case as this. Indeed, let $x \in K \cap (a_0 - A)$ be an arbitrary element. Then, $a_0 - x \in A$ and $a_0 - x = a - k - k_1 - x \in a - K - K_1$. Therefore, $a_0 - x \in a_0 + K_\varphi$, that is, $x \in K_\varphi$. For every $p_\alpha \in P$ we have $p_\alpha(-x) \leq \varphi(p_\alpha)(-x) = -\varphi(p_\alpha)(x) \leq 0$. Since $p_\alpha$ was arbitrary chosen in $P$ and $X$ is a Hausdorff locally convex space, it follows that $x = \theta$.

**Remark 8.** Clearly, the announced theorem represents *a significant result* concerning the possibilities of *multiple scalarization* for *the study* of the *efficiency programs in Hausdorff separated locally convex spaces*, as we can see also in the final comments of (*Isac, G., Bahya, A.O., 2002*) for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones.

**Open Problem 4.** *Apply the above research taking in place of $K_1$ any non-empty subset of an arbitrary ordered linear space, under proper hypotheses.*

**Remark 9.** In the next considerations, we offer significant examples and adequate remarks on Isac's cones with the mention that for the existence of the efficient points and important properties of the efficient points sets in separated locally convex spaces ordered by (weak) such as these convex cones, through the agency of the (weak) completeness instead of compactness, the reader is referred to ( *Isac, G., 1981, 1983, 1985, 1994, 1998; Isac, G., Postolică, V., 1993; Postolică, V., 1993, 1994, 1995, 1997, 1999, 2001, 2002, 2009; Truong, X. D. H., 1994 and the references therein*).

**Theorem 3.** *(Bahya, A. O., 1989). A convex and normal cone* K *in a Hausdorff locally convex space is supernormal if and only if every net of* K *weakly convergent to zero converges to zero in the locally convex topology.*

Let us consider some pertinent examples which can be also found in (*Postolică, V., 2014, 2015, 2016*).

1. *Any pointed, convex cone, in every Euclidean space $R^k$ (k $\in$ N*) is an Isac's cone.*

2. *In every locally convex space, any well-based convex cone is an Isac's cone.*

3. *A convex cone* is an Isac's cone *in a normed linear space if and only if it is well-based.*

4. Let $n \in N^*$ be *arbitrary fixed* and let $Y$ be the space of *all real symmetric (n, n) matrices ordered* by *the pointed, convex cone*

C = {A $\in$ Y: $x^T Ax \geq 0, \forall x \in R^n$}. Then, $Y$ is *a real Hilbert space* with respect to *the scalar product* defined by *<A, B>=trace (A. B)* for *all A, B $\in$ Y* and C is *well-based by B={A $\in$ C: <A, I>=1}*, where *I denotes the identity matrix.*

5. *Every pointed, locally or weakly locally compact convex cone in any Hausdorff locally convex space is an Isac's cone.*

6. *A convex cone is an Isac's cone in a nuclear space (Pietch, A., 1972) if and only if it is a normal cone.*

7. *In any Hausdorff locally convex space, a convex cone is an weakly Isac's cone if and only if* it is *weakly normal.*

8. In $L^p([a, b])$, $(p \geq)$ ,*the convex cone* $K_p = \{x \in L^p([a, b]) : x(t) \geq 0$ *almost everywhere}* is *an Isac's cone if and only if p=1, being well-based in this case* by *the set* $B = \{x \in K_1 : \int_a^b x(t) dt = 1\}$. Indeed, *if p>1*, then *the sequence $(x_n)$ defined by*

$$x_n(t) = \begin{cases} n^{\frac{1}{p}}, \ a \leq t \leq a+(b-a)/2n \\ 0, \ a+(b-a)/2n < t \leq b \end{cases} \quad n \in \mathbf{N}$$

*converges to 0 in the weak topology but not in the usual norm topology. Therefore, by virtue of Theorem 1, $K_p$ is not an Isac's cone. Generally, for every p>1, $K_p$ has a base $B = \{x \in K_p : \int_a^b x(t) dt = 1\}$ which is unbounded and any cone generated by a closed and bounded set $B_t = \{x \in B : \int_a^b |x(t)|^p dt \leq t\}$ with $t \geq 0$ is certainly an Isac's cone. A similar result holds for $L^p(R)$. Thus, if we consider a countable family $(A_n)$ of disjoint sets which covers R such that $\mu(A_n) = 1$ for all n in N, where $\mu$ is the Lebesgue measure, then the sequence $(y_n)$ given by $y_n(t) = 1$ if $t \in A_n$ and $y_n(t) = 0$ for $t \in R \backslash A_n$ converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above theorem, it follows that the usual positive cone in $L^p(R)$ is not an Isac's cone if p>1, that is, it is not well - based in all these cases. However, these cones are normal for every $p \geq 1$. The same conclusion concerning the non - supernormality is valid for the positive orthant convex cones in the usual Orlicz spaces.*

9. In $l^p (p \geq 1)$ *equipped with the usual norm* $\|\cdot\|_p$ *the positive cone*

$C_p = \{(x_n) \in l^p : x_n \geq 0$ *for all* $n \in N\}$ *is also normal with respect to the norm topology, but it is not an Isac's cone excepting the case p = 1. Indeed, for every p>1, the sequence $(e_n)$ having 1 at the n - th coordinate* and *zeros elsewhere converges to zero in the weak topology, but not in the norm topology and, by virtue of* **Theorem 3**, *it follows that $C_p$ is not an Isac's cone. For p = 1, $C_p$ is well - based* by *the set* $B = \{x \in C_1 : \|x\|_1 = 1\}$ *and* **Proposition 5** *(Isac, G., 1983) ensures that it is an Isac's cone. In this case, the locally convex topology defined by the seminorms* $p_n((x_k)) = \sum_{k=0}^{n} |x_k|$ *for every $(x_k)$ in $l^1$ and n*

$\in N$ is *weaker than its usual weak* topology. Consequently, *the usual positive cone remains an Isac's cone with respect to this topology (now it is normal in a nuclear space and one applies* **Proposition 6** of *(Isac, G., 1983)* but *it is not well - based. We now recall the concept* of *H - locally convex space introduced by Precupanu, T. in 1969*: *any Hausdorff locally convex space* with *the seminorms satisfying the parallelogram law. Taking into account it and the property that every nuclear space is also a H-locally convex space* with respect to *an equivalent system* of *seminorms (Pietch, A., 1972), the above example shows* that in *a H - locally convex space a proper convex cone may be an Isac's cone without to be well - based. Moreover, if we consider in $l^2$ the H - locally convex topology induced by the seminorms*

$$\tilde{p}_n((x_k)) = \left( \sum_{i \geq n} |x_i|^2 \right)^{\frac{1}{2}}, \ n \in N, \ (x_k) \in \ell^2,$$

*then the convex cone* $C_2 = \{(x_k) \in l^2 : x_k \geq 0$ *for all $k \in N\}$ is normal in the*

*H - locally convex space ($l^2$,{ $\tilde{p}_n$ }$_{n \in N}$ ), but it is not a supernormal cone* because *the same sequence ($e_k$) is weakly convergent to zero while ($\bar{p}_n(e_k)$ ) is convergent to 1 for each $n \in N$ and one applies again* **Theorem 3**. *Another interesting example* of *normal cone in a H - locally convex space which is not Isac's cone is the usual positive cone* in *the space $L^2_{loc}$ (R) of all functions from R to C which are square integrable over any finite interval of R, endowed with the system of seminorms* { $\bar{p}_n : n \in N$ } defined by $\bar{p}_n(x) = \left( \int_{-n}^{n} |x(t)|^2 \, dt \right)^{1/2}$ *for every x in $L^2_{loc}$ (R). In this case, the sequence ($x_k$) given by:*

$$x_k(t) = \begin{cases} 0, \ t \in (-\infty, 0) \cup (1/k, +\infty) \\ \sqrt{k}, \ t \in [0, 1/k] \end{cases}$$

*converges weakly to zero, but it is not convergent in the H - locally convex topology. The results follows* by **Theorem 3**. *It is clear that every weak topology is a H - locally convex topology and, in these cases, the supernormality of convex cones coincides with the normality thanks to the Corollary of* **Proposition 2** *in (Isac, G., 1983).*

10. In *the space C([a, b]) of all continuous, real valued functions defined on every non - trivial, compact interval [a, b] equipped with the usual supremum norm the convex cone*

*K = {x ∈ C([a, b]): x is concave, x(a)=x (b) = 0 and x(t)≥0 for all t ∈ [a, b]} is an Isac's cone, being well-based by the set {x ∈ K: x($t_0$) =1} for some arbitrary $t_0 \in$ [a,b]. The hypothesis that all x ∈ K are concave is essential for the supernormality.*

11. *The convex cone of all non-negative sequences in the space of all absolutely convergent sequences is the dual of the usual positive cone in the space of all convergent sequences. Consequently, it has a weak star compact base and hence it is a weak star Isac's cone.*

12. In $l^p$ or in $c_0$ equipped with the supremum norm, *the convex cone* consisting of *all sequences having all partial sums non - negative is not normal, hence it is not supernormal.*

13. *In every Hausdorff locally convex space any normal cone is an Isac's cone with respect to the weak topology.*

14. *In every locally convex lattice which is a (L) - space the ordering cone is supernormal* (*see also Example 7 given by Isac, G. in 1994*).

15. *If we consider the space of all locally integrable functions* on *a locally compact space Y with respect to a Radon measure $\mu$ endowed with the topology induced by the family of seminorms {$p_A$} where $p_A(f) = \int_A |f(x)| \, d\mu$ for every non-empty and compact subset A of Y and every locally integrable function f, then the convex cone K = {f: f(x)≥0, x ∈ Y} is supernormal.*

16. *If Z is any locally convex lattice ordered* by *an arbitrary convex cone K and $Z^*$ is its topological dual ordered by the corresponding dual cone $K^*$, then the cone K is an Isac's cone with respect* to *the locally convex topology defined on Z by the neighbourhood base at the origin {[-f, f]°}$_f$ .*

17 *In every regular vector space (E, K)* (*that is, the order dual $E^*$ separates the points of E*) *with the property that E = K – K, the convex cone K is supernormal with respect to the topology defined in the preceding example.*

18. *Any semicomplete cone in a Hausdorff locally convex space is an Isac's cone* (*see Example 11* of *Isac, G., 1994*).

**Remark 10.** Clearly, if a convex cone $K$ is supernormal in a normed space, then $K$ admits a strictly positive, linear and continuous functional, that is, there exists a linear, continuous functional $f$ such that $f(k) \geq 0$ for all $k \in K \backslash \{\theta\}$. $k \in K$ .Generally, *the converse is not true* even in *a Banach space* as we can see in the following examples.

19. *If one considers in the usual space $l^p$ $\left(1 \le p \le \infty\right)$ the convex cone $K = \ell_+^p = \left\{x = (x_i) \in \ell^p : x_i \ge 0 \text{ for every } i \in N \right\}$ of infinite vectors with non-negative components, then the functional $\varphi$ defined by $\varphi(k) = \sum_{i=1}^{\infty} k_i$ for any $k = (k_i) \in l^p$ is linear, continuous and strictly positive. But, as we have seen in the above considerations (Example 9), this cone is supernormal if and only if p = 1.*

20. *Let K be the usual positive cone*

*$L^p_+ = \{x \in L^p([a, b]) : x(t) \ge 0 \text{ almost everywhere}\}$ in $L^p([a, b])$, $\left(1 \le p \le \infty\right)$. Then, the linear and continuous functional $\psi$ on $L^p([a, b])$ given by $\psi(x) = \int_a^b x(t)dt$ for every $x \in L^p([a, b])$ is strictly positive on K while K is supernormal (see the above Example 8) if and only if p = 1. The same conclusion is valid for the space $L^p(R)$ ($\left(1 \le p \le \infty\right)$ ).Therefore, $l^1_+$ and $L^1_+$ are Isac's cones, with empty topological interiors, and for every $p \in (1, +\infty)$ it follows that $l^p_+$ and*

*$L^p_+$ are normal cones, with empty interiors, which are not supernormal. Hence, these convex cones are not well based. Very simple examples of Isac's cones having non-empty topological interiors are $R_+^n$ $(n \in N^*)$.*

**Remark 11.** In *the order complete vector lattice B([a, b]) of all bounded, real valued functions on a compact non - singleton interval [a, b] endowed with its usual norm the standard positive cone*

*$K = \{u \in B([a, b]): u(t) \ge 0 \text{ for all } t \in ([a, b])\}$ is normal but it has not a base, that is, it is not an Isac's cone. However, this cone has non-empty interior. If we consider the linear space $l^1$ endowed with the separated locally convex topology generated by the family $\{p_n : n \in N\}$ of seminorms defined by $p_n(x) = \sum_{k=0}^{n} |x_k|$ for every $x = (x_k) \in l^1$, then the convex cone*

*$K = \{ x = (x_k) \in l^1 : x_k \ge 0 \text{ whenever } k \in N\}$ is an Isac's cone but it is not*

*well-based.*

**Open Problem 5.** *We introduce Isac's sets in the following area:let $M$ be a non-empty subset of $X$ and let $cone(M)$ be the convex cone generated by $M$. Obviously, the quality of Isac's cone for $cone(M)$ is equivalent with the fact that each seminorm is majorized by at least a linear and continuous functional on $M$. Therefore, it is natural to call it "Isac set". So, another open problem is to discover the properties of these sets.*

In synthesis, the position *of Isac's Cones* in the w*orld* of the *Ordered Vector Spaces* is the following:
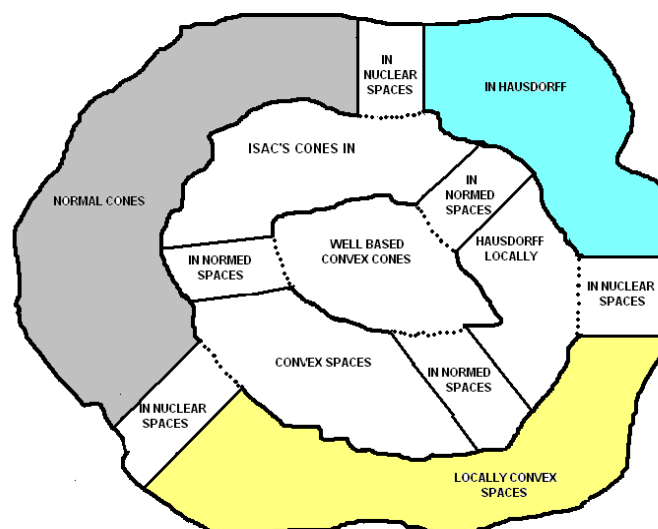
## 4. SOME CONCLUSIONS

*The natural context of supernormality* (nuclearity) *for convex cones* is *any separated locally convex space. Isac, G. introduced the concept of "nuclear cone" in 1981, published it* in *1983* and *he showed that in a normed space, a convex cone is nuclear* if and only if it is *well-based* or equivalently *iff* it is *"with plastering,"* the last concept being defined by *Krasnoselski, M. A.* in *fifties (see ,for example , Krasnoselski, M. A., 1964 and so on).* Such a convex cone was initially called *"nuclear cone"* by *Isac, G. (1981)* because *in every nuclear space (Pietch, A., 1972), any normal cone is a nuclear cone in Isac's sense (Proposition 6 of Isac, G., 1983). Later on, since the nuclear cone introduced by Isac appears as a reinforcement of the normal cone*, it was called *supernormal. Following the friendly approval of Professor Isac, we called and published it as "Isac's Cone" in 2009. The class of supernormal cones in Hausdorff locally convex spaces was initially imposed by the theory and the applications* of *the efficient (Pareto minimum type) points (especially existence conditions based on completeness instead of compactness were decisive together with the main properties of the efficient points sets). Afterwards, the study of critical points for dynamical systems and conical support points completed its importance, very well illustrated by final results, examples and comments, specified in our references and in all connected papers on this subject.* It is also *very significant to mention again* that *the concept of supernormality introduced by Isac, G. ( 1981) is not a simple generalization of the corresponding notion defined in normed linear spaces by Krasnoselski, M. A. and his colleagues in the fifties.* Thus, for example, *Isac's supernormality attached to the convex cones has his sense in every Hausdorff locally convex space identically with the well known Grothendieck's nuclearity.* By *analogy* with *the fact* that *a normed space is nuclear in Grothendieck's sense if and only if it is isomorphe with an usual Euclidean space*, a *convex cone is supernormal in a normed space iff it is well-based,* that is, *it is generated by a convex bounded set which does not contain the origin in its closure. Beside Pareto type optimization, we also mention Isac's significant contributions, through the agency of supernormal cones, to the convex cones in product linear spaces and Ekeland's variational type principles (Isac.G.,2003; Isac,G.,Tammer,Chr., 2003).* Therefore, *the more appropriate background for Isac's cones is any separated locally convex space.* On the other hand, *the extension in real or complex vector spaces of this notion can be done since any real or complex vector space can be considered as a locally convex space with respect to the locally convex topology generated by the family of the seminorms originated in the Minkowski functionals associated to the algebraic and geometric concepts of the convex hulls for the balanced (circled) and absorbing sets containing its origin (Postolică,V., 2014). The behaviour of Isac's cones in some variable locally convex topologies, especially in duality, is also interesting (Postolică,V., 2015). To draw a conclusion, the Best Ordering Mathematical Structure discovered till now for the study of the General Efficiency and its particular case represented by the Vectorial Optimization with the coresponding Applications is represented by Isac's Cones.*

**Definition 4.** *(Postolică,V., 1995).* *A nonempty set $B$ of a topological vector space $X$ ordered by a convex cone $K$ is called $K$ - bounded if there exists $B_0 \subseteq X$ , bounded, so that $B \subseteq B_0 \pm K$ and it is $K$ - closed if its conical extension $B \pm \bar{K}$ is closed, where $\bar{K}$ signifies the topological closure of K.*

A synthesis concerning the existence and significant properties of the efficient points are given in the next theorem.

**Theorem 4.** *(Postolică, V., 1995,1997, 2008 ).*

*(i) if $K$ is an arbitrary Isac's cone in any Hausdorff locally convex space $X$ , $A, B \subseteq X$ are every non-empty subsets positioned by $A \subseteq B \subseteq A + K$ ( $A \subseteq B \subseteq A - K$ ) and $B \cap (A_0 - K)$ ( $B \cap (A_0 + K)$ ) is a bounded and complete set for at least one non-empty set $A_0 \subseteq A$, then $MIN(A, K) \neq \varnothing$ ( $MAX(A, K) \neq \varnothing$ );*

*(ii) if $X$ is a quasi - complete Hausdorff separated locally convex space and $K$ is a closed Isac's cone in X, then for any $K$ - bounded and $K$ - closed*

*non - empty set $A$ of $X$ we have $MIN(A, K) \neq \varnothing$ , $A \subseteq MIN(A, K) + K$*

*( $A \subseteq MAX(A, K) - K$ )(the domination property), $MIN(A, K) + K = A + K$*

( $MAX(A,K) - K = A - K$ ) and $MIN(A,K)$ ( $MAX(A,K)$ ) is $K$ - bounded and

$K$ - closed;

(iii) according to the same hypotheses from (ii), $MIN(A,K) \neq \emptyset$ ( $MAX(A,K) \neq \emptyset$ ) for any non - empty subset $A$ with $B \cap (A_0 - K)$ ( $B \cap (A_0 + K)$ ) $K$ - bounded and $K$ - closed set whenever $A_0 \subseteq A \subseteq X$ and $A \subseteq B \subseteq A + K$ ( $A \subseteq B \subseteq A - K$ ),with the corresponding implications for $Eff(A,K)$.

**Open Problem 6.** *Extend the above theorem for the approximate efficient points given by non-empty subsets* $K_1$ *of* $K$. *If* $eff(A,K,K_1) \neq \emptyset$, *then there exist a Hausdorff locally convex space* $Y$, *an Isac's cone* $K_0$ *in* $Y, \emptyset \neq K_{01} \subseteq K_0$ *and a non - empty set* $A_0 \subset Y$ *with* $eff(A,K,K_1) = eff(A_0,K_0,K_{01})$ *(or, at least, $eff(A,K,K_1)$ to be dense in $eff(A_0,K_0,K_{01})$)?*

## 5. EFFICIENCY AND CHOQUET BOUNDARIES

The main connection between the general efficiency and the Potential Theory is briefly presented through the agency of the Choquet boundaries.

**Definition 5.**( *Postolică,V.,2008) A real function* $f : X \to R$ *is called* ($K + K_1$) - *increasing(* $K + K_1$ - *decreasing) if* $f(x_1) \geq f(x_2)$ *whenever* $x_1, x_2 \in X$ *and* $x_1 \in x_2 + K_1 + K$ ( $x_1 \in x_2 - K_1 - K$ ).

Obviously, every real increasing (decreasing) function defined on any linear space ordered by an arbitrary convex cone $K$ is ($K + K_1$) - increasing

( $K + K_1$ - decreasing), for each non - empty subset $K_1$ of $K$.

The next coincidence of the efficient points sets and the Choquet boundaries generalizes the main result given by *Bucur, I. and Postolică, V., 1994*, and can not be obtained as a consequence of the Axiomatic Potential Theory.

**Theorem 5.** *(Postolică,V., 2008, 2009). If A is any non-void, compact subset of every Hausdorff linear topological vector space* $X$ *and*

   (i) $K$ *is an arbitrary, closed, convex, pointed cone in* $X$ ;

   (ii) $K_1$ *is a non - empty subset of* $K$ *such that* $K + K_1$ *is closed with respect to the Hausdorff separated topology on* $X$.

*Then,* $MIN(A,K,K_1)$ ( $MAX(A,K,K_1)$ ) *coincides with the Choquet's boundary* $\partial_{S_1} A$ ( $\partial_{S_2} A$ ) *of A with respect to the convex cone* $S_1$ ( $S_2$ ) *of all*

$K + K_1$ - *increasing* ( $K + K_1$ - *decreasing) real continuous functions on A. Consequently, each of these sets, endowed with the corresponding trace topology, is a Baire space and a* $G_\delta$ - *subset of X whenever* $(A, \tau_A)$ *is metrizable. Moreover,* $eff(A,K,K_1) = \partial_{S_1} A \cup \partial_{S_2} A$.

**Corollary 5.1.** *(i) If for every* $f \in C(A)$ *one denotes* $\bar{f}(a) = \sup\{f(a') : a' \in A \cap a + K + K_1\}$ *and* $\tilde{f}(a) = \sup\{f(a'') : a' \in A \cap (a - K - K_1\}$, *then*

$$eff(A,K,K_1) = \bigcup_{a \in A} \{a \in A : f(a) = \bar{f}(a)(\tilde{f}(a)), \forall f \in C(A)\};$$

   *(ii)* $MIN(A,K,K_1)$ ( $MAX(A,K,K_1), MIN(A,K,K_1)$ (( $MAX(A,K,K_1)$)) $\bigcap \{a \in A : s(a) \leq 0\}$ ( $s \in S_1$ )
   *are compact sets with respect to Choquet's topology and as usual compact subsets of* $A$.

**Remark 12.** Generally, $MIN(A,K,K_1)(MIN(A,K,K_1))$ coincides with the corresponding Choquet boundary of A only with respect to the convex cone of all real, continuous and $K + K_1$ - increasing ( $K + K_1$ - decreasing) functions on A. Thus, for example, if $A$ is any non - empty, compact convex subset of every Hausdorff locally convex space and

$S = \{f : A \rightarrow R \,/\, f \text{ is continuous and concave}\}$, then its Choquet's boundary with respect to S coincides with the set $ex(A)$ of all extreme points x in $A$, that is, if $y, z \in A$ and there exists $\alpha \in (0,1)$ with $\alpha y + (1-\alpha)z = x$, then $y = z = x$. The hypothesis of concavity imposed on the functions is *essential* for the validity of this result. Generally, *even in the finite dimensional cases, an extreme point for a compact convex set is not necessary an efficient point and conversely.*

**Remark 13.** The largest class $\zeta$ of convex cones ensuring the existence for the efficient points in all non-empty compacts subsets of every separated topological vector space was defined by *Sterna – Karwat, A., 1986* as follows: if *V* is an arbitrary Hausdorff topological vector space, a convex cone *C* belongs to $\zeta$ iff for every closed vector subspace *L* of *V*, $C \cap L$ is a vector subspace whenever its closure $\overline{C \cap L}$ is a vector subspace. As we have already specified before Theorem 5, there exists more general conditions than compactness imposed upon a non-empty set A in a separated locally convex space ordered by a convex cone K ensuring that $eff(A,K) \neq \varnothing$. Perhaps our coincidence result suggests a natural extension of the Choquet boundary at least in these cases. Anyhow, **Theorem 5** represents an important link between the *Vector Optimization*, the *Potential Theory*, with *their Applications* and a new way for the study of the properties of the efficient points sets and the *Choquet boundaries*. Indeed, one of the main question in Potential Theory is to find the Choquet boundaries. This fact is relatively easy for particular cases but, in general, it is an unsolved problem. Since in a lot of cases the efficient points sets contain dense subsets which can be identified by adequate optimization methods, it is possible to determine the corresponding Choquet boundaries in all these situations. Consequently, our coincidence result has its practical consequences at first for the Axiomatic Theory of Potential and its applications. At the same time, by the above coincidence result, the Choquet boundaries offer important properties for the efficient points sets.We also note that, in general, *there exist no any connection between Choquet's boundary and Choquet's Integral*, probably thanks to *the introductions of these concepts in different temporal moments.*

The above context of research suggests immediately the following open problems.

**Open Problem 7.** *If $eff(A,K,K_1) \neq \varnothing$, there exist a separated locally convex space $\widehat{X}$, a (pointed), convex cone $\widehat{K}$ in $\widehat{X}$, , $\varnothing \neq \widehat{K_1} \subseteq \widehat{K}$ and a compact set $\widehat{A} \subseteq \widehat{X}$ such that $eff(A,K,K_1) = eff(\widehat{A},\widehat{K},\widehat{K_1})$ (at least $eff(A,K,K_1)$ to be dense in $eff(\widehat{A},\widehat{K},\widehat{K_1})$ or*

   *conversely) ?*

**Open Problem 8.** *If $T$ is a Hilbert space, $K$ is a closed, convex, pointed cone in $T$, $\varnothing \neq K_1 \subseteq K$ and $A$ is a non-empty, closed, convex subset of $T$ does $eff(A,K,K_1)$ preserve the property of coincidence with the corresponding Choquet's boundary as in the above theorem?*

**Open Problem 9.** *The same question for each of the following cases:*

   *(i) $T$ is a quasi - complete locally convex space, $K$ is an Isac's cone, $\varnothing \neq K_1 \subseteq K$, A is a $K$ - bounded and $K$ - closed set in $T$ (Postolică, V., 1995).*

   *(ii) $T$ is a quasi - complete locally convex space, the closure $\overline{K}$ of $K$ has the properties given in (Truong, X., D. H., 1994) and $A$ is a $K$ - bounded and $K$ - closed subset in $T$.*

**Open Problem 10.** *Generalize the concept of Choquet's boundary to non-empty complete sets and extend to non-compact sets the established strong connection by coincidence between the general efficient points sets and Choquet's boundaries.*

## 6. BEST APPROXIMATION IN LOCALLY CONVEX SPACES. EFFICIENCY BY SPLINES IN H-LOCALLY CONVEX SPACES

We conclude *this Research Report* with *Topics on the Best Approximation* (*Simultaneous* and *Vectorial*) in *Locally Convex Spaces*, *the Efficiency (Optimization) by Splines in H - Locally Convex Spaces* and *the Vectorial Optimization.*

Let $X$ *be a Hausdorff locally convex space with the topology induced by a family* $P = \{p_\alpha : \alpha \in I\}$ *of seminorms,* $x_0 \in X$ *and* $G$ *a non - empty subset of* $X$.

**Definition 6.** (*Isac, G., Postolică, V.,1993*) $g_0 \in G$ *is said to be a best simultaneous approximation for* $x_0$ *by the elements of* $G$ *with respect to the family* $P$ *(abbreviated* $g_0$ *is a* $P$ *- b.s.a. of* $x_0$ *) when*

$$p_\alpha(x_0 - g_0) \le p_\alpha(x_0 - g) \text{ for all } g \in G \text{ and } p_\alpha \in P.$$

*If, in addition, each element* $x \in X$ *possesses at least one* $P$ *- b.s.a. in* G, *then the set* $G$ *is called* $P$ *- simultaneous proximinal.*

**Definition 7.** *(Isac, G., Postolică, V., 1993)* $g_0 \in G$ *is said to be a best vectorial approximation of* $x_0$ *by* $G$ *with respect to* $P$ *(abbreviated* $g_0$ *is a*

$P$ *- b.v.a. of* $x_0$ *) if* $(p_\alpha(x_0 - g_0)) \in MIN(\{(p_\alpha(x_0 - g)) : g \in G\}, K)$ *where* $K = R_+^I$.

*If, in addition, each element* $x \in X$ *possesses at least one* $P$ *- b.v.a. in* $G$, *then the set* $G$ *is called* $P$ *- vectorial proximinal. Several original results and properties concerning these notions were given in our above mentioned book.Concerning the H - locally convex spaces, it is known that this concept was introduced and studied for the first time by Precupanu, T. (1969) and defined as any Hausdorff locally convex space with the seminorms satisfying the parallelogram law. At the same time, we introduced the notion of spline function in H - locally convex space (Postolică, V., 1981) and we established the basic properties of approximation and optimal interpolation for these splines. Our splines are natural extensions in H - locally convex spaces of the usual abstract splines which appear in any Hilbert space like the minimizing elements for a seminorm subject to the restrictions given by a set of linear continuous functionals.*

So, let now ( $X$, $P = \{p_\alpha : \alpha \in I\}$) *be a H - locally convex space* with *each seminorm* $p_\alpha$ *being induced by a scalar semiproduct* $(.,.)_a$ *($\alpha \in I$)* and $M$ *a closed linear subspace* of $X$ *for which there exist a H - locally convex space* $(Y, Q = \{q_\alpha : \alpha \in I\})$ *with each seminorm* $q_\alpha \in Q$ *generated by a scalar semiproduct* $<.,.>_a$ *($\alpha \in I$)* and *a linear (continuous) operator* U:X$\rightarrow$Y *such that* $M = \{ x \in X : (x,y)_a = <Ux, Uy>_a, \forall \alpha \in I \}$.

*The space of spline functions with respect to U was defined by*

*Postolică, V.( 1981) as the U-orthogonal of* $M$, *that is,*

$M^\perp = \{x \in X: <Ux, Uz>_a = 0, \forall z \in M, a \in I\}$. *Clearly,* $M^\perp$ *is the orthogonal of M in the H - locally convex sense, namely* (x,y) $\forall x \in M$, $y \in M^\perp$, $\in I$.

Let us consider *the direct sum* $X' = M \oplus M^\perp$ and *for every* $x \in X'$, *we denote its projection onto* $M^\perp$ *by* $s_x$. Then, taking into account *the **Theorem 4** obtained by Postolică, V. in 1981, it follows that this spline is a best simultaneous U - approximation of x with respect to* $M^\perp$ *since it satisfies all the next condition :* $p_\alpha(x - s_x) \le p_\alpha(x_0 - y)$, $\forall y \in M^\perp, p_\alpha \in P$.

Moreover, *following the definition of the approximate efficiency, the results given in Chapter 3* of *(Isac, G., Postolică, V., 1993)* and *the conclusions obtained* by *(Postolică, V.,1981,1993, 1998),* we have

**Theorem 6.**

*(i) For every $x \in X'$ the only elements of best simultaneous and vectorial approximation with respect to any family of seminorms which generates the H-locally convex topology on $X$ by the linear subspace of splines are the spline functions* $s_x$. *Moreover, if $M$ and $M^\perp$ supply an orthogonal decomposition for $X$, that is, $X = M \oplus M^\perp$, then $M^\perp$ is simultaneous and vectorial proximinal;*

*(ii) If $K = R_+^I$, then for each $s \in M^\perp$, every $\sigma \in M^\perp$ is the only one solution of the following optimization problem:*

$$MIN(\{(q_\alpha(U(\eta - s))) : \eta \in X', \eta - \sigma \in M\}, K);$$

*(iii) for every $x \in X'$ its spline function $s_x$ is the only solution for each of the next vectorial optimization problems:*

$$MIN(\{(q_\alpha(U(s - x))) : s \in M^\perp\}, K), \ MIN(\{(p_\alpha(x - y)) : y \in M^\perp\}, K),$$

$$MIN(\{(q_\alpha(Uy)) : y - x \in M\}, K).$$

*Finally*, let us consider *two numerical examples* in which, following

*Postolică,V.,(1981), Isac, G., Postolică, V.,(1993) and Postolică, V., (1998),* we specify *the expressions of splines* and $M$ and $M^\perp$ *realize orthogonal decompositions.*

**Example 5.2.** *Let $X=H^m(R)=\{f \in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2_{loc}(R)\}$, $m \geq 1$ endowed with the H - locally convex topology generated by the scalar semiproducts*

$$(x,y)_k = \sum_{h=0}^{m-1} [x^{(h)}(k)\, y^{(h)}(k) + x^{(h)}(-k)\, y^{(h)}(-k)] + \int_{-k}^{k} x^{(m)}(t)\, y^{(m)}(t)dt, \ k=0,1,2, \ ... \ and$$

*$Y = L^2_{loc}(R)$ with the H-locally convex topology induced by the scalar semiproducts $<x,y>_k = \int_{-k}^{k} x(t)\, y(t)dt, \ k=0,1,2, \ ...$*

*If $U:X \to Y$ is the derivation operator of order m, then*

*$M=\{x \in H^m(R): x^{(h)}(n)=0, \ \forall h=\overline{0, m-1}, \ n \in Z\}$*

*and $M^\perp = \{s \in H^m(R): \int_{-k}^{k} s^{(m)}(t)\, x^{(m)}(t)dt, \ \forall x \in M, \ k=0,1,2, \ ...\}$*

*We proved in (Postolică, V., 1981) that*

*$M^\perp = \{s \in H^m(R): s_{/(n, n+1)}$ is a polynomial function of degree 2m-1 at most} and if $y= (y_n)$, $y'= (y'_n)$, $y''= (y''_n)$, $y^{(m-1)}= (y^{(m-1)}_n)$ are m sequences of real numbers, then there exists an unique spline $S \in M^\perp$ satisfying the following conditions of interpolation: $S^{(h)}(n)= y^{(h)}(n)$ whenever $h=\overline{0, m-1}$ and $n \in Z$. Moreover, we observed in the paragraph 3 of (Isac, G., Postolică, V., 1993) that any spline function S such as this is defined by*

$$S(x) = p(x) + \sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1} + \sum_{h=0}^{m-1} c_2^{(h)}(x-2)_+^{2m-1} + \dots + \sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1} + \dots$$

*where $u_+=(|u|+u)/2$ for every real number $u$, $p$ is a polynomial function of degree $2m-1$ at most perfectly determined by the conditions $p^{(h)}(0)= y_0^{(h)}$ and $p^{(h)}(1)= y_1^{(h)}$ for all $h=\overline{0, m-1}$ and the coefficients $c_n^{(h)}$ ($h=\overline{0, m-1}$, $n \in Z$) are successively given by the general interpolation.*

*Therefore, for every function $f \in H^m(R)$, there exists an unique function denoted by $S_f \in M^\perp$ such that $S_f^{(h)}(n)= f^{(h)}(n)$, $\forall h=\overline{0, m-1}$ and $n \in Z$. Hence, in this case, $M$ and $M^\perp$ give an orthogonal decomposition for the space $H^m(R)$.*

**Example 2.** *Let $X=F_m=\{f \in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2(R)\}$ endowed with the H - locally convex topology induced by the scalar semiproducts*

$$(x,y)_n = x(n)y(n) + \int_R x^{(m)}(t)\, y^{(m)}(t)dt, \; n \in Z, \; Y= L^2(R) \text{ with the topology generated by the inner product}$$

$$(x,y)_n = \int_R x(t)\, y(t)dt, \; n \in Z \text{ and let } U:X \to Y \text{ be the derivation operator of order } m. \text{ Then, } M= \{x \in$$

*$F_m$: $x(n)=0$ for all $n \in Z\}$ and*

$$M^\perp = \{ s \in F_m: \int_R x^{(m)}(t)\, y^{(m)}(t)dt=0 \text{ for every } x \in M\}.$$

*In a similar manner as in Example 1 it may be proved that $M^\perp$ coincides with the class of all piecewise polynomial functions of order $2m$*

*(degree $2m-1$ at most) having their knots at the integer points. Moreover, for every function $f$ in $F_m$ there exists an unique spline function $S_f \in M^\perp$ which interpolates $f$ on the set $Z$ of all integer numbers, that is, $S_f$ satisfies the equalities $S_f(n)=f(n)$ for every $n \in Z$, being defined by*

$$S_f(x)= p(x) + a_1(x-1)_+^{2m-1} + a_2(x-2)_+^{2m-1} + \dots + a_0(-x)_+^{2m-1} + a_{-1}(-x-1)_+^{2m-1} + \dots$$

*where $u_+$ has the same signification as in Example 1, the coefficients $a_n$ ($n \in Z$) are successively and completely determined by the interpolation conditions $S_f(n)=f(n)$, $n \in Z\backslash\{0, 1\}$ and $p$ is a polynomial function satisfying the conditions $p(0)=f(0)$ and $p(1)=f(1)$. The uniqueness of $S_f$ is ensured in Theorem 2 given by (Postolică, V., 1981).*

*Thus, $M$ and $M^\perp$ give an orthogonal decomposition of the space $F_m$ and, as in the preceding example, $M^\perp$ is simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts.*

**Remark 14.** *Our examples show that the abstract construction of splines can be used to solve also several frequent problems of optimal interpolation and approximation, having the possibility to choose the spaces and the scalar semiproducts. It is obvious that for a given (closed) linear subspace of a H - locally convex space X, such a H - locally convex space Y (respectively, a linear (continuous) operator $U:X \to Y$) would not exist. Otherwise, the problem of best vectorial approximation by the corresponding orthogonal space of any (closed) linear subspace M for the elements in the direct sum $M \oplus M^\perp$ might be always reduced to the best simultaneous approximation. But, in general, such a possibility doesn't exist. Even in a H - locally convex space it is possible that there exist best vectorial approximations and the set of all best simultaneous approximations to be empty for some element of the space. We confine ourselves to mention the following simple example.*

**Example 3.** *Let* $X = R^N$ *endowed with the topology generated by the family* $P = \{p_i : i \in N\}$ *of seminorms defined by* $p_i(x) = |x_i|$ $(i \in N)$ *for every*

$$x = (x_i) \in X \ \text{and for each} \ c \geq 0 \ \text{let} \ G_c = \left\{(x_i) \in X : x_i \geq 0, \forall i \in N \ \& \ \sum_{i \in N} x_i = c\right\}.$$

$X$ *is a* $P$ *- simultaneous strictly convex ( Isac, G., Postolică, V., 1993)*

H - locally convex space. Nevertheless, every element of $G_c$ is a $P$ - b.v.a. for the origin while its

corresponding set of the best simultaneous approximations with respect to $P$ is empty.

## 7. PROJECTIONS IN VECTORIAL OPTIMIZATION

*Multi – Objective Programming* is an important component of the *General Optimization* in *the Mathematical Abstract Spaces* and a part of it is represented by *the Pareto Optimality* as *the first illustrative finite dimensional example of the Economical Efficiency*. Not even for *the market economies there exists no an universal mathematical model*. Pareto efficiency or Pareto optimality *(Pareto, Vilfredo, 1909)* was *a central theory* in *economics* with *broad applications in game theory, engineering and the social sciences*. It represents *the actual finite dimensional part of the Multiobjective Programming* in *Vector Optimization*. Thus, whenever a feasible deviation from a genuine solution S of an arbitrary multiobjective programme generates the improvement of at least one of the objectives while some other objectives degrade, any such a solution S as this is called *efficient or nondominated. A system in economics and in politics is called Pareto optimal* or *efficient* in the language of *the general efficiency* wheneve*r "no individual can be made better off without another being made worse off"* that is*, a social state is economically efficient, or Pareto optimal, provided that "no person in society can become better off without anyone else becoming worse off".*This characteristic of *Pareto type efficiency* has been pointed out in *(Arrow, K. J., 1963, Arrow, K. J., Hahn, F. M., 1971,Aubin, J. P., 1993, Debreu Gerard, 1959, Hansson, S. O., 2004, Pareto, Vilfredo, 1909, Stewart, D. J., 2004 and the others).* In terms of *the alternative allocations* this means that given *a set of alternative allocations* and *a set of individuals, any movement from one alternative allocation to another that can make at least one individual better off, without making any other individual worse off is called a Pareto improvement or a Pareto optimization. An allocation of resources is named Pareto optimal* whenever *no further Pareto improvements can be made*. If the allocation is *strictly preferred* by one person and no *other allocation would be as good for everyone*, then it is called *strongly Pareto optimal. A weakly Pareto optimal allocation is one where any feasible reallocation would be strictly preferred by all agents* (Zimmermann, H., 2000). Consequently, *Pareto type efficiency* is *an important approximate criterion* for *evaluating the economic systems* and *the political policies*, with *minimum assumptions on the interpersonal comparability*. We said *"approximate criterion"* because it asks *"the ideal"* which may not reflect, for example, *the workings of real economics*, thanks to the folowing *restrictive assumptions* necessary for *the existence* of Pareto efficient outcomes*: the markets exist for all possible goods*, being *perfectly competitive* and *the transaction costs are negligible*. In *the political policies*, *not every Pareto efficient outcome* is regarded as *"desirable"* (*see, for instance, the strategies based on the unilateral benefits*). For these reasons, Pareto optimality was accepted with *some or much uncertainty and controversy*, but, by *Arrow's renowned impossibility theorem* given in 1951 according to which *"no social preference ordering based on individual orderings only could satisfy a small set of very reasonable conditions"-the Pareto criterion* being one of them, it remains *"plausible and uncontroversial" (Rosencrantz, Holger, 2005).* Usually, *the concept of efficiency* replaces the notion of *optimality* in *multiple criteria optimization* because *whenever the solutions of a multiple – objectives program exist in Pareto'sense, they cannot be improved following the ordering induced by the cone*. Seemingly, the concept of *efficiency* is equivalent to *the* optimality but, in *reality, the optimality represents a particular case of the efficiency*, that is, *"the best approximation"* of all the efficient points. The next section is dedicated to the main kind of optimization problems in $R^n$. We also notice that *the finite dimensionality* of *the background space is essential* in contrast with *the general ordered vector spaces,* since *it controls* the results at least thanks to *the approximate numerical methods* for *the efficient points.*

## 8. MULTI – OBJECTIVE OPTIMIZATION PROGRAMS

Let $E$ be an arbitrary non – empty set and let $f = (f_1, f_2, ..., f_n) : E \to R^n (n \in N^*, n \geq 2)$ a vector - valued function. The main *general strong* $(P_1)$, $(P_2)$ and *vectorial* $(P_3)$, $(P_4)$ respectively, *Optimization Problems* that can be formulated in this context are the following:

$$(P_1) \quad SMIN\{f(x) : x \in E\}$$

where $x_0 \in E$ is a *strong minimum solution* if $f(x_0) \leq f(x), \forall x \in E$;

$$(P_2) \quad SMAX\{f(x) : x \in E\}$$

with $y_0 \in E$ *strong maximum solution* if $f(y_0) \geq f(x), \forall x \in E$;

$$(P_3) \quad VMIN\{f(x) : x \in E\};$$

$$(P_4) \quad VMAX\{f(x) : x \in E\}.$$

In order to define *the vectorial solutions*, we need to know the possibility of *"comparison"* in $R^n$. Let $R_+^n = \{(x_i) \in R^n : x_i \geq 0, \forall i = \overline{1,n}\}$ be the usual *ordering cone*. If $a = (a_i), b = (b_i) \in R^n$, then it is possible to have:

*1)* $a \leq b \Leftrightarrow a_i \leq b_i, \forall i = \overline{1,n} \Leftrightarrow b - a \in R_+^n$;

*2)* $a < b \Leftrightarrow a_i \leq b_i, \forall i = \overline{1,n}$ *with at least one of these inequalities being strict, that is*

$$b - a \in R_+^n \setminus \{\theta\}, \text{ where } \theta = (0,0,...,0) \in R_+^n;$$

*3)*

$$a \ll b \Leftrightarrow a_i < b_i, \forall i = \overline{1,n} \Leftrightarrow b - a \in \text{int}(R_+^n) =$$
$$= \{x_0 \in R_+^n : \exists B(x_0, \varepsilon > 0) = \{x \in R^n : \|x - x_0\| < \varepsilon\} \subset R_+^n\},$$

where $\|.\| : R^n \to R_+$ is the usual norm in $R^n$ given by $\|u\| = \sqrt{\sum_{i=1}^{n} |u_i|^2}$, $\forall u = (u_i) \in R^n$.

Thus, we are led to say that $x_0$ is *a vectorial minimum solution* of $(P_3)$ if *there exists no* $x \in E$ such that $f(x) < f(x_0)$, which is equivalent with each of the next properties:

*(i)* $f(E) \cap [f(x_0) - R_+^n] = \{f(x_0)\}$;

*(ii)* $f(E) \bigcap [f(x_0) - R_+^n \setminus \{\theta\}] = \varnothing$;

*(iii)* $R_+^n \bigcap [f(x_0) - f(E)] = \{\theta\}$;

*(iv)* $(R_+^n \setminus \{\theta\}] \bigcap [f(x_0) - f(E)] = \varnothing$;

$$(v) \quad [f(E) + R_+^n] \cap [f(x_0) - R_+^n \setminus \{\theta\}] = \varnothing.$$

These relations show that $f(x_0)$ is *a fixed point* for each of the following *multifunctions*:

$$F_1 : f(E) \to f(E), F_1(t) = \left\{ \alpha \in f(E) : f(E) \cap \left( \alpha - R_+^n \right) \subseteq t + R_+^n \right\},$$

$$F_2 : f(E) \to f(E), F_2(t) = \left\{ \alpha \in f(E) : f(E) \cap \left( t - R_+^n \right) \subseteq \alpha + R_+^n \right\},$$

$$F_3 : f(E) \to f(E), F_3(t) = \left\{ \alpha \in f(E) : \left( f(E) + R_+^n \right) \cap \left( \alpha - R_+^n \right) \subseteq t + R_+^n \right\},$$

$$F_4 : f(E) \to f(E), F_4(t) = \left\{ \alpha \in f(E) : \left( f(E) + R_+^n \right) \cap \left( t - R_+^n \right) \subseteq \alpha + R_+^n \right\},$$

that is, $f(x_0) \in F_i[f(x_0)]$ for every $i = \overline{1,4}$. By replacing $R_+^n$ with $(-R_+^n)$, one obtains the concept of *Vectorial Maximum Solution* for $(P_4)$. Clearly, any *vector optimization program* (which has its origin in the usual ordered Euclidean spaces programs thanks to the Pareto type Optimality of the vector - valued real functions) includes the corresponding *Strong Optimization Programme*. Moreover, it is possible to replace the usual convex cone $R_+^n$ by any other convenient convex cone in $R^n$. In this way, we are sent to examine for a set $\varnothing \neq A \subseteq R^n$ the following subsets

$\alpha)$ $SMIN(A) = \left\{ a_0 \in A : a - a_0 \in R_+^n, \forall a \in A \right\}$ *(the set of all strong minimum points);*

$\beta)$ $SMAX(A) = \left\{ \overline{a_0} \in A : \overline{a_0} - a \in R_+^n, \forall a \in A \right\}$ *(the set of all strong maximum points);*

$\gamma)$ $VMIN(A) = \left\{ b_0 \in A : A \cap (b_0 - R_+^n) = \{b_0\} \right\}$ *(the set of all vectorial minimum points);*

$\delta)$ $VMAX(A) = \left\{ \overline{b_0} \in A : A \cap (\overline{b_0} + R_+^n) = \{\overline{b_0}\} \right\}$ *(the set of all vectorial maximum points).*

All *the Strong Optimal Points* for $A$ are contained in the set $S(A) = SMIN(A) \cup SMAX(A)$ while the set of all *Efficient points (Vectorial or Strong)* is defined by

$V(A) = VMIN(A) \cup VMAX(A) = eff(A, R_+^n)$, the last equality being in accordance with (*Postolică, V., 2017*). Moreover, $a_0 \in eff(A, R_+^n)$ if and only if it is *a critical (ideal or balance) point* for *the generalized dynamical systems* $\Gamma_\mp : A \to 2^A$ defined by $\Gamma_\mp(a) = A \cap (a \mp R_+^n), \forall a \in A$. In this way, $eff(A, R_+^n)$ describes *the balance extremum moments* for $\Gamma_\mp$, which in *the natural market context*, expresses *the competitive equilibrium/non - equilibrium* consisting of *the general relation price/consumption*. The first main connection between the above mentioned sets is the following, established in the general conditions offered by any real or complex vector space ordered by an arbitrary convex cone. Following **Theorem 1**, if $S(A) \neq \varnothing$, then $V(A) = S(A)$. In fact, $SMIN(A) \neq \varnothing (SMAX(A) \neq \varnothing) \Rightarrow VMIN(A) = SMIN(A)(VMAX(A) = SMAX(A))$.

**Open Problem 11.** *Characterizations of the equivalence*

$$S(A) = \varnothing \Leftrightarrow V(A) = A.$$

As we mentioned in **Remark 4**, this result is also important for for *the numerical methods* since it says that, if there exist *"strong optimal points"*, then these elements are *the only "vectorial optimal points"*.

**Remark 15**. In our opinion, following the most general context of the ordered linear spaces, any optimization programme can be generally formulated as follows: *let $E$ be a non – empty set, let $X$ be a vector space ordered by a convex cone $K$ and let $f : E \rightarrow (X, K)$ be an objective function. The problem consists to look for the optimums of the set $E$ following the properties of its image* $f(E) = \{ f(x) : x \in E \}$ *in $X$ by $K$ and conversely.*

## 9. FUTURE RESEARCH DIRECTIONS

This research work can also be considered as a study in which the approximate solutions including those usual to vector optimization programs in ordered (topological) vector spaces are investigated. The listed properties of these solutions allow developments. Several new facts about the relationships between the approximate efficiency and the scalarized problems can be formulated. As we have specified , the family of Isac's cones represent the largest class of ordering cones in Hausdorff locally convex spaces ensuring the existence and the adequate properties of the general efficient points sets involved in optimization, following different completeness types instead of compactness. Consequently, one of the main goal of the next research is to identify new applications of Isac's cones and new links with other important classes of convex cones in separated locally convex spaces.

### ACKNOWLEDGEMENTS

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