International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 6, Issue 1, 2018, PP 16-20
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
DOI: http://dx.doi.org/10.20431/2347-3142.0601003
www.arcjournals.org

Fixed Point Theorems in 2-Metric Space for Some Contractive Conditions

Durdana Lateef

College of Science, Taibah University, Al-Madinah Al-Munawwarah, 41411, Kingdom of Saudi Arabia.

***Corresponding Author:** *Durdana Lateef, College of Science, Taibah University, Al-Madinah Al-Munawwarah, 41411, Kingdom of Saudi Arabia.*

Abstract: *In this paper, we shall prove a fixed point theorem in 2-metric space by using Nesic type contractive definition. This theorem is a version of many fixed point theorem in complete metric space given by many authors announced in literature.*

Keywords: *Fixed point, 2-Metric spaces, Contraction, 2010 MSC: 47H10, 54H25.*

1. Introduction

The concept of fixed point theory and contraction mapping was extended and elaborated with the introduction of Contraction principle by Banach [2]. Concept of 2-metric space was introduced by Gahler [3] having the area of triangle in R^2 as the inspirative example. It has been shown by Gahler that in 2-metric d is non-negative. After Gahler there was a flood of new results obtained by many authors in these spaces [4-8]. Military applications of fixed point theory in 2-metric spaces can be found, as well as applications in Medicine and Economics [9-11].

Then Naidu and Prasad [12] introduced the concept of weakly commuting pairs of self-mapping on a 2-metric space, then others [13] [14] and [15] have proved several common fixed point theorem by using these concept.

In this paper I proved a fixed point theorem in 2-metric space by using Nesic type contractive definition [1] and the result of Lohani and Badshah [16] also we shall use the Lemma of Singh [17].

Mathematical preliminaries

Definition 2.1 : A 2- metric space is a set X with non negative real Valued. function d on X x X x X satisfying the following conditions :

(M_1) for two distinct point x,y in X there exist a point z

in X such that $d(x,y,z) \neq 0$.

(M_2) $d(x,y,z) = 0$ if at least two of x, y, z are equal.

(M_3) $d(x,y,z) = d(x,z,y) = d(y, z, x)$

(M_4) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ $\forall$ x,y,z and u in X .

The function d is called 2 metric for the space X and (X,d) is called 2 – metric space. "Geometrically 2 – metric represent the area".

Example : Let a mapping $d : R^3 \to [0,+\infty)$ be defined by

$d(x, y, z) = \min\{|x-y|, |y-z|, |z-x|\}$.

Then d is a 2-metric on R, *i.e.*, the following inequality holds:

$d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$,

for arbitrary real numbers x, y, z, t.

Definition 2.2 :A sequence { x_n } in a 2-metric space (X,d)_ is said to be convergent to a point x ϵ X

$$\lim_{n\to\infty} x_n = x, \text{ if } \lim_{n\to\infty} d(x_n,x,z) = 0 \text{ for all } z \in X$$

The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 2. 3: A sequence $\{x_n\}$ in a 2-metric space (X, d) is called a Cauchy sequence if

$$\lim d(x_n,x_m,a) = 0 \text{ as } n,m \to \infty \text{ for all } a \in X.$$

Definition 2.4 : A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent .

Definition 2.5 : A 2-metric space (X,d) is called bounded if there exist a constant M such that

$$d(x, y, z) \leq M \text{ for all } x,y,z \in X.$$

Definition 2.6 : A mapping f in 2 – metric space is called orbitally continuous if for all a in X ,

$$d(f^n x, u, a) \to 0 \text{ as } n \to \infty$$

Implies

$$d(ff^n x, fu, a) \to 0 \text{ as } n \to \infty$$

Definition 2.7: A mapping S from a 2- metric space (X,d) into itself is said to be sequentially continuous at a point xϵX if every sequence { x_n} in X such that

$$\lim_{n \to \infty} d(x_n, x, z) = 0 \quad \text{for all } z \in X$$

$$\lim_{n \to \infty} d(Sx_n, Sx, z) = 0$$

Every convergent sequence in a 2-metric space is a cauchy sequence.

Definition 2.8: A 2 –metric space d which is continuous in all of its three arguments is called continuous.

Remarks:

(i) Every convergent sequence in 2- metric space is Cauchy.

(ii) Geometrically 2-metric space represents Area.

For proving our theorem we shall use the lemma of Singh [19].

Lemma 2.1: Let $\{x_n\}$ be a sequence in complete 2-metric space X. if there exists h ϵ [0,1] such that

$$d(x_n, x_{n+1}, a) \leq h d(x_{n-1}, x_n, a)$$

for some a ϵ X then $\{x_n)$ converges to point in X.

Drawing inspiration from Nesic type contractive definition [1] and the result of Lohani and Badshah [18] , we prove the following theorem in 2-metric spaces.

2. Main Results

Theorem 2.1: Let f be an orbitally continuous self –map from complete 2-metric space X into itself, if f satisfies.

$$[1+ pd(x,y,z) \leq p \max \{ \max \{ d(x, fx, a) . d(y, fy, a) , d(x, fy, a). d(y, fx, a)\}$$

$$+ q \max \{d(x, y, a) , d (x, fx, a) , d(y, fy, a)\} \quad .... (1.1)$$

for all x, y and a ϵ X and $p \geq 0$, $0<q<1$, then for each x ϵ X,

the sequence $\{T^n x\}$ converges to a unique fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point and we define $\{ x_n\}$ as

$x_1 = f(x_0), x_2 = f(x_1),----------, x_n = f(x_{n-1})$, …………(1.2)

Suppose $x_{2n} \neq x_{2n+1}$ for every x = 0,1,2,------- , then

$[1+p\, d (x_{2n}, x_{2n+1}, a)]\, d (f (x_{2n}), f(x_{2n+1}), a)$

$\leq p \max \{ d (x_{2n}, f (x_{2n}) , a). d(x_{2n+1}, f(x_{2n+1}), a)$,

$d(x_{2n}, f(x_{2n+1}), a) . d(x_{2n+1}, f(x_{2n}), a) \} +$

$q \max \{d(x_{2n}, x_{2n+1}, a),d(x_{2n},f(x_{2n}), a), d(x_{2n+1}, f(x_{2n+1},a)\}$

which implies,

$[1+p\, d (x_{2n}, x_{2n+1}, a)]\, d (x_{2n+1}, x_{2n+2}, a)$

$\leq p\max\{d(x_{2n}, x_{2n+1}, a).d(x_{2n+1},d(x_{2n+2}), a),d(x_{2n},x_{2n+2}, a)\}$

$+q \max\{d(x_{2n}, x_{2n+1},a), d(x_{2n}, x_{2n+1}, a), d(x_{2n+1}, x_{2n+2}, a)\}$

$d (x_{2n+1}, x_{2n+2}, a) \leq q \max \{d(x_{2n},x_{2n+1},a),d(x_{2n+1} ,x_{2n+2},a)\}$

since $q<1$, $0<q /(2-q) < q$, we have

$d(x_{2n+1}, x_{2n+2}, a) \leq q\, d(x_{2n}, x_{2n+1}, a)$ …………(1.3)

Now(1.3) hold for all $a \in x$. Hence in view of lemma 2.1, the sequence $\{x_n\}$ converges to some fixed point $u \in X$. then for all $a \in X$,

$\lim d(x_{2n}, u, a) = 0$ as $n\to \infty$

Which implies,

$\lim d (f^{2n} (x_0), u, a) = 0$ as $n \to \infty$

Since f is orbitally continuous, we have

$\lim d(f(f^{2n}(x_0)), f(u), a) = 0$ as $n \to \infty$

$\lim d(f^{2n+1} (x_0), f(u), a) = 0$ as $n \to \infty$

From the definition of 2-metric space,

$d(u,f(u),a) \leq d(u,f(u), f^{2n+1}(x_0)) + d(u, f^{2n+1} (x_0), f(u))$
$+ d(f^{2n+1} (x_0), f(u), a)$

which tends to zero as $n \to \infty$.

Consequently , $d (u, f(u), a) = 0 \Rightarrow f(u) = u$

Uniqueness : For uniqueness of u, suppose $v \in X$ be another common fixed point of f such that $v \neq u$. hence there exists a point

$a \in X$ such that , $d(u,v,a) \neq 0$ then from (1.1) we have

$[1+ P\, d(u, v, a)]\, d(fu, fv, a) \leq p \max \{d(u,fu,a). d(v, fv,a),$
$d(u, fv, a). d(v, fu, a)\}$
$+q \max \{d(u, v, a), d(u, fu, a) , d (v, fv, a)\}$

i.e. $[1+ p\, d (u, v, a)]\, d(u, v, a) \leq p \max \{d(u, u, a). d(v,v,a),$
$d(u, v, a). d(v, u, a) \}$
$+q \max \{ d(u, v, a) , d (u, u, a), d (v, v, a) \}$

$d (u, v, a) \leq q\, d (u, v, a)$

$d(u, v, a) < d(u, v, a)$ which is a Contradiction.

Hence , $d (u, v, a) = 0$ which implies that $u = v$.

The next result is the generalization of Iseki [12].

Theorem 2.2:

Let f_1 and f_2 be mapping of a complete bounding 2-metric space X into itself satisfying,

$p_1 d(f_1x, f_2y, a) + p_2 d(x, f_1x, a) + p_3 d(y, f_2y, a) =$

$$\min\{d(x, f_2y, a)\, d(y, f_1x, a)\} \leq q d(x, y, a) \ldots\ldots\ldots\ (2.2)$$

For all x,y,a ϵ X there exist p_1, p_2, p_3, q are real number such that

$$p_1 + p_2 + p_3 > q,\ q - p_2 \geq 0,\ q - p_3 \geq 0$$

Then f_1 and f_2 have a common fixed point.

Proof:

Let $x_o \,\epsilon$ X. we define $\{x_n\}$

by $x_{2n+1} = f_1(x_{2n})$

$x_{2n+2} = f_2(x_{2n+1})$

then we get,

$d(x_n, x_{n+1}, a) \leq \alpha^n d(x_0, x_1, a)$

where $\alpha = \left(\frac{q-p2}{p1+p3}\right) < 1$

which implies that $\{x_n\}$ is Cauchy sequence and has limit say u ϵ X. Then

$d(f_1u, u, a) \leq d(f_1u, u, x_{2n+2}) + d(f_1u, x_{2n+2}, a) + d(x_{2n+2}, u, a)$

from (2.2)

$p_1 d(f_1a, x_{2n+2}, a) + p_2 d(u, T_1, u, a) + p_3 d(x_{2n+1}, x_{2n+2}, a) - \min\{d(u, a_{2n+2}, a)$

$d(x_{2n+1}, f_1u, a)\} \leq q d(u, x_{2n+2}, a)$

by letting $n \to \infty$ we get

$(p_1 + p_2)\, d(f_1u, u, a) \leq 0$

We find $d(f_1u, u, q) = 0$

for all a ϵ X. hence $f_1u = u$.

similarly $f_2u = u$

Thus u is a common fixed point of f_1 and f_2.

REFERENCES

[1] Nesic.S.C, A theorem on contractive mappings. Mat. Vesnik 44(1992)51-54.

[2] Banach. S,: Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math. (1922), 133-181.

[3] Gahler, VS: 2-metrische Räume und ihre topologische Struktur. Math. Nachr. 26 (1963) 115-118.

[4] Cho, Y. J., Khan, M. S. and Singh, S. L. Common fixed points of weakly commuting mappings, Univ. Novomsadu, Zb. Rad. Prirod. Mt. Fak. Ser. Mat 18 (1) (1988) 129–142.

[5] Imdad, M., Khan, M. S. and Khan, M.D. A common fixed point theorem in 2-metric spaces,Math. Japonica 36 (5) (1991) 907–914.

[6] Iseki, K. A property of orbitally continuous mapping on 2-metric spaces, Math. Seminar Notes, Kobe Univ. 3(1975)131–132.

[7] Jungck, G. and Rhoades, B. E. Fixed points for set-valued function without continuity, Indian J. Pure Appl. Math., 29(1998) 227–238.

[8] Murthy, P.P., Chang, S.S., Cho, Y. J. and Sharma, B.K. Compatible mappings of type (A) and common fixed point theorems, Kyungpook Math. J., 32 (2)(1992) 203–216.

[9] Abd EL-Monsef, M. E., Abu-Donia, H. M. & AbdRabou, Kh. New types of common fixed point theorems 2-metric spaces. Chaos, Soliton and Fractals, 41(2009) 1435-1441.

[10] Border, K.C. Fixed point theorems with applications to economics and game theory (Cam-bridge Univ. Press, 1990).

[11] Namdeo, R.K., Dubey, S. and Tas, K. Coincidence and Fixed points of non-expansive type mappings on 2-metric spaces, Mathematical Forum 2 (16) (2007) 781–789.

[12] Naidu S.V.R and Prasad J.R., Fixed point theorems in 2-metric spaces, Indian J. Pure. Appl. Math, 17(8) (1986) 974-993.

[13] Constantin, A: Common fixed points of weakly commuting mappings in 2-metric space, Math. Japon., 36(3)(1991)507-513.

[14] Iseki, K: Fixed point theorem in 2-metric space, Math. Sem., Notes, Kobe Univ., 3 (1975)133-136.

[15] Zead Mustafa, Vahid Parvaneh, Jamal Rezaei Roshan and Zoran Kadelburg, *b*-Metric spaces and some fixed point Theorems, Fixed Point Theory and Applications (2014), 144-152.

[16] Lohani P.C & Badshah V.H, Fixed point theorem in 2- metric space. Acta Ciencia , Indica V.H., XX111(3)(1997) 243-245.

[17] S. L. Singh, "Some Contractive Type Principles on 2-Metric Spaces and Applications", Mathematics Seminar Notes (Kobe University), Vol. 7, No. 1, 1979, 1-11.

Citation: *D. Lateef, " Fixed Point Theorems in 2-Metric Space for Some Contractive Conditions ", International Journal of Scientific and Innovative Mathematical Research, vol. 6, no. 1, p. 16-20, 2018., http://dx.doi.org/10.20431/2347-3142.0601003*