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Γ-Semigroups in which Primary Γ- Ideals are Prime and Maximal

S. Savithri[1], A. Gangadhara Rao [2], L. Achala[3], J. M. Pradeep [4]

Dept. of Mathematics,
[1] YA Govt. Degree College for Women, Chirala, India
[2]V S R and N V R College, Tenali,[3] JKC College, Guntur, [4] AC College, Guntur, India

***Corresponding Author:** *Suragani Savithri, Lecturer in Mathematics, YAGovt. Degree College for Women, Chirala-523157, Prakasam District. India*

Abstract: *In this paper, the terms, Maximal Γ- ideal, Primary Γ-semigroup, prime Γ-ideal and simple Γ-semigroup are introduced. It is proved that if S is a Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. Then every non zero primary Γ-ideal is prime as well as maximal if and only if S \M is a 0-simple Γ-semigroup with either 1) M = (S \M) ΓaΓ (S \M) ∪{0}, a ∈ M and <a> Γ <a> = 0 or 2) M is a 0-simple Γ-semigroup. Also it is proved that if S is a duo Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. Then every non zero primary Γ-ideal is prime as well as maximal if and only if S is one of the following types 1) S = G ∪M where G is the Γ-group of units and M = {aγg : g ∈ G, aγa = 0, a ∈ M, γ ∈ Γ } ∪ {0}. 2) S is the union of two Γ-semigroups with 0-adjoined. Also it is proved that if S is a commutative Γ-semigroup with 0 and identity and with the maximal Γ-ideal M. Suppose that every non zero primary Γ-ideal is prime or every nonzero Γ-ideal is prime. Then S satisfies either one of the following conditions 1) S = G∪M, where G is the Γ-group of units in S and M = (a ΓG) ∪{0}, a ∈ M and a Γa = 0 2) $(M\Gamma)^{n-1}$ M = M for every positive integer n. Furthermore if S has maximum condition on Γ-ideals then for every m ∈ M, we have m ∈ M Γe, e being a proper idempotent and also proved that if S is a quasi commutative Noertherian Γ-semigroup containing identity. Suppose every primary Γ-ideal in S is prime. Then the following are equivalent 1) S is cancellative. 2) S has no proper Γ-idempotents. 3) S is a Γ-group.*

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1. Introduction

Γ- semigroup was introduced by Sen and Saha [8] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals, radicals and semi pseudo symmetric ideals in semigroups. Giri and Wazalwar [4] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [5], [6] initiated the study of prime Γ-radicals and primary and semiprimary Γ-ideals in Γ-semigroups. In this paper we characterize the Γ-semigroups containing 0 and identity in which non zero primary Γ-ideals are prime and maximal and also we study the Γ-semigroups in which primary Γ- ideals are prime.

2. Preliminaries

DEFINITION 2.1: Let S and Γ be any two non-empty sets. Then S is said to be a **Γ-*semigroup*** if there exist a mapping from S × Γ × S to S which maps $(a, \gamma, b) \rightarrow a\ \gamma\ b$ satisfying the condition : $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in$ S and $\alpha, \beta, \gamma \in \Gamma$.

NOTE 2.2: Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set { $a\gamma b$: $a \in$ A , $b \in$ B and $\gamma \in$ Γ } by AΓB.

DEFINITION 2.3: A Γ-semigroup S is said to be ***commutative* Γ-semigroup** provided $a\gamma b = b\gamma a$ for all $a, b \in S$ and $\gamma \in \Gamma$.

NOTE 2.4 : If S is a commutative Γ-semigroup then $a\,\Gamma b = b\,\Gamma a$ for all $a, b \in S$.

NOTE 2.5: Let S be a Γ-semigroup and *a, b* ∈ S and *α* ∈ Γ. Then *aαaαb* is denoted by $(a\alpha)^2 b$ and consequently *a α a α a α…..(n terms)b* is denoted by $(a\alpha)^n b$.

DEFINITION 2.6: A Γ-semigroup S is said to be ***quasi commutative*** provided for each *a, b* ∈ S, there exists a natural number *n* such that $a\gamma b = (b\gamma)^n a \ \forall \gamma \in \Gamma$.

NOTE 2.7: If a Γ-semigroup S is ***quasi commutative*** then for each *a, b* ∈ S , there exists a natural number *n* such that $a\Gamma b = (b\,\Gamma)^n a$.

DEFINITION 2.8: An element ***a*** of a Γ- semigroup S is said to be a ***left identity*** of S provided $a\alpha s$ = s for all s ∈ S and α ∈ Γ.

DEFINITION 2.9: An element ***a*** of a Γ-semigroup S is said to be a***right identity*** of S provided $s\alpha a$ = s for all s ∈ S and α ∈ Γ.

DEFINITION 2.10: An element ***a*** of a Γ- semigroup S is said to be a ***two sided identity*** or an identity provided it is both a left identity and a right identity of S.

NOTATION2.11: Let S be a Γ- semigroup. If S has an identity , let S^1 = S and if S does not have an identity, let S^1 be the Γ- semigroup S with identity adjoined, usually denoted by the symbol 1.

DEFINITION 2.12: An element ***a*** **of** a Γ- semigroup S is said to be a ***left zero*** of S provided $a\Gamma s = a$ for all s belongs S.

DEFINITION 2.13: An element ***a*** **of** a Γ- semigroup S is said to be a ***right zero*** of S provided $s\Gamma a = a$ for all s belongs S.

DEFINITION 2.14: An element ***a*** **of** a Γ- semigroup S is said to be a ***zero*** of S provided it is both left and right zero of S.

NOTATION2.15: Let S be a Γ- semigroup. If S has a zero, let S^0 = S and if S does not have a zero, let $\mathbf{S^0}$ be the Γ- semigroup S with **zero adjoined**, usually denoted by the symbol 0.

DEFINITION2.16: A non empty subset A of a Γ-semigroup S is said to be a ***left Γ-ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

NOTE2.17: A non empty subset A of a Γ-semigroup S is a ***leftΓ-ideal*** of S iff S Γ A ⊆ A.

DEFINITION2.18: A non empty subset A of a Γ-semigroupS is said to be a ***right Γ-ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

NOTE 2.19: A non empty subset A of a Γ-semigroup S is a ***right Γ-ideal*** of S iff AΓS ⊆ A.

DEFINITION 2.20: A non empty subset A of a Γ-semigroup S is said to be a ***two sided Γ-ideal*** or simply a ***Γ- ideal*** of S if *s* ∈ S, *a* ∈ A, α ∈ Γ imply $s\alpha a$ ∈ A, $a\alpha s$ ∈ A.

DEFINITION 2.21: A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal Γ-ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

DEFINITION 2.22: A Γ- ideal P of a Γ-semigroup S is said to be a ***prime Γ- ideal*** provided A, B are two Γ-ideals of S and AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.

DEFINITION 2.23: A Γ-ideal A of a Γ-semigroup S is said to be a ***semiprime Γ- ideal*** provided *x* ∈ S, $x\Gamma S^1 \Gamma x \subseteq A$ implies *x* ∈ A .

DEFINITION 2.24: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime Γ-radical*** or simply ***Γ-radical*** of A and it is denoted by **√A** or ***rad A***.

THEOREM 2.25 [5]: If A is a Γ-ideal of a Γ-semigroup S then √A is a semiprime Γ-ideal of S.

THEOREM 2.26 [5]: A Γ- ideal Q of Γ-semigroup S is a semiprime Γ- ideal of S iff √(Q) = Q implies $x\,\Gamma S^1 \Gamma y \subseteq$ A.

DEFINITION 2.27: An element ***a*** of a Γ- semigroup S is said to be ***left cancellative*** provided *a* Γ*x* = *a* Γ*y* for all *x, y* ∈ S implies *x* = *y*.

DEFINITION 2.28: An element ***a*** of a Γ- semigroup S is said to be ***right cancellative*** provided $x\,\Gamma a = y\,\Gamma a$ for all $x, y \in$ S implies $x = y$.

DEFINITOIN 2.29: An element ***a*** of a Γ- semigroup S is said to be ***cancellative*** provided it is both left and right cancellative element.

DEFINITION 2.30: A Γ-ideal A of a Γ-semigroup S is said to be a ***left primary Γ-ideal*** provided

1) If X, Y are two Γ-ideals of S such that X ΓY ⊆ A and Y ⊈ A then X ⊆ √A.

2) √A is a prime Γ-ideal of S.

DEFINITION 2.31: A Γ-ideal A of a Γ-semigroup S is said to be a ***right primary Γ-ideal*** provided

1) If X, Y are two Γ-ideals of S such that X Γ Y ⊆ A and X ⊈ A then Y ⊆ √A.

2) √A is a prime Γ-ideal of S.

EXAMPLE 2.32: Let S = {*a*, *b*, *c*} and Γ = {*x*, *y*, *z*}. Define a binary operation . in S as shown in the following table.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a maping S XΓ X S → S by $a\,\alpha\,b = ab$, for all $a, b \in$ S and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal $<a> = S^1\,\Gamma a\,\Gamma S^1 = \{a\}$. Let $p\,\Gamma q \subseteq <a>$, $p \notin <a>$ $\Rightarrow q \in \sqrt{<a>} \Rightarrow (q\,\Gamma)^{n-1}q \subseteq <s>$ for some $n \in$ N. Since $b\,\Gamma c \subseteq <a>$, $c \notin <a> \Rightarrow b \in <a>$. Therefore $<a>$ is left primary. If $b \notin <a>$ then $(c\,\Gamma)^{n-1}c \notin <a>$ for any $n \in$ N $\Rightarrow c \notin \sqrt{<a>}$. Therefore $<a>$ is not right primary.

DEFINITION 2.33: A Γ-ideal A of a Γ- semigroup S is said to be a **primary Γ-ideal** provided A is both left primary Γ-ideal and right primary Γ-ideal.

DEFINITION 2.34: A Γ-ideal A of a Γ- semigroup S is said to be a **principal Γ-ideal** provided A is a Γ-ideal generated by a single element ***a***. It is denoted by J[*a*] = <*a*>.

DEFINITION 2.35: An element ***a*** of a Γ-semigroup S with 1 is said to be ***left invertible*** or ***left unit*** provided there is an element $b \in$ S such that $b\Gamma a = 1$.

DEFINITION 2.36: An element ***a*** of a Γ-semigroup S with 1 is said to be ***right invertible*** or ***right unit*** provided there is an element $b \in$ S such that $a\,\Gamma b = 1$.

DEFINITION 2.37: An element ***a*** of a Γ-semigroup S is said to be ***invertible*** or a ***Unit*** in S provided it is both left and right invertible element in S.

DEFINITOIN 2.38: A Γ- semigroup S is said to be a ***simple* Γ- *semigroup*** provided S has no proper Γ- ideals.

DEFINITION 2.39: An element ***a*** of a Γ- semigroup S is said to be a **Γ-*idempotent*** provided $a\,\alpha\,a = a$ for all $\alpha \in \Gamma$.

NOTE 2.40: If an element ***a*** of a Γ- semigroup S is a **Γ-idempotent**, then $a\,\Gamma a = a$.

DEFINITION 2.41: A Γ- semigroup S is said to be an **idempotent Γ- semigroup** or a **band** provided every element in S is a Γ-idempotent.

DEFINITION 2.42: A Γ- semigroup S is said to be a **globally idempotent Γ- semigroup** provided S ΓS = S.

DEFINITION 2.43: A Γ- semigroup S is said to be a ***left duo* Γ- *semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.44: A Γ-semigroup S is said to be a ***right duo* Γ- *semigroup*** provided every right Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.45: A Γ- semigroup S is said to be a ***duo* Γ- *semigroup*** provided it is both a left duo Γ- semigroup and a right duo Γ- semigroup.

DEFINITOIN 2.46: An element ***a*** of a Γ-semigroup S is said to be ***regular*** provided $a = a\,\alpha\, x\, \beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\,\Gamma S\,\Gamma a$.

DEFINITION 2.47: A Γ- semigroup S is said to be a ***regular Γ- semigroup*** provided every element is regular.

DEFINITION 2.48: An element ***a*** of a Γ-semigroup S is said to be ***left regular*** provided $a = a\alpha a\beta x$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\,\Gamma a\,\Gamma S$.

DEFINITION 2.49: An element ***a*** of a Γ- semigroup S is said to be ***right regular*** provided $a = x\alpha a\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in S\,\Gamma a\,\Gamma a$.

DEFINITION 2.50: An element ***a*** of a Γ- semigroup S is said to be ***completely regular*** provided there exists an element $x \in S$ such that $a = a\alpha x\beta\, a$ for some $\alpha, \beta \in \Gamma$ and $a\alpha x = x\beta a$, for all $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma x\,\Gamma a$ and $a\Gamma x = x\,\Gamma a$.

DEFINITION2.51: A Γ-semigroup S is said to be a ***completely regular* Γ- *Semigroup*** provided every element is completely regular.

DEFINITION 2.52: An element ***a*** of a Γ-semigroup S is said to be ***intra regular*** provided $a = x\alpha a\beta a\,\gamma y$ for some $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

DEFINITION 2.53: An element ***a*** of a Γ- semigroup S is said to be ***semisimple*** provided $a \in \langle a\rangle\,\Gamma\,\langle a\rangle$, that is, $\langle a\rangle\,\Gamma\,\langle a\rangle = \langle a\rangle$.

DEFINITION 2.54: A Γ-semigroup S is said to be ***semisimple* Γ- *semigroup*** provided every element is a semisimple.

DEFINITION 2.55: A Γ-semigroup S is said to be **a Noetherian Γ**-semigroup provided every ascending chain of Γ-ideals becomes stationary.

THEOREM 2.56 [6]: Let S be a Γ-semigroup with identity and let M be the unique maximal Γ-ideal of S. If $\sqrt{A} = M$ for some Γ-ideal of S. Then A is a primary Γ-ideal.

THEOREM 2.57 [6]: If S is a duo Γ-semigroup, then the following are equivalent for any element $a \in S$.

1) ***a* is completely regular.**
2) ***a* is regular.**
3) ***a* is left regular.**
4) ***a* is right regular.**
5) ***a* is intra regular.**
6) ***a* is semisimple .**

THEOREM 2.58: Let S be a Γ- semigroup with identity. If (non zero, assume this if S has zero) proper prime Γ-ideals in S are maximal then S is primary Γ-semigroup.

Proof: Since S contains identity, S has a unique maximal Γ- ideal M, which is the union of all proper Γ- ideals in S. If A is a (non zero) proper Γ-ideal in S then $\sqrt{A} = M$ and hence by theorem 2.56, A is primary Γ- ideal. If S has zero and if $\langle 0\rangle$ is a prime Γ-ideal, then $\langle 0\rangle$ is primary and hence S is primary. If $\langle 0\rangle$ is not a prime Γ-ideal, then $\sqrt{\langle 0\rangle} = M$ and hence by theorem 2.56, $\langle 0\rangle$ is a primary Γ- ideal. Therefore S is a primary Γ- semigroup.

DEFINITION 2.59 : A Γ-semigroup S is said to be a **Γ-*group*** provided S has no left and right Γ-ideals.

3. Primary Γ-Ideals Are Prime and Maximal

THEOREM 3.1: Let S be a Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. Then every nonzero primary Γ-ideal is prime as well as maximal if and only if S\M is a 0-simple Γ-semigroup with either

1) $M = (S\backslash M)\,\Gamma\, a\,\Gamma(S\backslash M) \cup \{0\}$, $a \in M$ and $\langle a\rangle\,\Gamma\langle a\rangle = 0$

or

2) M is a 0-simple Γ-semigroup.

Proof: Suppose every nonzero primary Γ-ideal is prime and maximal. Since nonzero prime Γ-ideals are maximal, by theorem 2.58, S is a primary Γ-semigroup. If <0> is the maximal Γ-ideal in S, then the proof of this theorem is trivial.

Suppose S has nonzero maximal Γ-ideal M. Since S is a primary Γ-semigroup and every non zero primary Γ-ideal is maximal, we have M is the only nonzero proper Γ-ideal in S. Since M is a maximal Γ-ideal, S\M is a 0-simple Γ-semigroup. Now for every nonzero $a \in$ M, <*a*> = M. Since M ΓM is a Γ-ideal contained in M, either M ΓM = 0 or M ΓM = M. If M ΓM = 0 then for all *a, b* ∈ M, <*a*>Γ<*b*> = 0 and <*a*>Γ<*a*> = 0 for all *a* ∈ M. Since for all nonzero elements *a, b* ∈ M, <*a*> = <*b*> = M. We have *b* ∈ *g*Γ*a*Γh for some *g, h* ∈ S. If *g* or *h* ∈ M then by the above *b* = 0, this is a contradiction. So *g, h* ∈ S\M. Therefore M = (S\M)Γ*a*Γ(S\M) ∪ {0}, *a* ∈ M and <*a*>Γ<*a*> = 0. If MΓM = M then for every non zero *a* ∈ M. We have M Γ*a*ΓM = MΓSΓ*a*ΓSΓM = MΓMΓM = M. Therefore M is a 0-simple Γ-semigroup.

Conversly if S\M is a 0-simple Γ-semigroup with either M = (S\M) Γ*a* Γ(S\M) such that *a* ∈ M and <*a*>Γ<*a*> = 0 or M is a 0-simple Γ-semigroup, then clearly either M = <0> and S has no other Γ-ideals or M is the only nonzero Γ-ideal in S.

Case 1) : Suppose M = <0> implies S\M is a 0-simple implies <*a*> is a maximal Γ-ideal of S. Therefore S has no other nonzero Γ-ideals.

Case 2) : Suppose A is any nonzero proper Γ-ideal and A ⊆M. Let *a* ∈ A implies *a* ∈ M. Let *a* ≠ 0, *a* ∈ M implies <*a*> ⊆ M. M = (S\M) Γ*a* Γ (S\M) ⊆ S Γ*a* Γ S ⊆ <*a*>. Therefore M ⊆ <*a*> and clearly <*a*> ⊆ M. Therefore M = <*a*>. Therefore M is the only nonzero Γ-ideal in S.

NOTE 3.2: If S does not contain 0, then the case M ΓM = 0 in the above proof does not arise.

THEOREM 3.3:Let S be a Γ-semigroup containing identity and not containing 0. Then every primary Γ-ideal is prime as well as maximal if and only if S is either a simple Γ-semigroup or a 0-simple extension of a simple Γ-semigroup.

Proof: The proof can write by using theorem 3.1.

THEOREM 3.4: Let S be a duo Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. Then every nonzero primary Γ-ideal is prime as well as maximal if and only if S is one of the following types.

1) **S = G∪M where G is the Γ-group of units and M = { aγg : g ∈ G, aγa = 0, a ∈ M , γ ∈ Γ} ∪ {0} .**
2) **S is the union of two Γ-groups with 0 adjoined.**

Proof: Since S is a duo Γ-semigroup, we have S\M is a Γ-group consists of all units in S and the sets (S\M) Γ*a* Γ (S\M) ∪ {0} with *a* ∈ M and < *a*>Γ<*a*> = 0 and *a* Γ(S\M) ∪ {0} with *a* ∈ M and *a*Γ*a* = 0 are equal. Also if M is 0-simple, then M is a Γ-group with 0 adjoined. Thus by theorem 3.1, the proof of this theorem is trivial.

NOTE 3.5 : Every commutative Γ-semigroup is a duo Γ-semigroup.

THEOREM 3.6: Let S be a duo Γ-semigroup containing identity and not containing 0. Then every primary Γ-ideal is prime and maximal if and only if S is either a Γ-group or a union of two Γ-groups.

Proof: The proof of this theorem is an immediate consequence of theorem 3.4.

THEOREM 3.7: Let S be a Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. Suppose that every nonzero primary Γ-ideal is prime. Then S\M is a 0-simple Γ-semigroup such that either

1) M = (S\M) Γ*a*Γ (S\M) ∪ {0}, *a* ∈ M and < *a*>Γ<*a*> = 0

or

2) $(M\gamma)^{n-1}$ M = M for every natural number *n*.

Proof : Suppose every nonzero primary Γ-ideal is prime. If MΓM = 0 then M is the unique prime Γ-ideal in S. Now √<*a*> = M for any nonzero $a \in$ M and thus <*a*> is primary by theorem 2.56, MΓM is a primary Γ-ideal and hence MΓM is a prime Γ-ideal by hypothesis. Thus M = MΓM and hence $M = (M\gamma)^{n-1} M$ for every natural number *n*.

THEOREM 3.8: Let S be a Γ-semigroup containing identity and not containing 0 in which primary Γ-ideals are prime. Then S is a 0-simple Γ-semigroup extention of a globally idempotent Γ-semigroup.

Proof: The proof of this theorem is a direct consequence of theorem 3.7.

THEOREM 3.9: Let S be a duo Γ-semigroup containing 0 and identity with the maximal Γ-ideal M. If every nonzero primary Γ-ideal is prime, then S satisfies either one of the following conditions.

1) **S = G∪ M where G is the Γ-group of units in S and M = *a* ΓG∪ {0}, *a* ∈ M and *a*Γ*a* = 0.**
2) **$(M\Gamma)^{n-1}$ M = M for every natural number *n*. Furthermore if S is Noetherian and quasi commutative, then for every *a* ∈ M, we have *a* ∈ *a*Γ*e*, *e* being proper idempotent in S.**

Proof: By theorem 3.7, if every nonzero primary Γ-ideal is prime, then either 1)S = G∪M, where G is the Γ- group of units in S and M = (*a*ΓG) ∪ {0}, *a* ∈ M and *a*Γ*a* = 0, 2) $(M\Gamma)^{n-1}M = M$ for every natural number *n*. Suppose S is a Noetherian quasi commutative Γ-semigroup with MΓM = M. Since MΓM = M, every *x* ∈ M is of the form *a*Γ*b* where *a*, *b* ∈ M. Suppose there exists a nonzero element *a* ∈ M such that *a* cannot be a product of itself and some element in M, that is, let $a \in b_1\Gamma a_1$ where $a_1, b_1 \in$ M and $\neq a$. Then $a_1 \in b_2\Gamma a_2$ where $b_2, a_2 \in$ M and $\neq a_1$. Since otherwise $a_1 \in a_1\Gamma a_2$ implies $a \in b_1\Gamma a_1\Gamma a_2$ and so $a \in a\Gamma a_2$ this is a contradiction. Proceeding in this manner, we have $a_2 \in b_3\Gamma a_3$, . . . ,$a_k \in b_{k+1}\Gamma a_{k+1}$, ...Thus we obtain a strictly ascending chain of Γ-ideals $<a_1> \subset <a_2> \subset$ . . . Then since S is Noetherian, this chain terminates and hence we have $a_n \in b_{n+1}\Gamma a_{n+1}$ where $a_{n+1} \in s\Gamma a_n$. This implies $a_n \in b_{n+1}\Gamma s\Gamma a_n$, this is a contradiction. Therefore there does not exist a nonzero *a* ∈ M such that a cannot be a product of itself and some element in M. We claim that for every nonzero *a* ∈ M, *a* ∈ *a*Γ*e*, *e* = *e*Γ*e* ∈ M. Let us assume the contrary, that is, suppose that there exists *a* ∈ M such that a is not a product of *a* Γ- idempotent and itself. So $a \in a\Gamma b_1$ where b_1 is not a Γ- idempotent. Clearly $<a> \neq <b_1>$. Since otherwise $b_1 \in a\Gamma t$ and so $a \in (a\Gamma a)\Gamma t$ which implies by theorem 2.57, *a* is regular and hence *a* is a product of a Γ- idempotent and itself, which is a contradiction. So $<a> \subset <b_1>$. Proceeding in this manner we have $b_1 \in b_1\Gamma b_2, b_2 \in b_2\Gamma b_3$, ... Thus we have a strictly ascending chain of Γ-ideals $< a_1> \subset <b_1>. \subset$...... . Since S is Noetherian, Γ-semigroup, this chain terminates and hence $<b_n> = <b_{n+1}> =$.......... for some natural number *n*. Now we have b_n is a product of an idempotent and itself, this is a contradiction. Therefore *a* ∈ *a* Γ*e*, *e* ∈ *e* Γ*e*, *e* ∈ M.

THEOREM 3.10: Let S be a commutative Γ-semigroup with 0 and identity and with the maximal Γ-ideal M. Suppose that every nonzero primary Γ-ideal is prime or every non zero Γ-ideal is prime. Then S satisfies either one of the following conditions.

1) **S = G∪ M, where G is the Γ-group of units in S and M = (*a* ΓG) ∪ {0}, *a* ∈ M and *a* Γ*a* = 0.**
2) **$(M\Gamma)^{n-1}$M = M for every positive integer *n*. Furthermore if S has maximum condition on Γ-ideals then for every *m* ∈ M, we have *m* ∈ M Γ*e*, *e* being a proper idempotent.**

Proof: The proof of this theorem is an immediate consequence of theorem 3.9.

THEOREM 3.11: Let S be a quasi commutative Noetherian Γ-semigroup containing identity. Suppose every primary Γ-ideal in S is prime. Then the following are equivalent.

1) **S is cancellative.**
2) **S has no proper Γ- idempotents.**
3) **S is a Γ-group.**

Proof: 3) implies 1) is clear. Let *e* be a Γ-idempotent in S. Let *a* ∈ S. Now *aγe* = *aγeγe* implies *a* = *aγe*f or γ ∈ Γ. This is true for all *a* ∈ S, γ ∈ Γ. Similarly *e γa* = *a*. Therefore *e* is the identity in S. Therefore S has no proper idempotents. Therefore 1) implies 2). Assume 2). If S is not a Γ-group, then S has a unique maximal Γ-ideal M and hence theorem 3.9, for every $a \in$ M, $a = a\,\Gamma e$ for some proper idempotent e. This is a contradiction. Therefore 2) implies 3).

THEOREM 3.12: Let S be a quasi commutative Γ-semigroup with 0 and with out identity in which every nonzero Γ-ideal is prime. If S is Noetherian, then every element x in S of the form $x = x\,\Gamma t$, $t \in$ S or $x\,\Gamma x = (x\,\Gamma x)\Gamma e$ where e is an idempotent. Furthermore, if S is cancellative, then every $x \in$ S is of the form $x = x\,\Gamma t$, $t \in$ S.

Proof: If S has no proper nonzero Γ-ideals, then for any nonzero $x \in$ S, $x\,\Gamma$S = S. Thus $x = x\,\Gamma t$, $t \in$ S, $\gamma \in \Gamma$. If S has no proper Γ-ideals, then S is Noetherian, S contains maximal Γ-ideals. Suppose there exists a maximal Γ-ideal M such that MΓM = 0. Then for any prime Γ-ideal P, we have MΓM ⊆ P and hence M = P. So M is a unique nonzero Γ-ideal in S. Then $0 \neq x \in$ M implies $x\,\Gamma$S = M. Hence $x = x\,\Gamma t$ for some $t \in$ S. If $x \notin$ M, then since M is prime $x\,\Gamma x \not\subseteq$ M. So $x\,\Gamma$S = S. Thus $x \in x\,\Gamma t$ for some $t \in$ S. Now assume that MΓM ≠ 0 for any maximal Γ-ideal M. Let $x \in$ S. Then since S is Noetherian $x\Gamma$S contained in maximal Γ-ideal, say M.

Since MΓM ≠ 0, MΓM is prime and hence MΓM = M. Then it can be easily verified as in the proof of the theorem 3.9, that $x\,\Gamma x = x\,\Gamma x\,\Gamma e$ where e is a Γ- idempotent. Clearly if S is cancellative, then $x \in x\,\Gamma t$ for some $t \in$ S.

Conclusion: It is proved that if S is a quasi commutative Γ-semigroup with 0 and without identity in which every no-zero Γ-ideal is prime. If S is Noertherian, then every element x in S of the form x = x Γt, t ∈ S or x Γx = (x Γx) Γe where e is an idempotent. Furthermore, if S is cancellative, then every x ∈ S is of the form x = xΓt, t ∈ S.

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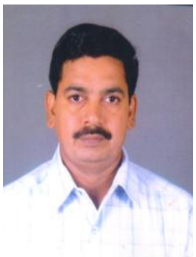
AUTHORS' BIOGRAPHY

Suragani Savithri, working as Lecturer in Mathematics, HOD, Department of Mathematics, YAGovt. Degree College for Women, Chirala, Prakasam District, AP, India.Pursuing PhD, in Mathematics at Acharya Nagarjuna University , Guntur with the award of Teacher ellowship during the Twelfth Plan Period (2012-2017) by the University Grants Commission. Member Indian Science Congress. Participated in National and International Seminars in Mathematics-6, Research Publications in International Journals-3.



Dr. A. Gangadhara Rao, working as Associate Professor in Mathematics H O D, Department of Mathematics, V S R and N V R College, Tenali, Affiliated to Acharya Nagarjuna University, Guntur (Dt.), A . P., India and Research director in Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, **Awards** : A P State Best Teacher Awardee in Mathematics for the year 2016. **Research Guidance** : Ph. D. awards – 01, Ph. D. submission – 01, Ph. D. persuing – 07, M. Phil. Awards – 02, M. Phil. Submission – 01,M. phil. Persuing -01, **Research publications** : Research publications in International Journals – 33, **National Seminars/Conferences/Workshops** : Delivered invited talks in National Seminars and workshops as Resource person – 03,Participated and presented papers – 35, Chair person in National seminars and workshops – 02,Organized National Seminars / workshops / conferences – 04.

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