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Bounded Projections on Fourier-Modified-Stieltjes Transforms

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Abstract: *We study certain algebraic projections on measure algebra (of a locally compact abelian group) which extends to bounded projections on the uniform closure of the Fourier-Modified-Stieltjes transforms. These projections arise by studying a Raikov system of subsets induced by locally compact subgroups. These results generalize the inequality* $\|\hat{\mu}_d\| \le \|\hat{\mu}\|_\infty$ *(where μ is measure algebra, μ $_d$ is discrete part of μ and* $\|\hat{\mu}\|_\infty$ *is the sup-norm of the Fourier-Modified-Stieltjes transforms.)*

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1. Introduction

Here H will be a locally compact abelian (LCA) group. The group H with discrete topology is denoted by H $_d$.This is same as giving H the topology induced from declaring the subgroup G = {0} ⊂ H to be open. The space of finite regular Borel measures on H is denoted M (H). For μ ∈ M (H), let μ $_d$ denote discrete part of μ. The ring homomorphism μ → μ $_d$ maps M (H) onto M (H $_d$), and this map is norm-non increasing in the measure norm; that is, $\|\mu\|_d \le \|\mu\|$,μ ∈ M (H). For μ ∈ M (H) we let μ denote Fourier-Modified-Stieltjes transform of μ; that is $\mu(\gamma) = \int_H (\gamma(x))^{-} d\mu(x)$

where γ∈H (dual of H). In the papers [3],[4] C. Dunkl and D. Ramirez showed (in more general setting)

$\|\hat{\mu}_d\| \le \|\hat{\mu}\|_\infty$ (Where $\|\cdot\|_\infty$ denotes the sup-norm) This further implies that

$\mathcal{M}$ (Ĥ) = $\mathcal{M}$ $_c$ (Ĥ) ⊕ $\mathcal{M}$ $_d$ (Ĥ) , where $\mathcal{M}$ (Ĥ), $\mathcal{M}$ $_c$ (Ĥ) and $\mathcal{M}$ $_d$ (Ĥ) are sup-norm closures on Ĥ of the Fourier-Modified-Stieltjes transform of measures from M(H), M $_c$ (H) (the space of continuous measures) and M $_d$ (H) respectively. Let Δ denote maximal ideal space of M (H) and let κ Ĥ denote Δ closure of Ĥ in Δ (Recall Ĥ ⊂Δ under the identification map from Ĥ to Δ by $\pi_\gamma(\mu) = \hat{\mu}(\gamma), \gamma \in \hat{H}$, $\mu \in M(H)$), we call the set κ Ĥ \ Ĥ the fringe of Ĥ. The result $\|\hat{\mu}_d\|_\infty \le \|\hat{\mu}\|_\infty$ (μ ∈ M(H)) implies that the fringe of Ĥ contains homomorphic copy of Bhor group βĤ of Ĥ (under the map χ→π$_x$ from βĤ to Δ given by $\pi_\gamma(\mu) = \int_H \bar{\gamma} d\mu$, μ ∈ M(G), χ ∈ β(Ĥ).

The setting of this paper is as follows. We let H be an LCA group with topology τ $_H$ and G be a subgroup of H which has an LCA group topology τ $_G$ such that injection (G, τ $_G$) → (H, τ $_H$) is continuous. For example G is the image under a continuous monomorphism of an LCA group. We let

H_G denote H with the topology induced by declaring the subgroup G with the τ_G topology to be open. We will assume that G is a nonopen subgroup of H so that $H \neq H_g$ topologically.

We now define the natural Projection P:M(H)→ M(H_G) by utilizing a Raikov system of subsets of H.(For basic facts connecting Raikov System see [6]). We choose F to be Raikov system generated by the family of compact subsets of G. Let R be the set of measures μ ϵ M(H) such that $|\mu|$ is connected on some element of F and let I be the set of measures μ ϵ M(H) such that $|\mu|(A) = 0$ for all A ϵ F. Then I is closed ideal in M (H) and R is the closed subalgebra of M(H). Further-more M(H) = R ⊕ I (see for example, [6,P151]).Now R can be identified with M(H_G) and thus the natural projection P:M(H)→ M(H_G) is induced by the given direct sum. For μ ϵ M (H) we write $\mu = \mu_G + \mu_I$ where $\mu_G \in M(H)$ an $\mu_I \in I$. Thus $P_\mu = \mu_G$, μ ϵ M(H) Observe that P is norm-bounded projection; that is , $|P_\mu| \le |\mu|$, μ ϵ M(H) .Our goal now is to show $\|\hat{P}_\mu\| \le \|\hat{\mu}\|_\infty$, μ ϵ M(H)

Let φ: H_G → H be the identity map and φ :Ĥ→ĤG the adjoint map (an injection). In paper [5] C. Dunkl and D. Ramirez showed that for any continuous homomorphism π: $G_1 \to G_2$ (G_1, G_2 LCA groups) that π is open if and only if π: G1 → G2 (the adjoint map) is proper (the inverse image of a compact set compact) . Thus since φ is not open, φ is not proper. The map φ induces continuous homomorphism φ* : M (H_G) → M (H). Since φ is one-to- one, φĤ is dense in $\hat{H}_G$. Indeed for any K⊂ Ĥ, φ (Ĥ\K) is dense in ĤG. For μ ϵ M (H), $\|\hat{\mu}\|_\infty$ is the supremum of $|\mu|$ over either φĤ or $\hat{H}_G$. For an LCA group L, we let P (L) denote space of continuous definite functions on L; we let P_c (L) be that f ∈P (L) with compact support. We will denote the Haar measure on H_G by λ . (The measure λ restricted to G is Haar measure on G)

2. Definitions

2.1. Fourier-Modified-Stieltjes Transform: Fourier –Modified – Stieltjes transform of complex valued smooth function (t, x) is defined by the convergent integral

$$F(s, y) = FT_{p+1}\{f(t, x)\} = \int_0^\infty \left(\int_0^\infty f(t,x) e^{-ist} (x+y)^{-p} \, dt \right) dx$$

2.2. Raikov System of Subsets: Let $\mathcal{F}$ denotes a family of σ – compact subsets of H such that (i) if A ϵ $\mathcal{F}$, B is σ – compact and B ⊂A then B ∈ $\mathcal{F}$ (ii) if $\{A_n\}_{n=1}^\infty \subset \mathcal{F}$ then $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$ (iii) A, B ∈ $\mathcal{F}$ then A+B ϵ $\mathcal{F}$ and (iv) A ϵ $\mathcal{F}$ and x ϵ H, then x + A ϵ $\mathcal{F}$ such a family of subsets of H is called Raikov System.

2. 3. Positive Definite Function: A function φ defined on G is said to be positive definite if the inequality $\sum_{n=1}^\infty c_n \bar{c}_m \varphi(x_n - x_m) \ge 0$ holds for every choice of x_1, x_2, --------, x_N in G for every choice of complex numbers c_1, c_2,----------,c_N .

3. Main Results

Mainly, the results of this section are from [1]

3.1. Proposition 1: Let f ∈ P_c (H_G) and let d μ =f d λ . If g ∈P_c (H) the g * μ (convolution in M (H)) is in P_c (H)

Proof: - Since f ∈ P_c (H_G), $\hat{f} \in L^1(\hat{H}_g)$ by inversion theorem [7, p.22], and $\hat{f} \ge 0$ by Bocher's theorem [7, p. 19]. Thus γ ∈ Ĥ ⊂ $\hat{H}_G$,

$$\hat{\mu}(\gamma) = \int_H \bar{\gamma} d\mu = \int_{H_G} \bar{\gamma} f d\lambda = \hat{f}(\gamma) \ge 0$$

Since g and μ have compact supports, g * μ is continuous function on H with compact support. Finally g * μ is positive definite since $(g \hat{*} \mu) = \hat{g}\hat{\mu} \ge 0$ on H □

An LCA group L is amenable, and thus satisfies the condition of Godement: the constant function 1 can be approximated uniformly on compact subsets of L by functions of form $K * \widetilde{K}$, where K is a continuous function with compact support and $\widetilde{K}(x) = (K(-x))^{-}, x \in L$ (see [6, p.168, 172]). Thus we have

3.2. Proposition 2: Let L be an LCA group and K⊂L a compact subset of L. Given ε > 0, there is p ∊ P_c (L) such that p (0) = 1 and │p - 1│< ε on K.

3.3. Proposition 3: Let K be compact subset of H_G, and let U be a relatively compact neighborhood of 0 in H_G. Then there is a neighborhood of V of 0 in H such that (x + V) ∩ K⊂ x + U for all x ∊ K.

Proof: - Since K is compact in H_G, K-K is also compact in H_G; and the inducted topology on K-K (since compact topologies are minimal Hausdorff) . Thus there is an H-open neighborhood of 0, V, such that V ∩ (K – K) ⊂ U ∩ (K- K). Thus for x ∊ K,

(x + V) ∩ K ⊂ x + (V ∩ (K – {x})) ⊂ x + (V ∩ (K – K)) ⊂ x + (U ∩ (K – K)) ⊂ x + U.

3.4. Proposition 4: Let ξ ∊ Ĥ $_G$, K a compact subset of H_G, ε > 0 be given. Then there exists γ ∊ Ĥ such that │γ - ξ│< ε on K.

Proof: - Recall that φĤ can be identified with Ĥ , and it is dense in Ĥ $_G$ .Finally the topology in Ĥ $_G$ is the compact –open topology .

3.5. Theorem:- Let P ; M(H) → M(H_G) , Then $\left\|(\hat{P}_\mu)\right\|_\infty \le \|\hat{\mu}\|_\infty$, $\mu \in M(H)$.

Proof: - Let μ ≠ 0 be in M (H), and let ξ ∊ Ĥ $_G$. We write μ = μ_G + μ_I where μ_G ϵ M (H_G) and μ_I ∊ I using the Raikov System. We will show that $|\hat{\mu}(\xi)| \le \|\hat{\mu}\|_\infty$.

We may assume that spt μ_G (spt denote the support) is compact in H_G. By Proposition 2, there is

p∈Pc(HG) such that p(0) =1 and $|p-1| < \varepsilon / \|\mu\|$ on spt μ_G.

Since $|\mu_I|$ (spt p) = 0, we assume that spt μ_I ∩ sptp = φ. Since p is uniformly continuous in the H_G topology, there is a H_G - open neighborhood of 0, U, such that for x ϵ H_G and y ∊ U,

│p(x + y) - p(x) │< ε / ‖μ‖. Let K = - K be a compact subset of H_G containing spt p and spt μ_G .By proposition 3 , choose V to be H-open neighborhood of 0 such that V = - V and (x + V) ∩K ⊂ x + U for all x ∊ K; we further assume that (spt p + V) ∩ (spt μ_I + V) = φ .

Now choose γ ∊ Ĥ by proposition 4 such that │γ - ξ│< ε / ‖μ‖ on K; and choose g ∊ P_c (H) with spt g ⊂ V, g ≥ 0, and $\int_U g d\lambda$ = 1. For any x ∊ K, │ (g*pdλ) (x) – p(x) │= $\left|\int_V g(y)p(x-y)d\lambda(y) - p(x)\right| = \left|\int_U g(y)(p(x-y)d\lambda(y) - p(x))d\lambda(y)\right| < \varepsilon / \|\mu\|$

(SinceV∩(x-K)⊂U, x∈spt p). Thus letting f = g *pdλ, spt f ⊂V+spt p and f ε P_c (H) (by proposition 1) .Also f(0) < p(0) + ε / μ = 1 + ε / μ and spt f ∩ spt μ_I = φ. For x ∊ spt μ_G,

$|f(x)-1| \le |f(x)-p(x)| + |p(x)-1| < 2\varepsilon\|\mu\|$

And

$$\left|\int_{H_G} \xi d\mu_G - \int_H \gamma f d\mu\right| \le \left|\int_{H_G} \xi d\mu_G - \int_{H_G} \gamma d\mu_G\right| + \left|\int_{H_G} \gamma d\mu_G - \int_H \gamma f d\mu_G\right| + \left|\int_H \gamma f d\mu_I\right| < (\varepsilon / \|\mu\|)\|\mu_G\| + (2\varepsilon / \|\mu\|)\|\mu_G\| + 0 \le 3\varepsilon$$

Now $\left|\int_H \gamma f d\mu\right| \le f(0)\|\mu\|_\infty < (1 + \varepsilon / \|\mu\|)\|\hat{\mu}\|_\infty$ (Since γf is positive definite).

Summarizing, given $\xi \in \hat{H}_G$,

$$|\hat{\mu}_G(\xi)| = \left| \int_{H_G} \xi d\mu_G \right| \le \left| \int_H \gamma f d\mu \right| + 3\varepsilon \le (1+\varepsilon/\|\mu\|)\|\hat{\mu}\|_\infty + 3\varepsilon \le \|\hat{\mu}\|_\infty + 4\varepsilon$$

And so $\|\hat{\mu}_G\|_\infty \le \|\mu\|_\infty$.

3.6. Corollary 1:- Let ($\hat{H}$), ($\hat{H}_G$) and τ denote the uniform closures of Fourier-Modified-Stieltjes transforms of M(H), M(H_G)and I respectively. Then ($\hat{H}$) = ($\hat{H}_G$) $\oplus$ τ.

3.7. Corollary 2:-If $\mu \in$ M (H) and $\mu \in$ ($\hat{H}_G$), then $\mu \in$ M (H_G).

3.8. Corollary 3:-Let $\hat{H}_G$ be embedded in $\kappa\,\hat{H}$ (The maximal ideal space of ($\hat{H}$); equivalently , the closure of $\hat{H}$ in Δ), by $\gamma \to \pi_\gamma$ from $\hat{H}_G$ to $\kappa\,\hat{H}$ where π_γ (μ) = $\hat{\mu}_G(\gamma)$ ($\gamma \in$ M(H)). Since π_γ (μ) = 0 for $\mu \in L^1$ (H) (recall G is nonopen in H), $\pi_\gamma \in \kappa\,\hat{H}/\,\hat{H}$ (the fringe of $\hat{H}$). In particular, $\mu \in$ M (H), $\|\hat{\mu}_G\|_\infty \le \lim\sup \|\hat{\mu}_G\|_\infty \le \|\hat{\mu}\|_\infty$.

These corollaries follow from the inequality $\|\hat{\mu}_G\|_\infty \le \|\hat{\mu}\|_\infty$ ($\mu \in$ M (H)).

Some interesting examples of LCA groups H with nonopen subgroup G are: (1) H nondiscrete and G = {0}. (2) G is noncompact and H= β G the Bhor Compactification of G, (3) G = R (the real numbers) and H a compact solenoidal group, and (4) certain local direct product groups embedded in the appropriate complete direct product groups.

4. Conclusions

This paper is concerned with bounded projections on Fourier-Modified-Stieltjes transform. In this paper certain algebraic projections on measure algebra of locally compact abelian groups were studied. This is extended to bounded projections on uniform closure of Fourier-Modified-Stieltjes transform.

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