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On a Method to Compute the Determinant of a 4×4 Matrix

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Abstract: *In this article, we will study an interesting method to compute the determinant of a square matrix of order 4.*

Keywords: *Determinant, Matrices of order 4, Duplex fraction, Dodgson's condensation.*

1. Introduction

In linear algebra and matrix theory, the determinant of a square matrix is very important for many sciences as Physics, Statistics, Engineering, etc. The determinant of an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n},$$

is denoted by $det(A)$ or $|A|$, and a basic formula to compute the determinant is

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum sgn(j_1, j_2, \dots, j_n)\, a_{1j_1} \cdots a_{nj_n},$$

where the summation is taken over all $n!$ permutations $j_1, j_2, \dots j_n$ of the set of integers $1, 2, \dots, n$. Also, the function $sgn(j_1, j_2, \dots, j_n)$ is defined as:

$$sgn(j_1, j_2, \dots, j_n) = \begin{cases} +1 & \text{, if } j_1, j_2, \dots, j_n \text{ is an even permutation} \\ -1 & \text{, if } j_1, j_2, \dots, j_n \text{ is an odd permutation} \end{cases}.$$

There are many methods and rules to compute the determinant of a square matrix and some well-known methods are Sarrus' rule, triangle's rule, Chio's condensation method, Dodgson's condensation method, etc. In 2016, R. Farhadian [2] established a new method to compute the determinant of a square matrix of order 4. In this article, we will study the Farhadian's method, we also present some new results.

For the next section, the following notation will be used:

(a) A_n denotes a square matrix of order n.

(b) $\mathbb{R}$ denotes the set of all real numbers.

2. Duplex Fraction method to compute the Determinant of a square Matrix of order 4

In 2016, R. Farhadian established the following definition (see [2, Definition 2.1]):

Definition 1. (*Duplex Fraction*). *Let* $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ *and* $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ *are two* 2×2 *determinants such that* $b_{11}, b_{12}, b_{21}, b_{22}$ *are nonzero numbers and* $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0$. *The duplex fraction of* $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ *on* $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ *is defined as:*

$$\frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}}.$$

For example, the duplex fraction of $\begin{vmatrix} 25 & 15 \\ 21 & 12 \end{vmatrix}$ on $\begin{vmatrix} 5 & 3 \\ 3 & 6 \end{vmatrix}$ is

$$\frac{\begin{vmatrix} 25 & 15 \\ 21 & 12 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 6 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{25}{5} & \frac{15}{3} \\ \frac{21}{3} & \frac{12}{6} \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 6 \end{vmatrix}} = \frac{\begin{vmatrix} 5 & 5 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 6 \end{vmatrix}} = \frac{-25}{21}.$$

Now, we shall know about the *Dodgson's condensation* of a matrix that was introduced by Charles Lutwidge Dodgson (1832-1898) in 1866 (see [1]):

Definition 2. (*Dodgson's condensation*). *The Dodgson's condensation of an* $n \times n$ *matrix* $A_n = [a_{ij}]_{n\times n}$ *is an* $(n-1)\times(n-1)$ *matrix such as* $[m_{ij}]_{(n-1)\times(n-1)}$ *such that*

$$m_{ij} = \begin{vmatrix} a_{ij} & a_{i(j+1)} \\ a_{(i+1)j} & a_{(i+1)(j+1)} \end{vmatrix}.$$

Henceforth the notation $DC(A_n)$ denotes the Dodgson's condensation of a matrix A_n, and the second condensation is $DC(DC(A_n))$ and so on.

Thus, for a 4×4 matrix

$$A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4},$$

the firs Dodgson's condensation is

$$DC(A_4) = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{bmatrix}_{3\times 3}. \quad (1)$$

and the second condensation is

$$DC\big(DC(A_4)\big) = \begin{bmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{bmatrix}_{2\times 2}. \qquad (2)$$

For example, let

$$A_4 = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 3 & 2 & 3 & 1 \\ 1 & 4 & 2 & 5 \\ 7 & 1 & 3 & 1 \end{bmatrix}_{4\times 4},$$

using (1), we have

$$DC(A_4) = \begin{bmatrix} 1 & -5 & -14 \\ 10 & -8 & 13 \\ -27 & 10 & -13 \end{bmatrix}_{3\times 3},$$

and using (2), we have

$$DC\big(DC(A_4)\big) = \begin{bmatrix} 42 & -177 \\ -116 & -26 \end{bmatrix}_{2\times 2}.$$

We have the following theorem to compute the determinant of a square matrix of order 4.

Theorem 1. *Given a* 4×4 *matrix*

$$A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4},$$

where $a_{22}, a_{23}, a_{32}, a_{33}$ *are nonzero numbers and* $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$. *Then*

$$|A_4| = \frac{\big|DC\big(DC(A_4)\big)\big|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{\begin{vmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}}{a_{22}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix}}{a_{23}} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix}}{a_{32}} & \dfrac{\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}}{a_{33}} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}.$$

Proof. See [2].

As an example of Theorem 1, consider the matrix

$$A_4 = \begin{bmatrix} 3 & 5 & 1 & 6 \\ 5 & 2 & 1 & 1 \\ 1 & 4 & 3 & 5 \\ 7 & 1 & 3 & 3 \end{bmatrix}_{4\times 4},$$

we have

$$\xrightarrow{DC(A_4)} \begin{bmatrix} -19 & 3 & -5 \\ 18 & 2 & 2 \\ -27 & 9 & -6 \end{bmatrix} \xrightarrow{DC\left(DC(A_4)\right)} \begin{bmatrix} -92 & 16 \\ 216 & -30 \end{bmatrix},$$

hence, the determinant of A_4 is equal to

$$|A_4| = \frac{\begin{vmatrix} -92 & 16 \\ 216 & -30 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}} = \frac{\begin{vmatrix} 21 & -59 \\ -29 & -13 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}} = \frac{-404}{2} = -202.$$

We shall now prove the following theorem:

Theorem 2. *Given a* 4×4 *matrix*

$$A_4 = \begin{bmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & b_3 & b_4 & 0 \\ a_3 & 0 & 0 & a_4 \end{bmatrix}_{4\times 4},$$

where b_1, b_2, b_3, b_4 *are nonzero numbers and* $\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix} \neq 0$. *Then*

$$|A_4| = (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3).$$

Proof. We have

$$DC(DC(A_4)) = \begin{bmatrix} (a_1b_1)(b_1b_4 - b_2b_3) & (a_2b_2)(b_1b_4 - b_2b_3) \\ (a_3b_3)(b_1b_4 - b_2b_3) & (a_4b_4)(b_1b_4 - b_2b_3) \end{bmatrix}_{2\times 2},$$

Using Theorem 1, we have

$$|A_4| = \frac{|DC(DC(A_4))|}{\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}} = \frac{\begin{vmatrix} (a_1b_1)(b_1b_4 - b_2b_3) & (a_2b_2)(b_1b_4 - b_2b_3) \\ (a_3b_3)(b_1b_4 - b_2b_3) & (a_4b_4)(b_1b_4 - b_2b_3) \end{vmatrix}}{\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \frac{(a_1b_1)(b_1b_4-b_2b_3)}{b_1} & \frac{(a_2b_2)(b_1b_4-b_2b_3)}{b_2} \\ \frac{(a_3b_3)(b_1b_4-b_2b_3)}{b_3} & \frac{(a_4b_4)(b_1b_4-b_2b_3)}{b_4} \end{vmatrix}}{\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}} = \frac{\begin{vmatrix} a_1(b_1b_4 - b_2b_3) & a_2(b_1b_4 - b_2b_3) \\ a_3(b_1b_4 - b_2b_3) & a_4(b_1b_4 - b_2b_3) \end{vmatrix}}{\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix}}$$

$$= \frac{a_1a_4(b_1b_4-b_2b_3)^2 - a_2a_3(b_1b_4-b_2b_3)^2}{b_1b_4-b_2b_3} = (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3).$$

Theorem 3. *We have*

$$\begin{vmatrix} a_1 & x & y & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & b_3 & b_4 & 0 \\ a_3 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 & a_2 \\ x & b_1 & b_2 & 0 \\ y & b_3 & b_4 & 0 \\ a_3 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & b_3 & b_4 & 0 \\ a_3 & x & y & a_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & x \\ 0 & b_3 & b_4 & y \\ a_3 & 0 & 0 & a_4 \end{vmatrix} = (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3).$$

where b_1, b_2, b_3, b_4 *are nonzero numbers and* $\begin{vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{vmatrix} \neq 0$ *and* $x, y \in \mathbb{R}$.

Proof. The proof is similar to the proof of Theorem 2.

REFERENCES

[1] C. L. Dodgson, Condensation of Determinants, Being a New and Brief Method for Computing their Arithmetic Values, Proc. Roy. Soc. Ser. A 15, (1866), 150–155.

[2] R. Farhadian, Duplex Fraction Method to Compute the Determinant of a 4 × 4 Matrix, 2016, available at http://vixra.org/abs/1610.0353.

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