International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 5, Issue 12, 2017, PP 8-27
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
DOI: http://dx.doi.org/10.20431/2347-3142.0512002
www.arcjournals.org

Transient Solution of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ Retrial queueing system with Priority services, Modified Bernoulli Vacation, Bernoulli Feedback, Negative arrival, Breakdown, Delaying repair, Setup time and Balking

G. Ayyappan [1], J. Udayageetha [2]

Department of Mathematics, Pondicherry Engineering College, Puducherry, India
Department of Mathematics, Perunthalaivar Kamarajar Arts College, *Puducherry, India*

***Corresponding Author:** *J. Udayageetha, Department of Mathematics, Perunthalaivar Kamarajar Arts College, Puducherry, India*

Abstract: *This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general Retrial G-queueing system with priority services. Two customers from different classes arrive at the system in two independent compound Poisson processes. Under the pre-emptive priority rule, the server provides a general service to the arriving customers subject to breakdown and Modified server vacation with general (arbitrary) vacation periods. If the system is not empty during a normal service period, the arrival of a negative customer removes the positive customer being in service from the system and causes server breakdown. The repair of the failed server starts after some time known as delay time. After completing the delay time the repair process will start. After completing vacation and repair process the server needs some time to set up the system. The delay time, repair time and set up time follows general distribution. The priority customers who find the server busy are queued and then are served in accordance with FCFS discipline. The arriving low-priority customers on finding the server busy cannot be queued and leave the service area and join the orbit as a retrial customer. They try their luck for service from the orbit. Moreover if the high priority customer is not satisfied with the service given they may join the tail of the queue as a feedback customer with probability p or leave the system with probability q. We consider balking to occur at the low priority customers during server's busy or idle period. The time dependent solutions are derived by using supplementary variable technique and numerical examples are presented.*

Keywords: *Preemptive Priority Queueing systems, Batch Arrival, Retrial queue, Orbit, Modified Server Vacations, Negative arrival, Setup time, Balking.*

Msc 2010 No.: $60K25, 68M30, 90B22$.

1. INTRODUCTION

The retrial queueing systems are characterized as a customer, arriving when all servers are busy, leaves the system, but after some time makes a demand to the service facility again. These models play a vital role in computer and telecommunication networks. For example, in a telephone system, a customer might receive a busy signal due to a lack of capacity. Such a customer is not allowed to queue, but will try their luck again after some random time. Between trials, the blocked customers join a pool of unsatisfied customers called 'orbit'.

Queueing models with negative customers, otherwise known as G-queues, were first introduced by Gelenbe(1989). In simple words, the arrival of a negative customer has the effect of removing a positive (ordinary) customer from the system. The characteristics of negative arrivals are, (i) arrival of a negative customer eliminates all the customers in the system (catastrophe), (ii) arrival of a negative customer that removes the customer in service, (iii) arrival of a negative customer that deletes the customer at the end of queue. Negative arrivals have been interpreted as virus, orders or inhibitor signals. In this paper we consider the negative arrival of type (ii).

Retrial queues can be applied in telecommunication networks, switching systems and computer networks, and there has been an increasing interest in the analysis of retrial systems in recent years.

Retrial queues are described by the feature that the arriving customers who find the server busy join the retrial orbit to try again for their requests. A review of retrial queue literature can be found in Artalejo et.al (1994) and Yang et.al(1994)and the later one derived the stochastic decomposition of retrial queues.

Classical and constant retrial policies in M/G/1 queueing model with active breakdowns of the server were discussed by Atencia et al. (2005). Bulk arrival retrial queue with unreliable server and priority subscribers was given by Jain et. al (2008). Kirupa et.al (2014) deal with the batch arrival retrial G-queue and an unreliable server with delayed repair. Madhu Jain et.al(2014) analyse the batch arrival priority queueing model with second Optional service and server breakdown. Recently, Ayyappan et.al(2016) discuss the transient solution of non-preemptive priority retrial queueing system with negative arrival, two kinds of vacations, breakdown, delayed repair, balking, reneging and feedback.

For priority queues, one must distinguish pre-emptive service from non-pre-emptive service. A service discipline is said to be non-pre-emptive if, once the service to a customer is started, it is not disrupted until the whole service requirement is completed. Thus, only at the end of each service time one of the waiting customers of the highest priority class is selected for the next service. Among the customers with the same class, the tie is broken by a usual rule for low priority queues, such as first-come-first-served (FCFS), last-come-first-served (LCFS), and random order for service (ROS). In a pre-emptive service queue, the service is given to one of the customers of the highest priority class present in the system at all times. The service of the low priority customer is immediately pre-empted by the arrival of a customer of higher priority class.

Queues with impatient customers have attracted the attention of many researchers and we see significant contribution by numerous researchers in this area. One of the important works on balking and reneging was studied by Haight [1957] and Barrer [1957], who were the first to introduce reneging in which they studied deterministic reneging with single server Markovian arrival and service rates. Subha Rao, S. (1967) described the Queueing with balking and reneging in M/G/1 systems.

In this paper we propose a retrial queueing model accepting two types of positive customers and negative arrivals, with the additional characteristics of server's abnormal breakdown, delaying repair, Bernoulli feedback, balking and modified Bernoulli vacations. When a negative customer arrives it not only eliminates the customer in service but also causes server's failure, this process is called "abnormal breakdown". Models with this behaviour of a negative arrival can be used to analyse computer networks with virus affection and breakdowns due to a reset order.

This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general Retrial G-queueing system with priority services. Two different sorts of customers arrive at the system in two independent compound Poisson processes. Under the pre-emptive priority rule, the server provides general service to the arriving customers is subject to breakdown and modified server vacation with general vacation time. Arriving high priority customer who find the server busy with high(low) priority customer are queued(pre-empts the low priority service) and then are served in accordance with FCFS discipline. The arriving low-priority customer on finding the server busy cannot be queued and leave the service area and join the orbit as a retrial customer. They try their luck for service from the orbit. If the system is not empty during a normal service time, the arrival of a negative customer removes the positive customer being in service from the system and causes the server breakdown. The repair of the failed server starts after some time known as delay time. After completing the delay time the repair process will start. We assume that the server may take a vacation with probability θ, but no vacation is allowed if there is even a single high priority customer present in the system. After completing the service to each low priority customer the server can take a vacation with probability θ or continue the next service with probability $1-\theta$, if any. After completing vacation and repair process the server need some time to set up the system. Also, if the high priority customer is not satisfied with the service given they may join the tail of the queue as a feedback customer. We consider balking to occur at the low priority customers during server's busy or idle period.

The summary of the paper is as follows. Section 1 is an introduction to priority retrial queueing discipline and comprises literature review. Section 2 deals with model description, notations used, mathematical formulation and governing equations of the model. Section 3 elucidates the steady state solutions of the system. Section 4 delineates the stochastic decomposition law. Section 5 shows the performance measures of the model. In Section 6 the numerical results and graphs are computed following which the conclusion is given.

Following assumptions have been made about the queueing system :

2. Model Description

We consider an unreliable single server retrial queueing model with two types of customers namely, high priority and low-priority customers. The basic operation of the model can be described as:

Arrival and retrial process: Two class of customers namely high priority and low-priority customers arrive at the system in two independent compound Poisson processes with arrival rate λ_1 and λ_2 respectively. Let $\lambda_1 c_{1i}\, dt$ and $\lambda_2 c_{2i}\, dt\, (i = 1,2,3,...)$ be the first order probability that a batch of 'i' customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \le c_{1i} \le 1, \sum_{i=1}^{\infty} c_{1i} = 1$, $0 \le c_{2i} \le 1, \sum_{i=1}^{\infty} c_{2i} = 1$ and $\lambda_1 > 0, \lambda_2 > 0$. The arrival of negative customer follows Poisson distribution with arrival rate $\bar{\lambda}$. The high priority customer who find the server busy is queued and then is served. The arriving low-priority customer on finding the server busy cannot be queued and leave the service area and join the orbit as a retrial customer. The low-priority customers retry for their service from the orbit and the retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$. Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x+dx]$ given that the elapsed retrial time is x, so that

$$\eta(x) = \frac{i(x)}{1 - I(x)},$$

and,

$$i(s) = \eta(s) e^{-\int_0^s \eta(x)dx}.$$

Service process: If a high priority customer arrives in batch and finds a low priority customer in service, they pre-empt the low priority customer who is undergoing service. If a high priority customer is not satisfied with the service given, he may join the tail of the queue as a feedback customer with probability p or permanently leaves the system with probability $q\,(=1-p)$, thus the service of the pre-empted low priority customer begins only after the completion of service of all high priority customers present in the system. The service times for the high priority and low priority customers are generally(arbitrary) distributed with distribution functions $B_i(s)$ and the density functions $b_i(s)$, i = 1,2 respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of the high priority and low priority customers service during the interval $(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},$$

and,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}.$$

Breakdown state: Existence of negative arrival during busy period will lead the system to break down and removes the customer who is currently in service. This type of breakdown is referred as "abnormal breakdown".

Delaying Repair and Repair Process: The broken down server does not send for repair immediately. There is a delay time to start the repair process. After the completion of delay time the repair process will start so as to regain its functionality. Immediately after returning from the repair, the server starts to serve high priority/low priority customers. The delay time to repair and the repair process are generally distributed with distribution functions $D(s)$ and $R(s)$ and density functions $d(s)$ and $r(s)$ respectively. Let $\phi(x)dx$ and $\gamma(x)dx$ be the conditional probability of a completion of delay time and repair time during the interval $(x, x+dx]$ given that the elapsed time is x, so that

$$\phi(x) = \frac{d(x)}{1-D(x)},$$

$$d(s) = \phi(s)e^{-\int_0^s \phi(x)dx}.$$

and

$$\gamma(x) = \frac{r(x)}{1-R(x)},$$

$$r(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}.$$

Modified Bernoulli Vacation: If all the high priority customers are served then the server can take a modified Bernoulli vacation with probability θ or continue the service to the low priority customers with probability $1-\theta$. Also, with every service completion to the low priority customer the server may take a vacation with probability θ or continue the service to the next customer with probability $1-\theta$. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$. Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{v(x)}{1-V(x)},$$

and,

$$v(s) = \beta(s)e^{-\int_0^s \beta(x)dx}.$$

Setup time: After returning from vacation or repair process the server takes some time to set up the system to increase the efficiency of the service. Set up time is generally distributed with distribution function $M(s)$ and the density function $m(s)$. Let $\delta(x)dx$ be the conditional probability of a completion of setup time during the interval $(x, x+dx]$ given that the elapsed setup time is x, so that

$$\delta(x) = \frac{m(x)}{1-M(x)},$$

and,

$$m(s) = \delta(s)e^{-\int_0^s \delta(x)dx}.$$

Balking: If the server is busy or unavailable in the system, an arriving low-priority customers either join the orbit with probability b or balks (do not join the orbit) with probability (1 - b).

Idle State: After doing the setup if there is any customer waiting in the system the server starts doing the service. Otherwise the server is simply present in the system for the customers to arrive.

2.1. Definitions And Notations

1. $P_{m,n}^{(1)}(x,t) =$ Probability that at time t, the server is actively providing service to the high priority customer and there are $m\,(\geq 0)$ high priority customers in the queue and $n\,(\geq 0)$ low priority customers in the orbit excluding the one high priority customer in service with elapsed service time

for this customer is x. $P^{(1)}_{m,n}(t) = \int_0^\infty P^{(1)}_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit excluding one high priority customer in service without regard to the elapsed service time x of a high priority customer.

2. $V_{m,n}(x,t) =$ Probability that at time t, the server is on vacation with elapsed vacation time x and there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit. $V_{m,n}(t) = \int_0^\infty V_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit, without regard to the elapsed vacation time x.

3. $P^{(2)}_{0,n}(x,t) =$ Probability that at time t, the server is actively providing service and there are $n(\geq 0)$ low priority customers in the orbit excluding the one low priority customer in service with elapsed service time for this customer is x. $P^{(2)}_{0,n}(t) = \int_0^\infty P^{(2)}_{0,n}(x,t)dx$ denotes the probability that at time t there are $n(\geq 0)$ low priority customers in the orbit excluding one low priority customer in service without regard to the elapsed service time x.

4. $D_{m,n}(x,t) =$ Probability that at time t, the server is on breakdown with elapsed delay time to start repair x and there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit. $D_{m,n}(t) = \int_0^\infty D_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit, without regard to the elapsed delay time to repair, x.

5. $R_{m,n}(x,t) =$ Probability that at time t, the server is undergone repair process with elapsed repair time x and there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit. $R_{m,n}(t) = \int_0^\infty R_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit, without regard to the elapsed repair time x.

6. $M_{m,n}(x,t) =$ Probability that at time t, the server is set up the system with elapsed setup time x and there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit. $M_{m,n}(t) = \int_0^\infty M_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority customers in the queue and $n(\geq 0)$ low priority customers in the orbit, without regard to the elapsed setup time x.

7. $I_{0,n}$ (t) = Probability that at time t, there are no high priority customers in the queue and $n(\geq 0)$ low-priority customers in the orbit and the server is idle but available in the system.

2.2. Equations Governing The System

The Kolmogorov forward equations which governs the model:

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -(\lambda_1 + \lambda_2 + \bar{\lambda} + \mu_1(x))P^{(1)}_{m,n}(x,t) + \lambda_1 \sum_{i=1}^{m} c_{1i} P^{(1)}_{m-i,n}(x,t)$$

$$+ \lambda_2 b \sum_{i=1}^{n} c_{2i} P^{(1)}_{m,n-i}(x,t) + \lambda_2 (1-b) P^{(1)}_{m,n}(x,t); m \geq 1, n \geq 1, \tag{1}$$

$$\frac{\partial}{\partial t}P_{m,0}^{(1)}(x,t)+\frac{\partial}{\partial x}P_{m,0}^{(1)}(x,t)=-(\lambda_1+\lambda_2+\bar{\lambda}+\mu_1(x))P_{m,0}^{(1)}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}P_{m-i,0}^{(1)}(x,t)$$
$$+\lambda_2(1-b)P_{m,0}^{(1)}(x,t); m\geq 1, \tag{2}$$

$$\frac{\partial}{\partial t}P_{0,n}^{(1)}(x,t)+\frac{\partial}{\partial x}P_{0,n}^{(1)}(x,t)=-(\lambda_1+\lambda_2+\bar{\lambda}+\mu_1(x))P_{0,n}^{(1)}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}P_{0,n-i}^{(1)}(x,t)$$
$$+\lambda_2(1-b)P_{0,n}^{(1)}(x,t); n\geq 1, \tag{3}$$

$$\frac{\partial}{\partial t}P_{0,0}^{(1)}(x,t)+\frac{\partial}{\partial x}P_{0,0}^{(1)}(x,t)=-(\lambda_1+\lambda_2+\bar{\lambda}+\mu_1(x))P_{0,0}^{(1)}(x,t)+\lambda_2(1-b)P_{0,0}^{(1)}(x,t), \tag{4}$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t)+\frac{\partial}{\partial x}V_{m,n}(x,t)=-(\lambda_1+\lambda_2+\beta(x))V_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}V_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}c_{2i}V_{m,n-i}(x,t)+\lambda_2(1-b)V_{m,n}(x,t); m\geq 1, n\geq 1, \tag{5}$$

$$\frac{\partial}{\partial t}V_{m,0}(x,t)+\frac{\partial}{\partial x}V_{m,0}(x,t)=-(\lambda_1+\lambda_2+\beta(x))V_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}V_{m-i,0}(x,t)$$
$$+\lambda_2(1-b)V_{m,0}(x,t);\ m\geq 1, \tag{6}$$

$$\frac{\partial}{\partial t}V_{0,n}(x,t)+\frac{\partial}{\partial x}V_{0,n}(x,t)=-(\lambda_1+\lambda_2+\beta(x))V_{0,n}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}V_{0,n-i}(x,t)$$
$$+\lambda_2(1-b)V_{0,n}(x,t);\ n\geq 1, \tag{7}$$

$$\frac{\partial}{\partial t}V_{0,0}(x,t)+\frac{\partial}{\partial x}V_{0,0}(x,t)=-(\lambda_1+\lambda_2+\beta(x))V_{0,0}(x,t)+\lambda_2(1-b)V_{0,0}(x,t), \tag{8}$$

$$\frac{\partial}{\partial t}P_{0,n}^{(2)}(x,t)+\frac{\partial}{\partial x}P_{0,n}^{(2)}(x,t)=-(\lambda_1+\lambda_2+\bar{\lambda}+\mu_2(x))P_{0,n}^{(2)}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}P_{0,n-i}^{(2)}(x,t)$$
$$+\lambda_2(1-b)P_{0,n}^{(2)}(x,t); n\geq 1, \tag{9}$$

$$\frac{\partial}{\partial t}P_{0,0}^{(2)}(x,t)+\frac{\partial}{\partial x}P_{0,0}^{(2)}(x,t)=-(\lambda_1+\lambda_2+\bar{\lambda}+\mu_2(x))P_{0,0}^{(2)}(x,t), \tag{10}$$

$$\frac{\partial}{\partial t}D_{m,n}(x,t)+\frac{\partial}{\partial x}D_{m,n}(x,t)=-(\lambda_1+\lambda_2+\phi(x))D_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}D_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}c_{2i}D_{m,n-i}(x,t)+\lambda_2(1-b)D_{m,n}(x,t); m\geq 1, n\geq 1, \tag{11}$$

$$\frac{\partial}{\partial t}D_{m,0}(x,t)+\frac{\partial}{\partial x}D_{m,0}(x,t)=-(\lambda_1+\lambda_2+\phi(x))D_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}D_{m-i,0}(x,t)$$
$$+\lambda_2(1-b)D_{m,0}(x,t);\ m\geq 1, \tag{12}$$

$$\frac{\partial}{\partial t}D_{0,n}(x,t)+\frac{\partial}{\partial x}D_{0,n}(x,t)=-(\lambda_1+\lambda_2+\phi(x))D_{0,n}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}D_{0,n-i}(x,t)$$
$$+\lambda_2(1-b)D_{0,n}(x,t);\ n\geq 1, \tag{13}$$

$$\frac{\partial}{\partial t}D_{0,0}(x,t)+\frac{\partial}{\partial x}D_{0,0}(x,t)=-(\lambda_1+\lambda_2+\phi(x))D_{0,0}(x,t)+\lambda_2(1-b)D_{0,0}(x,t), \tag{14}$$

$$\frac{\partial}{\partial t}R_{m,n}(x,t)+\frac{\partial}{\partial x}R_{m,n}(x,t)=-(\lambda_1+\lambda_2+\gamma(x))R_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}R_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}c_{2i}R_{m,n-i}(x,t)+\lambda_2(1-b)R_{m,n}(x,t);m\ge 1,n\ge 1, \tag{15}$$

$$\frac{\partial}{\partial t}R_{m,0}(x,t)+\frac{\partial}{\partial x}R_{m,0}(x,t)=-(\lambda_1+\lambda_2+\gamma(x))R_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}R_{m-i,0}(x,t)$$
$$+\lambda_2(1-b)R_{m,0}(x,t);\ m\ge 1, \tag{16}$$

$$\frac{\partial}{\partial t}R_{0,n}(x,t)+\frac{\partial}{\partial x}R_{0,n}(x,t)=-(\lambda_1+\lambda_2+\gamma(x))R_{0,n}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}R_{0,n-i}(x,t)$$
$$+\lambda_2(1-b)R_{0,n}(x,t);\ n\ge 1, \tag{17}$$

$$\frac{\partial}{\partial t}R_{0,0}(x,t)+\frac{\partial}{\partial x}R_{0,0}(x,t)=-(\lambda_1+\lambda_2+\gamma(x))R_{0,0}(x,t)+\lambda_2(1-b)R_{0,0}(x,t), \tag{18}$$

$$\frac{\partial}{\partial t}M_{m,n}(x,t)+\frac{\partial}{\partial x}M_{m,n}(x,t)=-(\lambda_1+\lambda_2+\delta(x))M_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}M_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{i=1}^{n}c_{2i}M_{m,n-i}(x,t)+\lambda_2(1-b)M_{m,n}(x,t);m\ge 1,n\ge 1, \tag{19}$$

$$\frac{\partial}{\partial t}M_{m,0}(x,t)+\frac{\partial}{\partial x}M_{m,0}(x,t)=-(\lambda_1+\lambda_2+\delta(x))M_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1i}M_{m-i,0}(x,t)$$
$$+\lambda_2(1-b)M_{m,0}(x,t);\ m\ge 1, \tag{20}$$

$$\frac{\partial}{\partial t}M_{0,n}(x,t)+\frac{\partial}{\partial x}M_{0,n}(x,t)=-(\lambda_1+\lambda_2+\delta(x))M_{0,n}(x,t)+\lambda_2 b\sum_{i=1}^{n}c_{2i}M_{0,n-i}(x,t)$$
$$+\lambda_2(1-b)M_{0,n}(x,t);\ n\ge 1, \tag{21}$$

$$\frac{\partial}{\partial t}M_{0,0}(x,t)+\frac{\partial}{\partial x}M_{0,0}(x,t)=-(\lambda_1+\lambda_2+\delta(x))M_{0,0}(x,t)+\lambda_2(1-b)M_{0,0}(x,t), \tag{22}$$

$$\frac{d}{dt}I_{0,0}(t)=-(\lambda_1+\lambda_2)I_{0,0}(t)+(1-\theta)q\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx+\int_0^\infty M_{0,0}(x,t)\delta(x)dx$$
$$+(1-\theta)\int_0^\infty P_{0,0}^{(2)}(x,t)\mu_2(x)dx. \tag{23}$$

$$\frac{\partial}{\partial t}I_{0,n}(x,t)+\frac{\partial}{\partial x}I_{0,n}(x,t)=-(\lambda_1+\lambda_2+\eta(x))I_{0,n}(x,t);n\ge 1. \tag{24}$$

The above set of equations are to be solved under the following boundary conditions at x=0.

$$I_{0,n}(0,t)=(1-\theta)q\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx+(1-\theta)\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty M_{0,n}(x,t)\delta(x)dx;\ n\ge 1, \tag{25}$$

$$P_{m,n}^{(1)}(0,t) = \lambda_1 c_{1m+1} I_{0,n}(t) + q\int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx + \lambda_1 c_{1m+1}\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \int_0^\infty M_{m+1,n}(x,t)\delta(x)dx;\ m\ge 1,\ n\ge 1, \tag{26}$$

$$P_{m,0}^{(1)}(0,t) = \lambda_1 c_{1m+1} I_{0,0}(t) + q\int_0^\infty P_{m+1,0}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{m,0}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty M_{m+1,0}(x,t)\delta(x)dx;\ m\ge 1, \tag{27}$$

$$P_{0,n}^{(1)}(0,t) = \lambda_1 c_{11} I_{0,n}(t) + q\int_0^\infty P_{1,n}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty M_{1,n}(x,t)\delta(x)dx + \lambda_1 c_{11}\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx;\ n\ge 1, \tag{28}$$

$$P_{0,0}^{(1)}(0,t) = \lambda_1 c_{11} I_{0,0}(t) + q\int_0^\infty P_{1,0}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty M_{1,0}(x,t)\delta(x)dx, \tag{29}$$

$$V_{0,n}(0,t) = \theta q\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \theta\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx;\ n\ge 0, \tag{30}$$

$$P_{0,0}^{(2)}(0,t) = \int_0^\infty I_{0,1}(x,t)\eta(x)dx + \lambda_2 b c_{21} I_{0,0}(t), \tag{31}$$

$$P_{0,n}^{(2)}(0,t) = \int_0^\infty I_{0,n+1}(x,t)\eta(x)dx + \lambda_2 b c_{2n+1} I_{0,0}(t) + \lambda_2 b\sum_{i=1}^{n} c_{2i}\int_0^\infty I_{0,n+1-i}(x,t)dx;\ n\ge 1, \tag{32}$$

$$D_{m,n}(0,t) = \bar{\lambda}\int_0^\infty P_{m,n}^{(1)}(x,t)dx;\ m\ge 1,\ n\ge 0, \tag{33}$$

$$D_{0,n}(0,t) = \bar{\lambda}\int_0^\infty P_{0,n}^{(1)}(x,t)dx + \bar{\lambda}\int_0^\infty P_{0,n}^{(2)}(x,t)dx;\ n\ge 1, \tag{34}$$

$$D_{0,0}(0,t) = \bar{\lambda}\int_0^\infty P_{0,0}^{(1)}(x,t)dx + \bar{\lambda}\int_0^\infty P_{0,0}^{(2)}(x,t)dx;\ n\ge 1, \tag{35}$$

$$R_{m,n}(0,t) = \int_0^\infty D_{m,n}(x,t)\phi(x)dx;\ m\ge 1,\ n\ge 0, \tag{36}$$

$$R_{m,0}(0,t) = \int_0^\infty D_{m,0}(x,t)\phi(x)dx;\ m\ge 1, \tag{37}$$

$$R_{0,n}(0,t) = \int_0^\infty D_{0,n}(x,t)\phi(x)dx;\ n\ge 1, \tag{38}$$

$$R_{0,0}(0,t) = \int_0^\infty D_{0,0}(x,t)\phi(x)dx, \tag{39}$$

$$M_{m,n}(0,t) = \int_0^\infty V_{m,n}(x,t)\beta(x)dx + \int_0^\infty R_{m,n}(x,t)\gamma(x)dx;\ m\ge 1,\ n\ge 1, \tag{40}$$

$$M_{m,0}(0,t) = \int_0^\infty V_{m,0}(x,t)\beta(x)dx + \int_0^\infty R_{m,0}(x,t)\gamma(x)dx;\ m\ge 1, \tag{41}$$

$$M_{0,n}(0,t) = \int_0^\infty V_{0,n}(x,t)\beta(x)dx + \int_0^\infty R_{0,n}(x,t)\gamma(x)dx;\ n\ge 1, \tag{42}$$

$$M_{0,0}(0,t) = \int_0^\infty V_{0,0}(x,t)\beta(x)dx + \int_0^\infty R_{0,0}(x,t)\gamma(x)dx. \tag{43}$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$P_{m,n}^{(1)}(0) = P_{m,0}^{(1)}(0) = P_{0,n}^{(1)}(0) = P_{0,0}^{(1)}(0) = P_{0,n}^{(2)}(0) = P_{0,0}^{(2)}(0) = V_{m,n}(0) = V_{m,0}(0) = V_{0,n}(0)$$

$$= V_{0,0}(0) = D_{m,n}(0) = D_{m,0}(0) = D_{0,n}(0) = D_{0,0}(0) = R_{m,n}(0) = R_{m,0}(0) = R_{0,n}(0)$$

$$= R_{0,0}(0) = M_{m,n}(0) = M_{m,0}(0) = M_{0,n}(0) = M_{0,0}(0) = I_{0,n}(0) = 0 \text{ and } I_{(0,0)}(0) = 1. \quad (44)$$

The Probability Generating Function(PGF) of this model:

$$I(x, z_2, t) = \sum_{n=1}^{\infty} z_2^n I_{0,n}(x,t), \quad A(x, z_1, z_2, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(x,t) \quad (45)$$

where $A = P^{(i)}, V, D, R, M$, which are convergent inside the circle given by $|z_1| \le 1, |z_2| \le 1$.

The Laplace transform of a function $f(t)$ is given by,

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt.$$

By taking Laplace transforms from (1) to (24) and solve the equations, we get,

$$\bar{I}_0(x,s,z_2) = \bar{I}_0(0,s,z_2)[1-\bar{I}(f(a,s))]e^{-f(a,s)x}, \quad (46)$$

$$\bar{P}^{(1)}(x,s,z_1,z_2) = \bar{P}^{(1)}(0,s,z_1,z_2)[1-\bar{B}_1(f_1(s,z_1,z_2))]e^{-f_1(s,z_1,z_2)x}, \quad (47)$$

$$\bar{P}^{(2)}(x,s,z_2) = \bar{P}^{(2)}(0,s,z_2)[1-\bar{B}_2(f_1(s,z_2))]e^{-f_1(s,z_2)x}, \quad (48)$$

$$\bar{V}(x,s,z_1,z_2) = \bar{V}_0(0,s,z_2)[1-\bar{V}(f_2(s,z_1,z_2))]e^{-f_2(s,z_1,z_2)x}, \quad (49)$$

$$\bar{D}(x,s,z_1,z_2) = \bar{D}(0,s,z_1,z_2)[1-\bar{D}(f_2(s,z_1,z_2))]e^{-f_2(s,z_1,z_2)x}, \quad (50)$$

$$\bar{R}(x,s,z_1,z_2) = \bar{R}(0,s,z_1,z_2)[1-\bar{R}(f_2(s,z_1,z_2))]e^{-f_2(s,z_1,z_2)x}, \quad (51)$$

$$\bar{M}(x,s,z_1,z_2) = \bar{M}(0,s,z_1,z_2)[1-\bar{M}(f_2(s,z_1,z_2))]e^{-f_2(s,z_1,z_2)x}. \quad (52)$$

where,

$$f(a,s) = s + \lambda_1 + \lambda_2$$
$$f_1(s,z_1,z_2) = s + \lambda_1[1-C_1(z_1)] + \lambda_2 b[1-C_2(z_2)] + \bar{\lambda},$$
$$f_1(s,z_2) = s + \lambda_1 + \lambda_2 b[1-C_2(z_2)] + \bar{\lambda},$$
$$f_2(s,z_1,z_2) = s + \lambda_1[1-C_1(z_1)] + \lambda_2 b[1-C_2(z_2)],$$
$$f_2(s,z_2) = s + \lambda_1 + \lambda_2 b[1-C_2(z_2)].$$

Similarly for the boundary conditions we can get,

$$z_1\bar{P}^{(1)}(0,s,z_1,z_2) = \lambda_1 C_1(z_1)\bar{I}_0(x,s,z_2) + (q+pz_1)\int_0^{\infty}\bar{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx$$
$$+ \lambda_1 C_1(z_1) z_2 \int_0^{\infty}\bar{P}_0^{(2)}(x,s,z_2)dx - q\int_0^{\infty}\bar{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx$$
$$+ \int_0^{\infty}\bar{M}(x,s,z_1,z_2)\delta(x)dx - \int_0^{\infty}\bar{M}_0(x,s,z_2)\delta(x)dx. \quad (53)$$

$$\bar{I}_0(0,s,z_2) = 1 - (s+\lambda_1+\lambda_2)\bar{I}_{0,0}(s) + (1-\theta)q\int_0^{\infty}\bar{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx$$
$$+ (1-\theta)\int_0^{\infty}\bar{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx + \int_0^{\infty}\bar{M}_0(x,s,z_2)\delta(x)dx. \quad (54)$$

$$z_2\bar{P}_0^{(2)}(0,s,z_2) = \bar{I}_0(0,s,z_2)(\bar{I}(f(a,s)) + \lambda_2 b C_2(z_2)[\frac{1-\bar{I}(f(a,s))}{f(a,s)}]) + \lambda_2 b C_2(z_2)\bar{I}_{0,0}(s) \quad (55)$$

By applying Rouche's theorem on (53), we get,

$$\bar{P}_0^{(1)}(0,s,z_2) = \frac{\lambda_1 C_1(g(z_2))\bar{I}_0(0,s,z_2)[\frac{1-\bar{I}(f(a,s))}{f(a,s)}] + \bar{P}_0^{(2)}(0,s,z_2)\bar{G}_4(s,z_1,z_2)}{\bar{G}_1(s,z_1,z_2)} \quad (56)$$

By substituting (56) in required equations we get,

$$\overline{P}_0^{(1)}(0,s,z_1,z_2) = \frac{\left[\overline{I}_0[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\{\lambda_1 C_1(z_1)\overline{G}_1(s,z_1,z_2) - \lambda_1 C_1(g(z_2))\overline{G}_2(s,z_1,z_2)\} + \overline{P}_0^{(2)}(0,s,z_2)\overline{G}_5(s,z_1,z_2)\right]}{\left[\overline{G}_1(s,z_1,z_2)\{z_1-(q+pz_1)\overline{B}_1(f_1(s,z_1,z_2)) - \overline{\lambda}[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}]\overline{D}(f_2(s,z_1,z_2))\overline{R}(f_2(s,z_1,z_2))\overline{M}(f_2(s,z_1,z_2))\}\right]}, \quad (57)$$

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left[(1-(s+\lambda_1+\lambda_2 b[1-C_2(z_2)])\overline{I}_{(0,0)}(s))\overline{G}_1\{\overline{I}(0,s,z_2)+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}}{f(a,s)}]\} + \lambda_2 bC_2(z_2)\overline{I}_{(0,0)}(s)\{\overline{G}_1(s,z_1,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_3(s,z_1,z_2)\}\right]}{\left[z_2\{\overline{G}_1(s,z_1,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_3(s,z_1,z_2)\} - \overline{G}_5(s,z_1,z_2)\{\overline{I}(f(a,s))+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\}\right]} \quad (58)$$

$$\overline{I}_0(0,s,z_2) = \frac{\left[-z_2(1-(s+\lambda_1+\lambda_2)\overline{I}_{(0,0)}(s))\overline{G}_1(s,z_1,z_2)+\lambda_2 bC_2(z_2)\overline{I}_{(0,0)}(s)\overline{G}_5(s,z_1,z_2)\right]}{\left[z_2\{\overline{G}_1(s,z_1,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_3(s,z_1,z_2)\} - \overline{G}_5(s,z_1,z_2)\{\overline{I}(f(a,s))+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\}\right]} \quad (59)$$

$$\overline{V}(0,s,z_1,z_2) = \theta q\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(f_1(s,z_2)) + \theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(f_1(s,z_2)), \quad (60)$$

$$\overline{D}(0,s,z_1,z_2) = \overline{\lambda}\,\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(z_1,z_2)}] + \overline{\lambda}\,\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}], \quad (61)$$

$$\overline{R}(0,s,z_1,z_2) = \{\overline{\lambda}\,\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}] + \overline{\lambda}\,\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\} \times \overline{D}(f_2(s,z_1,z_2)). \quad (62)$$

$$\overline{M}(0,s,z_1,z_2) = \{\overline{\lambda}\,\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}] + \overline{\lambda}\,\overline{P}_0^{(2)}(0,s,z_2) [\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\}\overline{D}(f_2(s,z_1,z_2))\overline{R}(f_2(s,z_1,z_2)) + \theta q\overline{P}_0^{(1)}(0,s,z_2) \overline{B}_1(f_1(s,z_2))\overline{V}(f_2(s,z_1,z_2)) + \theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(f_1(s,z_2))\overline{V}(f_2(s,z_1,z_2)) \quad (63)$$

where,

$$\overline{G}_1(s,z_1,z_2) = q\overline{B}_1(f_1(s,z_2))\{1-\theta\overline{V}(f_2(s,g(z_2)))\overline{M}(f_2(s,g(z_2)))+\theta\overline{V}(f_2(s,z_2))$$

$$\overline{M}(f_2(s,z_2))\}+\overline{\lambda}[\frac{1-\overline{B}_1(f_1(s,z_2))}{f_1(s,z_2)}]\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2))\overline{M}(f_2(s,z_2)) \tag{64}$$

$$\overline{G}_2(s,z_1,z_2)=q\overline{B}_1(f_1(s,z_2))\{1-\theta\overline{V}(f_2(s,z_1,z_2))\overline{M}(f_2(s,z_1,z_2))+\theta\overline{V}(f_2(s,z_2))$$

$$\overline{M}(f_2(s,z_2))\}+\overline{\lambda}[\frac{1-\overline{B}_1(f_1(s,z_2))}{f_1(s,z_2)}]\overline{D}(f_2(s,s_1,z_2))\overline{R}(f_2(s,z_1,z_2))\overline{M}(f_2(s,z_1,z_2)) \tag{65}$$

$$\overline{G}_3(s,z_1,z_2)=\{q\overline{B}_1(f_1(s,z_2))[1-\theta+\theta\overline{V}(f_2(s,z_2))\overline{M}(f_2(s,z_2))]+\overline{\lambda}[\frac{1-\overline{B}_1 f_1(s,z_2)}{f_1(s,z_2)}]$$
$$\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2))\overline{M}(f_2(s,z_2))\} \tag{66}$$

$$\overline{G}_4(s,z_1,z_2)=\{\lambda_1 C_1(g(z_2))z_2[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}][\overline{D}(f_2(s,g(z_2)))$$
$$\overline{R}(f_2(s,g(z_2)))\overline{M}(f_2(s,g(z_2)))-\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2))\overline{M}(f_2(s,z_2))]$$
$$+\theta\overline{B}_2(f_1(s,z_2))[\overline{V}(f_2(s,g(z_2)))\overline{M}(f_2(s,g(z_2)))-\overline{V}(f_2(s,z_2))\overline{M}(f_2(s,z_2))]\} \tag{67}$$

$$\overline{G}_5(s,z_1,z_2)=\overline{G}_3(s,z_1,z_2)\overline{G}_4(s,z_1,z_2)+\overline{G}_1(s,z_1,z_2)\{[1-\theta+\theta\overline{V}(f_2(s,z_2))$$
$$\overline{M}(f_2(s,z_2))]+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2))\overline{M}(f_2(s,z_2))\} \tag{68}$$

$$\overline{G}_6(s,z_1,z_2)=\frac{1-\overline{B}_1(f_1(s,z_2))}{f_1(s,z_2)} \tag{69}$$

$$\overline{G}_7(s,z_1,z_2)=\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}$$

2.3 Theorem:

The inequality $P^{(1)}(1,1)+P^{(2)}(0,1)=\rho<1$, is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state, queue size and orbit size distributions are given by,

$$\overline{I}_{(0)}(s,z_2)=\overline{I}_0(0,s,z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}], \tag{70}$$

$$\overline{P}^{(1)}(s,z_1,z_2)=\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}], \tag{71}$$

$$\overline{V}(s,z_1,z_2)=\{\theta q\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(f_1(s,z_2))+\theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(f_1(s,z_2))\}$$
$$\times[\frac{1-\overline{V}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}], \tag{72}$$

$$\overline{P}_0^{(2)}(s,z_2)=\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}], \tag{73}$$

$$\bar{D}(s,z_1,z_2)=\{\bar{\lambda}\,\bar{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\bar{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}]+\bar{\lambda}\,\bar{P}_0^{(2)}(0,s,z_2)[\frac{1-\bar{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\}\times[\frac{1-\bar{D}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}], \quad (74)$$

$$\bar{R}(s,z_1,z_2)=\{\bar{\lambda}\,\bar{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\bar{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}]+\bar{\lambda}\,\bar{P}_0^{(2)}(0,s,z_2)[\frac{1-\bar{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\}\times\bar{D}(f_2(s,z_1,z_2))[\frac{1-\bar{R}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}]. \quad (75)$$

$$\bar{M}(s,z_1,z_2)=\{\{\bar{\lambda}\,\bar{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\bar{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}]+\bar{\lambda}\,\bar{P}_0^{(2)}(0,s,z_2)[\frac{1-\bar{B}_2(f_1(s,z_2))}{f_1(s,z_2)}]\}\bar{D}(f_2(s,z_1,z_2))\bar{R}(f_2(s,z_1,z_2))+\theta q\bar{P}_0^{(1)}(0,s,z_2)\bar{B}_1(f_1(s,z_2))\bar{V}(f_2(s,z_1,z_2))+\theta\bar{P}_0^{(2)}(0,s,z_2)\bar{B}_2(f_1(s,z_2))\bar{V}(f_2(s,z_1,z_2))\}\times[\frac{1-\bar{M}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}] \quad (76)$$

3. Steady State Analysis: Limiting Behaviour

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s)=\lim_{t\to\infty} f(t).$$

to the above equations, we obtain the Steady- State solutions of this model.

$$I_{(0)}(z_2)=I(0,z_2)[\frac{1-\bar{I}(f(a))}{f(a)}], \quad (77)$$

$$P^{(1)}(z_1,z_2)=P^{(1)}(0,z_1,z_2)[\frac{1-\bar{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}], \quad (78)$$

$$V(z_1,z_2)=\{\theta qP_0^{(1)}(0,z_2)\bar{B}_1(f_1(z_2))+\theta P_0^{(2)}(0,z_2)\bar{B}_2(f_1(z_2))\}\times[\frac{1-\bar{V}f_2(z_1,z_2)}{f_2(z_1,z_2)}], \quad (79)$$

$$P_0^{(2)}(z_2)=P_0^{(2)}(0,z_2)[\frac{1-\bar{B}_2(f_1(z_2))}{f_1(z_2)}], \quad (80)$$

$$D(z_1,z_2)=\{\bar{\lambda}P^{(1)}(0,z_1,z_2)[\frac{1-\bar{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}]+\bar{\lambda}P_0^{(2)}(0,z_2)[\frac{1-\bar{B}_2(f_1(z_2))}{f_1(z_2)}]\}\times[\frac{1-\bar{D}(f_2(z_1,z_2))}{f_2(z_1,z_2)}], \quad (81)$$

$$R(z_1,z_2)=\{\bar{\lambda}P^{(1)}(0,z_1,z_2)[\frac{1-\bar{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}]+\bar{\lambda}P_0^{(2)}(0,z_2)[\frac{1-\bar{B}_2(f_1(z_2))}{f_1(z_2)}]\}\times\bar{D}(f_2(z_1,z_2))[\frac{1-\bar{R}(f_2(z_1,z_2))}{f_2(z_1,z_2)}]. \quad (82)$$

$$M(z_1,z_2) = \{\{\bar{\lambda}P^{(1)}(0,z_1,z_2)[\frac{1-\bar{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}] + \bar{\lambda}P_0^{(2)}(0,z_2)[\frac{1-\bar{B}_2(f_1(z_2))}{f_1(z_2)}]\} \bar{D}(f_2(z_1,z_2))\bar{R}(f_2(z_1,z_2)) + \theta q P_0^{(1)}(0,z_2)\bar{B}_1(f_1(z_2))\bar{V}(f_2(z_1,z_2)) + \theta P_0^{(2)}(0,z_2)\bar{B}_2(f_1(z_2))\bar{V}(f_2(z_1,z_2))\}[\frac{1-\bar{M}(f_2(z_1,z_2))}{f_2(z_1,z_2)}]. \tag{83}$$

where,

$$P^{(1)}(0,z_1,z_2) = \frac{\left[I_0[\frac{1-\bar{I}(f(a))}{f(a)}]\{\lambda_1 C_1(z_1)\bar{G}_1(z_1,z_2) - \lambda_1 C_1(g(z_2))\bar{G}_2(z_1,z_2)\} + P_0^{(2)}(0,z_2)\bar{G}_5(z_1,z_2) \right]}{\left[\bar{G}_1(z_1,z_2)\{z_1 - (q+pz_1)\bar{B}_1(f_1(z_1,z_2)) - \bar{\lambda}[\frac{1-\bar{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}]\bar{D}(f_2(z_1,z_2))\bar{R}(f_2(z_1,z_2))\bar{M}(f_2(z_1,z_2))\} \right]}, \tag{84}$$

$$P_0^{(2)}(0,z_2) = \frac{\left[(-(\lambda_1+\lambda_2 b[1-C_2(z_2)])I_{(0,0)})\bar{G}_1(z_1,z_2)\{I(0,z_2) + \lambda_2 b C_2(z_2)[\frac{1-\bar{I}}{f(a)}]\} + \lambda_2 b C_2(z_2) I_{(0,0)}\{\bar{G}_1(z_1,z_2) - \lambda_1 C_1(g(z_2))[\frac{1-\bar{I}(f(a))}{f(a)}]\bar{G}_3(z_1,z_2)\} \right]}{\left[z_2\{\bar{G}_1(z_1,z_2) - \lambda_1 C_1(g(z_2))[\frac{1-\bar{I}(f(a))}{f(a)}]\bar{G}_3(z_1,z_2)\} - \bar{G}_5(z_1,z_2)\{\bar{I}(f(a)) + \lambda_2 b C_2(z_2)[\frac{1-\bar{I}(f(a))}{f(a)}]\} \right]} \tag{85}$$

$$I_0(0,z_2) = \frac{\left[-z_2(-(\lambda_1+\lambda_2)I_{(0,0)})\bar{G}_1(z_1,z_2) + \lambda_2 b C_2(z_2) I_{(0,0)}\bar{G}_5(z_1,z_2) \right]}{\left[z_2\{\bar{G}_1(z_1,z_2) - \lambda_1 C_1(g(z_2))[\frac{1-\bar{I}(f(a))}{f(a)}]\bar{G}_3(z_1,z_2)\} - \bar{G}_5(z_1,z_2)\{\bar{I}(f(a)) + \lambda_2 b C_2(z_2)[\frac{1-\bar{I}(f(a))}{f(a)}]\} \right]} \tag{86}$$

In order to determine $I_{(0,0)}$, we use the normalizing condition

$$P^{(1)}(1,1) + V(1,1) + P^{(2)}(0,1) + D(1,1) + R(1,1) + M(1,1) + I_0(0,1) + I_{0,0} = 1.$$

For this, let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations from (77) to (83), we obtain,

$$P_q(z_1,z_2) = P^{(1)}(z_1,z_2) + V(z_1,z_2) + P^{(2)}(0,z_2) + D(z_1,z_2) + R(z_1,z_2) + M(z_1,z_2), \tag{87}$$

$$P_q(z_1,z_2) = \frac{N_1(z_1,z_2)}{D_1(z_1,z_2)} + \frac{N_2(z_1,z_2)}{D_1(z_1,z_2)} + \frac{N_3(z_1,z_2)}{D_2(z_1,z_2)},$$

where

$$N_1(z_1,z_2) = I_0(0,z_2)[\frac{1-\bar{I}(f(a))}{f(a)}]\{G_1(z_1,z_2)f_2(z_1,z_2) + \lambda_1 C_1(g(z_2)) \{q\theta\bar{B}_1(f_1(z_2))(1-\theta\bar{V}(f_2(g(z_2)))\bar{M}(f_2(g(z_2))))\}\},$$

$$N_2(z_1,z_2) = P_0^{(2)}(0,z_2)\{\{[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]f_2(z_1,z_2)+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]$$
$$[1-\overline{D}(f_2(z_1,z_2))\,\overline{R}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2))]+\theta\overline{B}_2(f_1(z_2))[1-\overline{V}(f_2(z_1,z_2))$$
$$\overline{M}(f_2(z_1,z_2))]\}G_1(z_1,z_2)+\{\lambda_1 C_1(g(z_2))z_2[\frac{(1-\overline{B}_2(f_1(z_2)))}{f_1(z_2)}]$$
$$+\theta\overline{B}_2(f_1(z_1,z_2))[\overline{V}(f_2(g(z_2)))\overline{M}(f_2(g(z_2)))-\overline{V}(f_2(z_2))\overline{M}(f_2(z_2))]$$
$$+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]\{\overline{D}(f_2(g(z_2)))\overline{R}(f_2(g(z_2)))\overline{M}(f_2(g(z_2)))-\overline{D}(f_2(z_2))$$
$$\overline{R}(f_2(z_2))\overline{M}(f_2(z_2))\}\}\{q\theta\overline{B}_1(f_1(z_2))(1-\overline{V}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2)))\}\}$$

$$N_3(z_1,z_2) = P^{(1)}(0,z_1,z_2)\{(1-\overline{B}_1(f_1(z_1,z_2)))f_2(z_1,z_2)+\overline{\lambda}(1-\overline{B}_1(f_1(z_1,z_2)))$$
$$(1-\overline{D}(f_2(z_1,z_2))\overline{R}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2)))\},$$

$$D_1(z_1,z_2) = G_1(z_1,z_2)\,f_2(z_1,z_2),$$

$$D_2(z_1,z_2) = f_1(z_1,z_2)\,f_2(z_1,z_2)$$

In order to obtain the Probability of idle time $I_{0,0}$, we use the normalizing condition,

$$P_q^{(1)}(1,1)+I_{0,0} = 1.$$

$$I_{0,0} = \frac{\overline{\lambda}\,G_1(1,1)}{Dr}, \tag{88}$$

$$Dr = \overline{\lambda}\,G_1(1,1) + \overline{I}_0(0,1)[\frac{1-\overline{I}(f(a))}{f(a)}]\overline{\lambda}\{G_1(1,1) + \lambda_1\theta q\overline{B}_1(\lambda_1+\overline{\lambda})[E(V)+E(M)]\}$$
$$+P_0^{(2)}(0,1)\{G_6(1,1)+\theta\overline{B}_2(\lambda_1+\overline{\lambda})[E(V)+E(M)]$$
$$+G_4(1,1)\,\theta q\overline{B}_1(\lambda_1+\overline{\lambda})(E(V)+E(M))\}$$
$$+P^{(1)}(0,1,1)\,G_1(1,1)\{(1-\overline{B}_1(\overline{\lambda}))[1+\overline{\lambda}[E(R)+E(D)+E(M)]]\}$$

4. Stochastic Decomposition

Theorem:

The number of customers in the system under steady state can be decomposed into two independent Probability generating functions, one of which is the PGF of the queue size distribution in the priority arrival classical G-queue and unreliable server with delayed repair and the other is the PGF of the conditional distribution of the number of customer in the orbit given that the system is idle. The existence of the stochastic decomposition property for our model can be demonstrated easily by showing that

$$\Omega(z) = \Pi(z)\Psi(z) \tag{89}$$

Proof: The probability generating function $\Pi(z)$ of the system size in the classical priority arrival G-queue and unreliable server with delayed repair,setup time and balking is given by,

$$\Pi(z) = \frac{nr}{dr} \tag{90}$$

$$nr = T_1(z_1,z_2)\{\{[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]f_2(z_1,z_2)+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]$$
$$[1-\overline{D}(f_2(z_1,z_2))\,\overline{R}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2))]+\theta\overline{B}_2(f_1(z_2))[1-\overline{V}(f_2(z_1,z_2))$$

$$\overline{M}(f_2(z_1,z_2))]\}+\{\lambda_1 C_1(g(z_2))z_2[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]$$
$$+\theta\overline{B}_2(f_1(z_1,z_2))[\overline{V}(f_2(g(,z_2)))\overline{M}(f_2(g(z_2)))-\overline{V}(f_2(z_2))\overline{M}(f_2(z_2))]$$
$$+\overline{\lambda}[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]\{\overline{D}(f_2(g(z_2)))\overline{R}(f_2(g(z_2)))\overline{M}(f_2(g(z_2)))-\overline{D}(f_2(z_2))$$
$$\overline{R}(f_2(z_2))\overline{M}(f_2(z_2))\}\}\{q\theta\overline{B}_1(f_1(z_2))(1-\overline{V}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2)))\}\}\ f_1(z_1,z_2)$$
$$+T_2(z_1,z_2)\{(1-\overline{B}_1(f_1(z_1,z_2)))\ f_2(z_1,z_2)+\overline{\lambda}(1-\overline{B}_1(f_1(z_1,z_2)))$$
$$(1-\overline{D}(f_2(z_1,z_2))\overline{R}(f_2(z_1,z_2))\overline{M}(f_2(z_1,z_2)))\}G_1(z_1,z_2),$$

$$dr=G_1(z_1,z_2)\,f_1(z_1,z_2)\,f_2(z_1,z_2),$$

where,

$$T_1(z_1,z_2)=\frac{P_0^{(2)}(0,z_2)\overline{G}_5(z_1,z_2)}{\left[\begin{array}{l}\overline{G}_1(z_1,z_2)\{z_1-(q+pz_1)\overline{B}_1(f_1(z_1,z_2))\overline{\lambda}[\frac{1-\overline{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}]\\ \overline{D}(f_2(z_1,z_2))\,\overline{R}(f_2(z_1,z_2))\,\overline{M}(f_2(z_1,z_2))\}\end{array}\right]},\tag{91}$$

$$T_2(z_1,z_2)=\frac{\left[\{(-(\lambda_1+\lambda_2 b[1-C_2(z_2)])I_{0,0})+\lambda_2 bC_2(z_2)I_{0,0}\}\overline{G}_1(z_1,z_2)\right]}{\left[z_2\overline{G}_1(z_1,z_2)-\overline{G}_5(z_1,z_2)\right]}\tag{92}$$

The probability generating function $\psi(z)$ of the number of customers in the orbit when the system is idle is given by

$$\Psi(z)=\frac{I_{0,0}+I(0,z_2)}{I_{0,0}+I(0,1)}\tag{93}$$

From equation (90) and (93), we see that $\Omega(z)=\Pi(z)\Psi(z)$

5. The Average Queue Length

The Mean number of customers in the queue and in the orbit under the steady state condition is,

$$L_{q_1}=\frac{d}{dz_1}P_{q_1}(z_1,1)\,|_{z_1=1}\,.$$
$$L_{q_2}=\frac{d}{dz_2}P_{q_2}(1,z_2)\,|_{z_2=1}\tag{94}$$

then,

$$L_{q_1}=\frac{D_1'(1)N_1''(1)-D_1''(1)N_1'(1)}{2(D_1'(1))^2}+\frac{D_1'(1)N_2''(1)-D_1''(1)N_2'(1)}{2(D_1'(1))^2}$$
$$+\frac{D_2'(1)N_3''(1)-D_2''(1)N_3'(1)}{2(D_2'(1))^2},$$
$$L_{q_2}=\frac{d_1'(1)n_1''(1)-d_1''(1)n_1'(1)}{2(d_1'(1))^2}+\frac{d_1'(1)n_2''(1)-d_1''(1)n_2'(1)}{2(d_1'(1))^2}$$
$$+\frac{d_2'(1)n_3''(1)-d_2''(1)n_3'(1)}{2(d_2'(1))^2},$$

5.1. The Average Waiting Time In The Queue:

Average waiting time of a customer in the high priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}. \tag{95}$$

Average waiting time of a customer in the low priority queue is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}. \tag{96}$$

where L_{q_1} and L_{q_2} have been found in above equations.

5.2. Particular Cases:

Case: 1 $M/G/1$ Queueing model:

If there are no priority arrival, no vacation, no negative arrival, no balking, no retrial and single arrival. i.e., $\lambda_1 = 0, \bar{\lambda} = 0, \theta = 0, b = 1, \bar{I} = 1, E(I_2) = 1, E(I_2(I_2 - 1)) = 0,$.

The model under study becomes classical $M/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{-(1 - \bar{B}(\lambda - \lambda z))I_{0,0}}{z - \bar{B}(\lambda - \lambda z)} \tag{97}$$

Case: 2 $M^X/G/1$ Queueing model:

If there are no priority arrival, no vacation, no negative arrival, no balking, no retrial and batch arrival (i.e $\lambda_1 = 0, \bar{\lambda} = 0, \theta = 0, b = 1, \bar{I} = 1$) .

The model under study becomes classical $M^X/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{-(1 - \bar{B}(\lambda - \lambda C(z)))I_{0,0}}{z - \bar{B}(\lambda - \lambda C(z))} \tag{98}$$

The above two results are coincide with the results of Gross.D and Harris.M (1985).

Case: 3 If there are no priority arrivals, no balking, no vacation (i.e $\lambda_1 = 0, \theta = 0, b = 1.$)

$$I(z) = \frac{\left[\{C(z)\bar{B}(f(z))f(z) + \bar{\lambda}C(z)(1 - \bar{B}(f(z)))\bar{D}(B(z))\bar{R}(B(z)) - zf(z)\}(1 - \bar{I})I_0\right]}{\left[zf(z) - \{\bar{B}(f(z))[f(z)] + \bar{\lambda}(1 - \bar{B}(f(z)))\bar{D}(B(z))\bar{R}(B(z))\}(C(z) + \bar{I}(1 - C(z)))\right]},$$

$$P(z) = \frac{\left[\{\lambda(C(z) - 1)\}(1 - \bar{B}(f(z)))\bar{I}I_0\right]}{\left[zf(z) - \{\bar{B}(f(z))[f(z)] + \bar{\lambda}(1 - \bar{B}(f(z)))\bar{D}(B(z))\bar{R}(B(z))\}(C(z) + \bar{I}(1 - C(z)))\right]},$$

$$D(z) = \frac{\left[-I_0\{\bar{\lambda}(1 - \bar{B}(f(z)))\bar{I}\}(1 - \bar{D}(B(z)))\right]}{\left[zf(z) - \{\bar{B}(f(z))[f(z)] + \bar{\lambda}(1 - \bar{B}(f(z)))\bar{D}(B(z))\bar{R}(B(z))\}(C(z) + \bar{I}(1 - C(z)))\right]},$$

$$D(z) = \frac{\left[-I_0\{\bar{\lambda}\bar{D}(B(z))(1 - \bar{B}(f(z)))\bar{I}\}(1 - \bar{R}(B(z)))\right]}{\left[zf(z) - \{\bar{B}(f(z))[f(z)] + \bar{\lambda}(1 - \bar{B}(f(z)))\bar{D}(B(z))\bar{R}(B(z))\}(C(z) + \bar{I}(1 - C(z)))\right]},$$

This result coincides with Kirupa. K and Udaya Chandrika. K (2014)

6. Numerical Results

In order to see the effect of different parameters on the different states of the server, the utilization factor and proportion of idle time, we compute some numerical results. We consider the service time, vacation time and repair time to be exponentially distributed to numerically illustrate the feasibility of our results. Giving the suitable values which satisfies the stability condition , we compute the following table values.

For Table 1 $(\lambda_2, \mu_1, \mu_2, \eta, \theta, \bar{\lambda}, \beta, \phi, \gamma, \delta, p, b) = (0.7, 9, 9, 6, 0.75, 0.8, 5, 5, 5, 5, 0.7, 0.2)$.

For Table 2, $(\lambda_1, \mu_1, \mu_2, \eta, \theta, \bar{\lambda}, \beta, \phi, \gamma, \delta, p, b) = (0.7, 10, 10, 10, 0.4, 0.1, 5, 5, 5, 5, 0.7, 0.1)$.

For Table 3, $(\lambda_1, \lambda_2, \eta, \theta, \bar{\lambda}, \beta, \phi, \gamma, \delta, p, b) = (0.8, 0.7, 6, 0.75, 0.8, 5, 5, 5, 5, 0.7, 0.2)$.

Table 1. *Effect of* λ_1 *on various queue characteristics*

λ_1	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.4	0.6969	0.3031	0.0059	0.5688	0.0146	0.8125
0.5	0.6281	0.3719	0.0105	0.7302	0.0211	0.0431
0.6	0.5610	0.4390	0.0169	0.8861	0.0282	1.2658
0.7	0.4972	0.5028	0.0249	1.0288	0.0356	1.4697
0.8	0.4379	0.5621	0.0345	1.1529	0.0431	1.6470
0.9	0.3838	0.6162	0.0453	1.2549	0.0503	1.7927
1.0	0.3351	0.6649	0.0570	1.3334	0.0570	1.9048
1.1	0.2918	0.7082	0.0695	1.3883	0.0632	1.9833
1.2	0.2536	0.7464	0.0824	1.4209	0.0687	2.0299
1.3	0.2203	0.7797	0.0955	1.4330	0.0734	2.0472

Table 2. *Effect of* λ_2 *on various queue characteristics*

λ_2	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.4	0.2658	0.7342	0.0079	0.0299	0.0113	0.0748
0.5	0.2637	0.7363	0.0079	0.1769	0.0113	0.3537
0.6	0.2618	0.7382	0.0079	0.3224	0.0113	0.5373
0.7	0.2600	0.7400	0.0079	0.4684	0.0113	0.6691
0.8	0.2582	0.7418	0.0079	0.6157	0.0113	0.7696
0.9	0.2566	0.7434	0.0079	0.7648	0.0113	0.8497
1.0	0.2551	0.7449	0.0079	0.9157	0.0113	0.9157
1.1	0.2537	0.7463	0.0079	1.0686	0.0113	0.9715
1.2	0.2523	0.7477	0.0079	1.2234	0.0113	1.0195
1.3	0.2511	0.7489	0.0079	1.3799	0.0113	1.0615

Table 3. *Effect of* μ *on various queue characteristics*

μ	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
5.1	0.3559	0.6441	0.0458	1.5667	0.0573	2.2382
5.2	0.3586	0.6414	0.0453	1.5441	0.0567	2.2059
5.3	0.3612	0.6388	0.0448	1.5228	0.0560	2.1755
5.4	0.3638	0.6362	0.0443	1.5027	0.0554	2.1468
5.5	0.3664	0.6336	0.0439	1.4838	0.0548	2.1197
5.6	0.3689	0.6311	0.0434	1.4658	0.0543	2.0940
5.7	0.3715	0.6285	0.0430	1.4487	0.0537	2.0696
5.8	0.3739	0.6261	0.0426	1.4326	0.0532	2.0465
5.9	0.3764	0.6236	0.0422	1.4172	0.0527	2.0245
6.0	0.3788	0.6212	0.0418	1.4025	0.0522	2.0036

In Table 1 it is clearly shows that as long as increasing the arrival rate of high priority customers the servers idle time decreases while the utilisation factor, average queue length for both high priority and low priority customers are also increases.

In Table 2 it is clearly shows that as long as increasing the arrival rate of low priority customers the servers idle time decreases and the utilisation factor, average queue length for low priority customers are increases. Since this paper construct under pre-emptive priority rule, the arrival of low priority customer does not affect the service of high priority customer. Therfore, the average queue size and waiting time of a high priority customer is always constant.

Table 3 shows that as long as increase in service rate, idle time increases while the utilisation factor, queue size and waiting time of both high priority and low priority customers are decreases.

6.1. Graphical Study

We can plot the above data graphically to illustrate the feasibility of our results.

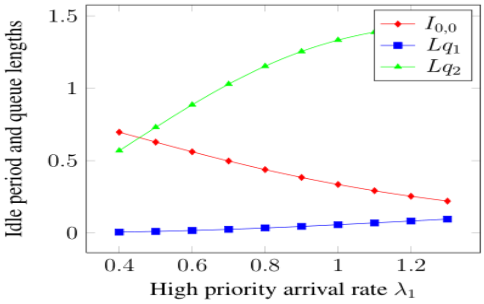

Figure 1: Average queue sizes Vs High priority arrival rate λ_1

Fig. 1 graphically represent the effect of high priority arrival rate over the idle period and queue length of the model. It is clear from the figure that if arrival rate increases with respect to all other parameters the queue length of both priority queues and busy period increases but idle time decreases.

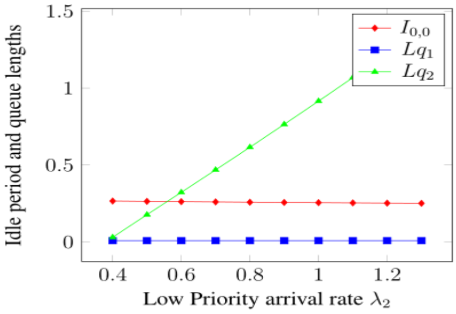

Figure 2: Average queue sizes Vs Low priority arrival rate λ_2

Fig. 2 graphically represent the effect of low priority arrival rate over the idle period and queue length of the model. It is clearly shows that as long as increasing the arrival rate of low priority customers the servers idle time decreases and the utilisation factor, average queue length for low priority customers are increases. Though there is no impact on high priority arrivals, the average queue size and waiting time of a high priority customer is always constant.

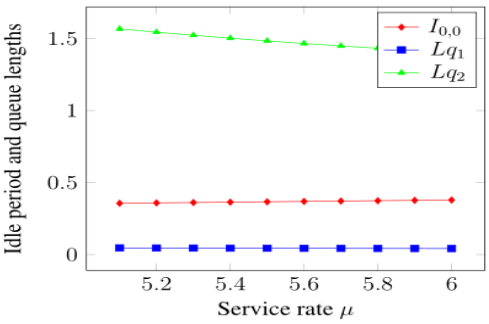

Figure 3: Average queue sizes Vs Service rate μ

Fig. 3 graphically represent the effect of service rate over the idle period and queue length of the model. Due to increase in server rate, the queue length of high and low priority customers are decreases, the proportion of idle time of the server increases and utilization factor or busy period decreases.

7. Conclusion

In this paper we analysed a $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue with negative customers and priority under modified Bernoulli vacation schedule subject to the server breakdowns, repairs and setup time is investigated. In addition, the effect of balking on a service system or more complex system may also be a problem of interest. Essentially, a breakdown is represented by a negative customer arriving at the server which removes the customer being in service when the server is busy. For the classical retrial strategy, the necessary and sufficient condition for the system to be stable is obtained. The joint distribution of the number of customers in the queue and the number of customers in the orbit is derived. In addition, we obtain the stochastic decomposition law. Numerical examples have been carried out to observe the trend of the mean number of customers in the system for varying parametric values.

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AUTHORS' BIOGRAPHY

Dr.G.Ayyappan is a professor and head, department of mathematics in Pondicherry Engineering college. He has completed Phd in Annamalai university and having 23 years of teaching experience. He has published more than 100 papers in reputed journals and published more than 30 papers in National and International conferences. His area of interests are Queueing theory, Reliability Analysis and Inventory models.

Ms.J.Udayageetha is an assistant professor in department of mathematics in Perunthalaivar Kamarajar Arts College. She has completed PG in Pondicherry University and having 6 years of teaching experience.

Citation: *G. Ayyappan, J. Udayageetha, " Transient Solution of* $M^{[X_1]}, M^{[X_2]} / G_1, G_2 / 1$ *Retrial queueing system with Priority services, Modified Bernoulli Vacation, Bernoulli Feedback, Negative arrival, Breakdown, Delaying repair, Setup time and Balking ", International Journal of Scientific and Innovative Mathematical Research, vol. 5, no. 12, p. 8-27, 2017., http://dx.doi.org/10.20431/2347-3142.0512002*