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Analysis of Batch Arrival Bulk Service Queue with Two Fluctuating Modes of Service Closedown Multiple Vacation and State Dependent Arrival Rate

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Abstract: *In this paper, an* $M^{[X]}/(G_1(a,b),G_2(a,b))/1$ *queueing model with closedown, multiple vacation and state dependent arrival rate is considered. The single server provides two modes of service. The server provides mode 1 service with probability* α_1 *or mode 2 service with probability* α_2*, where* $\alpha_2=1-\alpha_1$*. It is considered that the arrival rate depends on the state of server. Using supplementary variable technique, the steady-state probability generating function of the system size at an arbitrary time is obtained. The performance measures and cost model are also derived. Numerical illustrations are presented to visualize the effect of system parameters.*

Keywords: *Batch arrival, Bulk service, Closedown, Multiple vacations, State dependent arrival rate.*

1. Introduction

Queueing models where the server performs closedown work and resumes vacation when there is no sufficient batch size (less than the minimum threshold) for service, is quite common in various practical situations related to manufacturing systems, service systems, etc. Neuts [1] initiated the concept of bulk queues and analyzed a general class of such models. A literature survey on vacation queueing models can be found in Doshi [2] and Takagi [3] which include some applications. Lee [4] developed a systematic procedure to calculate the system size probabilities for a bulk queueing model. Krishna Reddy et al. [5] considered the $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Anabosi and Madan [6] considered a single server queueing model in which the server provides two types of heterogeneous service and the customers may opt any one. They also derived the steady state probability generating functions of queue size, system size and the performance measures like expected system length, expected waiting time, etc.

Arumuganathan and Jeyakumar [7] obtained the probability generating function of queue length distributions at an arbitrary time epoch for the bulk queueing model with multiple vacation and closedown times. Also they have developed a cost model with a numerical study for their queueing model. Madan et al. [8] considered the $M^{[X]}/(G_1,G_2)/1$ queueing model where the batch of customers can choose any one of the two service and then after the completion of service, they may opt for re-service. They derived the steady state probability generating functions for the number of customers in the queue and the system. Arumuganathan and Jeyakumar [9] obtained the probability generating function of queue size distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, setup times and N-policy. Arumuganathan and Ramaswami [10] derived the steady state system size probabilities for a batch arrival bulk service queueing model where the server undergoes multiple vacations and the arrival rate is assumed as state dependent.

Ke [11] investigated an $M^{[X]}/G/1$ queueing model with vacation policies, breakdown and startup/closedown times where the vacation, startup, closedown and repair times are generally distributed. Parthasarathy and Sudhesh [12] derived the transient system size probabilities and the duration of busy period for a single server queueing model using continued fraction where the system alternates between arrivals and service with state dependent rates. Balasubramanian and Arumuganathan [13] considered an $M^{[X]}/G(a,b)/1$ queueing model and obtained its queue size distribution for the steady state. They also derived the average length of busy and idle periods, expected queue length and waiting time. Ayyappan and Shyamala [14] derived the PGF of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time for both steady state and transient cases.

The rest of the paper is organized as follows. In section 2, an $M^{[X]}/(G_1(a,b),G_2(a,b))/1$ queueing model with closedown, multiple vacation and state dependent arrival rate is described and the steady-state system size equations are considered. In section 3, using supplementary variable technique, the probability generating function of the queue size are derived and some particular cases are provided. In section 4, performance measures like expected length of busy and idle periods, expected queue length and waiting time are obtained. In section 5, the cost model is provided. In section 6, numerical illustrations are presented to validate the analytical results. In section 7, this research work is concluded with the proposed future work.

2. Model Description

In this paper, we analyze an $M^{[X]}/(G_1(a,b),G_2(a,b))/1$ queueing model with closedown, multiple vacation and state dependent arrival rate. Arrival follows compound Poisson process with rate λ when the server is busy and λ_0 when the server is idle. The single server provides two modes of service. The server provides mode 1 service with probability α_1 or mode 2 service with probability α_2, where $\alpha_1+\alpha_2=1$. After the completion of service (first or second), customers leave the system. If the queue length is less than 'a', the server performs closedown work. After that, the server leaves for multiple vacation of random length. After this vacation, if the queue length is still less than 'a', the server leaves for another vacation and so on until he finds at least 'a' customers waiting in the queue. After the vacation completion, if the server finds at least 'a' customers waiting for service, he serves a batch of ξ customers, where $a \le \xi \le b$. The customers are served on the first come first serve (FCFS) basis. For the model under consideration, the PGF of the queue size at an arbitrary time and various performance measures are obtained. Cost model for the system is also derived.

2.1 Notations

The following notations are used in this paper.

λ - Arrival rate,

X - Group size random variable,

$g_k - Pr\{X=k\}$,

$X(z)$ - Probability generating function (PGF) of X.

Here $S_1(.), S_2(.), V(.)$ and $C(.)$ represent the cumulative distribution function (CDF) of first mode of service time, second mode of service time, vacation time and closedown time and their corresponding probability density functions are $s_1(x), s_2(x), v(x)$ and $c(x)$ respectively. $S_1^0(t), S_2^0(t), V^0(t)$ and $C^0(t)$ represent the remaining service time of mode 1 service of a batch, mode 2 service time of a batch, vacation time and closedown time at time t respectively. $\widetilde{S_1}(\theta), \widetilde{S_2}(\theta), \tilde{V}(\theta)$ and $\tilde{C}(\theta)$ represent the Laplace-Stieltjes transform of S_1, S_2, V and C respectively.

The supplementary variables $S_1^0(t), S_2^0(t), V^0(t)$ and $C^0(t)$ are introduced in order to obtain the bivariate Markov process $\{N(t), Y(t)\}$, where $N(t)=\{N_q(t)\cup N_s(t)\}$ and

$Y(t)=(0)[1]\{2\}\langle 3\rangle$, if the server is on(mode 1 service)[mode 2 service] $\{$vacation$\}\langle$closedown time$\rangle$.
$Z(t)=j$, if the server is on jth *vacation*. $N_s(t)$ = Number of customers in the service at time t.
$N_q(t)$ = Number of customers in the queue.

Define the probabilities as,

$$P_{i,j}^{(1)}(x,t)dt = P\{N_s(t)=i, N_q(t)=j, x\le S_1^0(t)\le x+dx, Y(t)=0\}, a\le i\le b,\ j\ge 0.$$
$$P_{i,j}^{(2)}(x,t)dt = P\{N_s(t)=i, N_q(t)=j, x\le S_2^0(t)\le x+dx, Y(t)=1\}, a\le i\le b,\ j\ge 0.$$
$$Q_{j,n}(x,t)dt = P\{N_q(t)=n, x\le V^0(t)\le x+dx, Y(t)=2, Z(t)=j\}, n\ge 0,\ j\ge 1.$$
$$C_n(x,t)dt = P\{N_q(t)=n, x\le C^0(t)\le x+dx, Y(t)=3\}, n\ge 0.$$

The supplementary variable technique was introduced by Cox [15]. Using supplementary variables one can convert non-Markovian models into Markovian models.

The steady-state system size equations are obtained as follows:

$$-P_{i,0}'^{(1)}(x) = -\lambda P_{i,0}^{(1)}(x)+\alpha_1\sum_{m=a}^{b}P_{m,i}^{(1)}(0)s_1(x)+\alpha_1\sum_{m=a}^{b}P_{m,i}^{(2)}(0)s_1(x)+\alpha_1\sum_{l=1}^{\infty}Q_{l,i}(0)s_1(x),\quad a\le i\le b, \tag{1}$$

$$-P_{i,j}'^{(1)}(x) = -\lambda P_{i,j}^{(1)}(x)+\sum_{k=1}^{j}P_{i,j-k}^{(1)}(x)\lambda g_k,\quad j\ge 1, a\le i\le b-1, \tag{2}$$

$$-P_{b,j}'^{(1)}(x) = -\lambda P_{b,j}^{(1)}(x)+\alpha_1\sum_{m=a}^{b}P_{m,b+j}^{(1)}(0)s_1(x)+\sum_{k=1}^{j}P_{b,j-k}^{(1)}(x)\lambda g_k$$
$$+\alpha_1\sum_{m=a}^{b}P_{m,b+j}^{(2)}(0)s_1(x)+\alpha_1\sum_{l=1}^{\infty}Q_{l,b+j}(0)s_1(x),\quad j\ge 1, \tag{3}$$

$$-P_{i,0}'^{(2)}(x) = -\lambda P_{i,0}^{(2)}(x)+\alpha_2\sum_{m=a}^{b}P_{m,i}^{(2)}(0)s_2(x)+\alpha_2\sum_{m=a}^{b}P_{m,i}^{(1)}(0)s_2(x)+\alpha_2\sum_{l=1}^{\infty}Q_{l,i}(0)s_2(x),\quad a\le i\le b, \tag{4}$$

$$-P_{i,j}'^{(2)}(x) = -\lambda P_{i,j}^{(2)}(x)+\sum_{k=1}^{j}P_{i,j-k}^{(2)}(x)\lambda g_k,\quad j\ge 1, a\le i\le b-1, \tag{5}$$

$$-P_{b,j}'^{(2)}(x) = -\lambda P_{b,j}^{(2)}(x)+\alpha_2\sum_{m=a}^{b}P_{m,b+j}^{(2)}(0)s_2(x)+\sum_{k=1}^{j}P_{b,j-k}^{(2)}(x)\lambda g_k$$
$$+\alpha_2\sum_{m=a}^{b}P_{m,b+j}^{(1)}(0)s_2(x)+\alpha_2\sum_{l=1}^{\infty}Q_{l,i}(0)s_2(x),\quad j\ge 1, \tag{6}$$

$$-C_n'(x) = -\lambda_0 C_n(x)+\sum_{m=a}^{b}P_{m,n}^{(1)}(0)c(x)+\sum_{m=a}^{b}P_{m,n}^{(2)}(0)c(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda_0 g_k,\quad n\le a-1, \tag{7}$$

$$-C_n'(x) = -\lambda_0 C_n(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda_0 g_k,\quad n\ge a, \tag{8}$$

$$-Q_{1,0}'(x) = -\lambda_0 Q_{1,0}(x)+C_0(0)v(x), \tag{9}$$

$$-Q_{1,n}'(x) = -\lambda_0 Q_{1,n}(x)+C_n(0)v(x)+\sum_{k=1}^{n}Q_{1,n-k}(x)\lambda_0 g_k,\quad n\ge 1, \tag{10}$$

$$-Q'_{j,0}(x) = -\lambda_0 Q_{j,0}(x) + Q_{j-1,0}(0)v(x), \quad j \geq 2, \tag{11}$$

$$-Q'_{j,n}(x) = -\lambda_0 Q_{j,n}(x) + Q_{j-1,n}(0)v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_0 g_k, \; j \geq 2, 1 \leq n < a, \tag{12}$$

$$-Q'_{j,n}(x) = -\lambda_0 Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_0 g_k, \; j \geq 2, n \geq a. \tag{13}$$

The Laplace-Stieltjes transform of $P^{(1)}_{i,j}(x), P^{(2)}_{i,j}(x), C_n(x), Q_{j,n}(x),$ are defined as follows:

$$\tilde{P}^{(1)}_{i,j}(\theta) = \int_0^\infty e^{-\theta x} P^{(1)}_{i,j}(x)dx, \quad \tilde{P}^{(2)}_{i,j}(\theta) = \int_0^\infty e^{-\theta x} P^{(2)}_{i,j}(x)dx,$$

$$\tilde{C}_n(\theta) = \int_0^\infty e^{-\theta x} C_n(x)dx, \quad \tilde{Q}_{j,n}(\theta) = \int_0^\infty e^{-\theta x} Q_{j,n}(x)dx.$$

Taking Laplace-Stieltjes transform from (1) to (13), we get

$$\theta\tilde{P}^{(1)}_{i,0}(\theta) - P^{(1)}_{i,0}(0) = \lambda\tilde{P}^{(1)}_{i,0}(\theta) - \alpha_1\tilde{S}_1(\theta)\Big[\sum_{m=a}^{b} P^{(1)}_{m,i}(0) + \sum_{m=a}^{b} P^{(2)}_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\Big], a \leq i \leq b, \tag{14}$$

$$\theta\tilde{P}^{(1)}_{i,j}(\theta) - P^{(1)}_{i,j}(0) = \lambda\tilde{P}^{(1)}_{i,j}(\theta) - \sum_{k=1}^{j} \tilde{P}^{(1)}_{i,j-k}(\theta)\lambda g_k, \tag{15}$$

$$\theta\tilde{P}^{(1)}_{b,j}(\theta) - P^{(1)}_{b,j}(0) = \lambda\tilde{P}^{(1)}_{b,j}(\theta) - \sum_{k=1}^{j} \tilde{P}^{(1)}_{b,j-k}(\theta)\lambda g_k - \tilde{S}_1(\theta)\alpha_1\Big[\sum_{m=a}^{b} P^{(1)}_{m,b+j}(0)$$

$$+\sum_{m=a}^{b} P^{(2)}_{m,b+j}(0) + \sum_{l=1}^{\infty} Q_{l,b+j}(0)\Big], \; j \geq 1, \tag{16}$$

$$\theta\tilde{P}^{(2)}_{i,0}(\theta) - P^{(2)}_{i,0}(0) = \lambda\tilde{P}^{(2)}_{i,0}(\theta) - \tilde{S}_2(\theta)\alpha_2\Big[\sum_{m=a}^{b} P^{(2)}_{m,i}(0) + \sum_{m=a}^{b} P^{(1)}_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\Big], a \leq i \leq b, \tag{17}$$

$$\theta\tilde{P}^{(2)}_{i,j}(\theta) - P^{(2)}_{i,j}(0) = \lambda\tilde{P}^{(2)}_{i,j}(\theta) - \sum_{k=1}^{j} \tilde{P}^{(2)}_{i,j-k}(\theta)\lambda g_k, \tag{18}$$

$$\theta\tilde{P}^{(2)}_{b,j}(\theta) - P^{(2)}_{b,j}(0) = \lambda\tilde{P}^{(2)}_{b,j}(\theta) - \sum_{k=1}^{j} \tilde{P}^{(2)}_{b,j-k}(\theta)\lambda g_k - \tilde{S}_2(\theta)\alpha_2\Big[\sum_{m=a}^{b} P^{(2)}_{m,b+j}(0)$$

$$+\sum_{m=a}^{b} P^{(1)}_{m,b+j}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0)\Big], \; j \geq 1, \tag{19}$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda_0\tilde{C}_n(\theta) - \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta)\lambda g_k - \tilde{C}(\theta)\Big[\sum_{m=a}^{b} P^{(1)}_{m,n}(0) + \sum_{m=a}^{b} P^{(2)}_{m,n}(0)\Big], n \leq a-1, \tag{20}$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda_0\tilde{C}_n(\theta) - \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta)\lambda_0 g_k, \tag{21}$$

$$\theta\tilde{Q}_{1,0}(\theta) - Q_{1,0}(0) = \lambda_0\tilde{Q}_{1,0}(\theta) - \tilde{V}(\theta)C_0(0), \tag{22}$$

$$\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) = \lambda_0\tilde{Q}_{1,n}(\theta) - \tilde{V}(\theta)C_n(0) - \sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta)\lambda_0 g_k, \; n \geq 1, \tag{23}$$

$$\theta\tilde{Q}_{j,0}(\theta) - Q_{j,0}(0) = \lambda_0\tilde{Q}_{j,0}(\theta) - \tilde{V}(\theta)Q_{j-1,0}(0), \tag{24}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda_0\tilde{Q}_{j,n}(\theta) - \tilde{V}(\theta)Q_{j-1,n}(0) - \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda_0 g_k, \tag{25}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda_0\tilde{Q}_{j,n}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda_0 g_k. \tag{26}$$

To find the probability generating function (PGF) of queue size, we define the following PGFs:

$$\tilde{P}_i^{(1)}(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{i,j}^{(1)}(\theta)z^j,\quad P_i^{(1)}(z,0)=\sum_{j=0}^{\infty}P_{i,j}^{(1)}(0)z^j,\quad a\le i\le b,$$

$$\tilde{P}_i^{(2)}(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{i,j}^{(2)}(\theta)z^j,\quad P_i^{(2)}(z,0)=\sum_{j=0}^{\infty}P_{i,j}^{(2)}(0)z^j,\quad a\le i\le b,$$

$$\tilde{C}(z,\theta)=\sum_{n=0}^{\infty}\tilde{C}_n(\theta)z^n,\quad C(z,0)=\sum_{n=0}^{\infty}C_n(0)z^n,$$

$$\tilde{Q}_j(z,\theta)=\sum_{n=0}^{\infty}\tilde{Q}_{j,n}(\theta)z^n,\quad Q_j(z,0)=\sum_{n=0}^{\infty}Q_{j,n}(0)z^j,\quad j\ge 1. \tag{27}$$

By multiplying the equations from (14) to (26) by suitable power of z^n and summing over n, ($n=0$ to ∞) and using (27),

$$(\theta-\lambda_0+\lambda_0X(z))\tilde{Q}_1(z,\theta)=Q_1(z,0)-C(z,0)\tilde{V}(\theta), \tag{28}$$

$$(\theta-\lambda_0+\lambda_0X(z))\tilde{Q}_j(z,\theta)=Q_j(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\ge 2, \tag{29}$$

$$(\theta-\lambda_0+\lambda_0X(z))\tilde{C}(z,\theta)=C(z,0)-\tilde{C}(\theta)\Big[\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(1)}(0)z^n+\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(2)}(0)z^n\Big], \tag{30}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_i^{(1)}(z,\theta)=P_i^{(1)}(z,0)-\tilde{S}_1(\theta)\alpha_1\Big[\sum_{m=a}^{b}P_{m,i}^{(1)}(0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big],a\le i\le b-1, \tag{31}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_b^{(1)}(z,\theta)=P_b^{(1)}(z,0)-\frac{\tilde{S}_1(\theta)}{z^b}\alpha_1\Big[\sum_{m=a}^{b}P_m^{(1)}(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^{(1)}(0)z^j$$
$$+\sum_{m=a}^{b}P_m^{(2)}(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j\Big], \tag{32}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_i^{(2)}(z,\theta)=P_i^{(2)}(z,0)-\tilde{S}_2(\theta)\alpha_2\Big[\sum_{m=a}^{b}P_{m,i}^{(1)}(0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big], \tag{33}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_b^{(2)}(z,\theta)=P_b^{(2)}(z,0)-\frac{\tilde{S}_2(\theta)}{z^b}\alpha_2\Big[\sum_{m=a}^{b}P_m^{(1)}(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^{(1)}(0)z^j$$
$$+\sum_{m=a}^{b}P_m^{(2)}(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j\Big]. \tag{34}$$

By Substituting $\theta=\lambda_0-\lambda_0X(z)$ in (28) to (30), we get

$$Q_1(z,0)=\tilde{V}(\lambda-\lambda X(z))C(z,0), \tag{35}$$

$$Q_j(z,0)=\tilde{V}(\lambda-\lambda X(z))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\ge 2, \tag{36}$$

$$C(z,0)=\tilde{C}(\lambda-\lambda X(z))\Big[\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(1)}(0)z^n+\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(2)}(0)z^n\Big], \tag{37}$$

By Substituting $\theta=\lambda-\lambda X(z)$ in (31) to (34), we get

$$P_i^{(1)}(z,0)=\tilde{S}_1(\lambda-\lambda X(z))\alpha_1\Big[\sum_{m=a}^{b}P_{m,i}^{(1)}(0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big],a\le i\le b-1, \tag{38}$$

$$z^bP_b^{(1)}(z,0)=\tilde{S}_1(\lambda-\lambda X(z))\alpha_1\Big[\sum_{m=a}^{b}P_m^{(1)}(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}^{(1)}(0)z^j+\sum_{m=a}^{b}P_m^{(2)}(z,0)$$
$$-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{j=0}^{b-1}\sum_{l=1}^{\infty}Q_{l,j}(0)z^j\Big],a\le i\le b-1, \tag{39}$$

$$z^b P_b^{(1)}(z,0)-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))P_b^{(1)}(z,0)-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))P_b^{(2)}(z,0)$$
$$=\tilde{S}_1(\lambda-\lambda X(z))\alpha_1\Big[\sum_{m=a}^{b-1}P_m^{(1)}(z,0)+\sum_{m=a}^{b-1}P_m^{(2)}(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)$$
$$-\sum_{j=0}^{b-1}\Big[\sum_{m=a}^{b}P_{m,j}^{(1)}(0)z^j+\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_{l,j}(0)z^j\Big]\Big]. \quad (40)$$

$$P_i^{(2)}(z,0)=\tilde{S}_2(\lambda-\lambda X(z))\alpha_2\Big[\sum_{m=a}^{b}P_{m,i}^{(1)}(0)+\sum_{m=a}^{b}P_{m,i}^{(2)}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)\Big],\ a\le i\le b-1, \quad (41)$$

$$z^b P_b^{(2)}(z,0)=\tilde{S}_2(\lambda-\lambda X(z))\alpha_2\Big[\sum_{m=a}^{b}P_m^{(1)}(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}^{(1)}(0)z^j+\sum_{m=a}^{b}P_m^{(2)}(z,0)$$
$$-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{j=0}^{b-1}\sum_{l=1}^{\infty}Q_{l,j}(0)z^j\Big], \quad (42)$$

$$z^b P_b^{(2)}(z,0)-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))P_b^{(2)}(z,0)-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))P_b^{(1)}(z,0)$$
$$=\tilde{S}_2(\lambda-\lambda X(z))\alpha_2\Big[\sum_{m=a}^{b-1}P_m^{(1)}(z,0)+\sum_{m=a}^{b-1}P_m^{(2)}(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)$$
$$-\sum_{j=0}^{b-1}\Big[\sum_{m=a}^{b}P_{m,j}^{(1)}(0)z^j+\sum_{m=a}^{b}P_{m,j}^{(2)}(0)z^j+\sum_{l=1}^{\infty}Q_{l,j}(0)z^j\Big]\Big]. \quad (43)$$

Solving (40) and (43), we get

$$P_b^{(2)}(z,0)=\frac{\alpha_2\tilde{S}_2(\lambda-\lambda X(z))f(z)}{z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))}, \quad (44)$$

$$P_b^{(1)}(z,0)=\frac{\alpha_1\tilde{S}_1(\lambda-\lambda X(z))f(z)}{z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))}, \quad (45)$$

where $f(z)=\sum_{m=a}^{b-1}P_m^{(1)}(z,0)+\sum_{m=a}^{b-1}P_m^{(2)}(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{n=0}^{b-1}g_n z^n.$

Here $p_i^{(1)}=\sum_{m=a}^{b}P_{m,i}^{(1)}(0),\quad p_i^{(2)}=\sum_{m=a}^{b}P_{m,i}^{(2)}(0),$

$$q_i=\sum_{l=1}^{\infty}Q_{l,i}(0),\quad g_i=p_i^{(1)}+p_i^{(2)},\quad k_i=g_i+q_i \quad (46)$$

Substitute (35) to (37), (38), (41), (44), (45), we get

$$\tilde{Q}_1(z,\theta)=\frac{\left[\tilde{V}(\lambda_0-\lambda_0X(z))-\tilde{V}(\theta)\right]C(z,0)}{(\theta-\lambda_0+\lambda_0X(z))}, \quad (47)$$

$$\tilde{Q}_j(z,\theta)=\frac{\left[\tilde{V}(\lambda_0-\lambda_0X(z))-\tilde{V}(\theta)\right]\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n}{(\theta-\lambda_0+\lambda_0X(z))},\ j\ge 2, \quad (48)$$

$$\tilde{C}(z,\theta)=\frac{\left(\tilde{C}(\lambda_0-\lambda_0X(z))-\tilde{C}(\theta)\right)\sum_{n=0}^{b-1}g_n z^n}{(\theta-\lambda_0+\lambda_0X(z))}, \quad (49)$$

$$\tilde{P}_i^{(1)}(z,\theta)=\frac{\left(\tilde{S}_1(\lambda-\lambda X(z))-\tilde{S}_1(\theta)\right)\alpha_1 k_i}{(\theta-\lambda+\lambda X(z))}, \quad (50)$$

$$\tilde{P}_b^{(1)}(z,\theta)=\frac{\left[\tilde{S}_1(\lambda-\lambda X(z))-\tilde{S}_1(\theta)\right]\alpha_1 f(z)}{(\theta-\lambda+\lambda X(z))\left[z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))\right]}, \quad (51)$$

$$\tilde{P}_i^{(2)}(z,\theta)=\frac{\left(\tilde{S}_2(\lambda-\lambda X(z))-\tilde{S}_2(\theta)\right)k_i}{(\theta-\lambda+\lambda X(z))}, \tag{52}$$

$$\tilde{P}_b^{(2)}(z,\theta)=\frac{\left[\tilde{S}_2(\lambda-\lambda X(z))-\tilde{S}_2(\theta)\right]\alpha_2 f(z)}{(\theta-\lambda+\lambda X(z))\left[z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))\right]}. \tag{53}$$

3. Probability Generating Function of Queue Size

In this section, the PGF, $P(z)$ of the queue size at an arbitrary time epoch is derived.

3.1 PGF of Queue Size at an Arbitrary Time Epoch

If P(z) be the PGF of the queue size at an arbitrary time epoch, then

$$P(z)=\sum_{m=a}^{b-1}\tilde{P}_m^{(1)}(z,0)+\tilde{P}_b^{(1)}(z,0)+\sum_{m=a}^{b-1}\tilde{P}_m^{(2)}(z,0)+\tilde{P}_b^{(2)}(z,0)+\tilde{C}(z,0)+\sum_{l=1}^{\infty}\tilde{Q}_l(z,0). \tag{54}$$

By substituting $\theta=0$ into the equations (47) to (53), then the equation (54) becomes

$$P(z)=\frac{\begin{aligned}&\Big\{(\alpha_1\tilde{S}_1(\lambda-\lambda X(z))+\alpha_2\tilde{S}_2(\lambda-\lambda X(z))-1)(-\lambda_0+\lambda_0X(z))\sum_{n=a}^{b-1}(z^b-z^n)k_n\\&+(\tilde{V}(\lambda_0-\lambda_0X(z))\tilde{C}(\lambda_0-\lambda_0X(z))-1)\big[\big[\alpha_1\tilde{S}_1(\lambda-\lambda X(z))\\&+\alpha_2\tilde{S}_2(\lambda-\lambda X(z))-1\big](-\lambda_0+\lambda_0X(z))+\big[z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))\\&-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))\big](-\lambda+\lambda X(z))\big]\sum_{n=0}^{a-1}g_nz^n+(\tilde{V}(\lambda_0-\lambda_0X(z))-1)\\&\times\big[\left(\alpha_1\tilde{S}_1(\lambda-\lambda X(z))+\alpha_2\tilde{S}_2(\lambda-\lambda X(z))-1\right)(-\lambda_0+\lambda_0X(z))\\&+\left(z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))\right)(-\lambda+\lambda X(z))\big]\sum_{n=0}^{a-1}q_nz^n\Big\}\end{aligned}}{(-\lambda+\lambda X(z))(-\lambda_0+\lambda_0X(z))\left[z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))\right]} \tag{55}$$

Equation (55) has $a+b$ unknowns $k_a,k_{a+1},\ldots,k_{b-1},g_0,g_1,\ldots,g_{a-1},q_0,q_1,\ldots,q_{a-1}$. Using the following theorem, we express q_i in terms of g_i in such a way that numerator has only b constants. Now equation (55) gives the PGF of the number of customers involving only 'b' unknowns.

By Rouche's theorem of complex variables, it can be proved that $z^b-\alpha_1\tilde{S}_1(\lambda-\lambda X(z))-\alpha_2\tilde{S}_2(\lambda-\lambda X(z))$ has $b-1$ zeros inside and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations in b unknowns. These equations can be solved by any suitable numerical technique.

3.2 Steady State Condition

Using P(1)=1, the steady state condition is derived as

$$\rho=\lambda E(X)\left[\alpha_1E(S_1)+\alpha_2E(S_2)\right]/b.$$

Theorem 1:

Let q_i can be expressed in terms of g_i as

$$q_n=\sum_{i=0}^{n}L_ig_{n-i},\quad n=0,1,2,\ldots,a-1, \tag{56}$$

where

$$L_n = \frac{h_n + \sum_{i=1}^{n} \eta_i L_{n-i}}{1-\xi_0}, \quad n = 1, 2, 3, \ , a-1, \tag{57}$$

with

$$h_n = \sum_{i=0}^{n} \eta_i \beta_{n-i}, \quad L_0 = \frac{\eta_0 \beta_0}{1-\eta_0}, \tag{58}$$

η_i's and β_i's are the probabilities of the i customers arrive during vacation time and closedown time respectively.

Proof:

From equations (35) and (36) , we have

$$\sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\lambda - \lambda X(z)) \tilde{C}(\lambda - \lambda X(z)) \Big[\sum_{n=0}^{a-1} p_n^{(1)} z^n + \sum_{n=0}^{a-1} p_n^{(2)} z^n \Big] + \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} q_n z^n$$

$$= \sum_{n=0}^{\infty} \eta_n z^n \left[\sum_{n=0}^{\infty} \beta_i z^i \sum_{n=0}^{a-1} \left[p_n^{(1)} + p_n^{(2)} \right] z^n + \sum_{n=0}^{a-1} q_n z^n \right]$$

$$= \sum_{n=0}^{\infty} \eta_n z^n \left[\sum_{n=0}^{\infty} \beta_i z^i \sum_{n=0}^{a-1} g_n z^n + \sum_{n=0}^{a-1} q_n z^n \right]. \tag{59}$$

Equating the coefficient of $z^n, n = 0, 1, 2, \ \ldots, a-1$ on both sides of (72), we get

$$q_n = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \eta_i \beta_{n-i-j} g_j + \sum_{i=0}^{n-1} \eta_{n-i} q_i + \eta_0 q_n,$$

$$q_n = \frac{\sum_{j=0}^{n} \sum_{i=0}^{n-j} \eta_i \beta_{n-i-j} g_j + \sum_{i=0}^{n-1} \eta_{n-i} q_i}{1-\eta_0}.$$

Co-efficient of g_n in q_n is

$$\frac{\eta_0 \beta_0}{1-\eta_0} = L_0 \ (say).$$

Co-efficient of g_{n-1} in q_n is

$$\frac{(\eta_0 \beta_1 + \eta_1 \beta_0) + \eta_1 \left(\frac{\eta_0 \beta_0}{1-\eta_0} \right)}{1-\eta_0}$$

$$\frac{h_1 + \eta_1 L_0}{1-\eta_0} = L_1 \ (say),$$

where $h_1 = \eta_0 \beta_1 + \eta_1 \beta_0.$

By induction

$$L_n = \frac{h_n + \sum_{i=1}^{n} \eta_i L_{n-i}}{1-\eta_0}, n = 0, 1, 2, \ \ldots, a-1,$$

$$L_0 = \frac{\eta_0 \beta_0}{1-\eta_0}, h_n = \sum_{i=0}^{n} \eta_i \beta_{n-i}.$$

3.3 Particular Cases

Case (i): When the server provides only one type of service and there is no closedown.

$$P(z)=\frac{\left\{\left(\tilde{S}(\lambda-\lambda X(z))-1\right)(-\lambda_0+\lambda_0 X(z))\sum_{n=a}^{b-1}\left(z^b-z^n\right)k_n+\left(\tilde{V}(\lambda_0-\lambda_0 X(z))-1\right)\left[\left(\tilde{S}(\lambda-\lambda X(z))-1\right)(-\lambda_0+\lambda_0 X(z))+(z^b-\tilde{S}(\lambda-\lambda X(z)))(-\lambda+\lambda X(z))\right]\sum_{n=0}^{a-1}(g_n+q_n)z^n\right\}}{(-\lambda+\lambda X(z))(-\lambda_0+\lambda_0 X(z))(z^b-\tilde{S}(\lambda-\lambda X(z))),} \quad (60)$$

which coincides with the PGF of Balasubramanian and Arumuganathan [6] if the number of vacations are infinite, (i.e) $M=\infty$.

Case (ii): When the server provides only one type of service, both the arrival rates are same, (i.e) $\lambda=\lambda_0$

$$P(z)=\frac{\left\{\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum_{n=a}^{b-1}\left(z^b-z^n\right)k_n+\left(\tilde{V}(\lambda-\lambda X(z))\tilde{C}(\lambda-\lambda X(z))-1\right)\times(z^b-1)\sum_{n=0}^{a-1}g_n z^n+\left(\tilde{V}(\lambda-\lambda X(z))-1\right)(z^b-1)\sum_{n=0}^{a-1}q_n z^n\right\}}{(-\lambda+\lambda X(z))(z^b-\tilde{S}(\lambda-\lambda X(z))),} \quad (61)$$

which coincides with the PGF of Arumuganathan and Jeyakumar [2] if the set up is zero and $N=a$.

4. Performance Measures

3.4 Expected Queue Length

The expected queue length E(Q) at an arbitrary epoch is obtained by differentiating P(z) at z=1 and is given by

$$E(Q)=\left\{f_1(X,S_1,S_2)\left[\sum_{n=a}^{b-1}[b(b-1)-n(n-1)]k_n\right]+f_2(X,S_1,S_2)\sum_{n=a}^{b-1}(b-n)k_n+f_3(X,S_1,S_2,V)\sum_{n=0}^{a-1}(g_n+q_n)+f_4(X,S_1,S_2,V,C)\sum_{n=0}^{a-1}g_n+f_5(X,S_1,S_2,V)\sum_{n=0}^{a-1}(ng_n+nq_n)+f_6(X,S_1,S_2,V,C)\sum_{n=0}^{a-1}ng_n\right\}\Big/2.\left[(\lambda.X_1).(\lambda_0.X_1).(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right]^2 \quad (62)$$

$f_1(X,S_1,S_2)=H_3.H_1,$

$f_2(X,S_1,S_2)=H_1.H_4-H_2.H_3,$

$f_3(X,S_1,S_2,V)=V^{(2)}\left[\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)(\lambda_0.X_1)+\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right)(\lambda.X_1)\right].H_1$

$+V^{(1)}\left[(\lambda_0.X_1)\left(\alpha_1.S_1^{(2)}+\alpha_2.S_2^{(2)}\right)+(\lambda_0.X_2)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\right.$

$$+\left(b(b-1)-\alpha_1.S_1^{(2)}-\alpha_2.S_2^{(2)}\right)(\lambda.X_1)\ +\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)})(\lambda.X_2\right)\Big].H_1$$

$$-V^{(1)}\Big[(\lambda_0.X_1)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\ +(\lambda.X_1)\left(b-\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\Big].H_2$$

$$f_4(X,S_1,S_2,V,C)=\left[C^{(2)}+2.C^{(1)}.V^{(1)}\right].\Big[\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)(\lambda_0.X_1)$$

$$+\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right)(\lambda.X_1)\Big].H_1$$

$$+C^{(1)}\Big[(\lambda_0.X_1)\left(\alpha_1.S_1^{(2)}+\alpha_2.S_2^{(2)}\right)+(\lambda_0.X_2)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)$$

$$+\left(b(b-1)-\alpha_1.S_1^{(2)}-\alpha_2.S_2^{(2)}\right)(\lambda.X_1)+\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right)(\lambda.X_2)\Big].H_1$$

$$-C^{(1)}\Big[(\lambda_0.X_1)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)+(\lambda.X_1)\left(b-\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\Big].H_2,$$

$$f_5(X,S_1,S_2,V,C)=2.V^{(1)}\Big[(\lambda_0.X_1)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)+(\lambda.X_1)\left(b-\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\Big].H_1$$

$$f_6(X,S_1,S_2,V,C)=2.C^{(1)}\Big[(\lambda_0.X_1)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)+(\lambda.X_1)\left(b-\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)\Big].H_1$$

where

$$H_1=(\lambda.X_1).(\lambda_0.X_1).\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right),$$

$$H_2=(\lambda.X_2).(\lambda_0.X_1).\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right)+(\lambda.X_1).(\lambda_0.X_2).\left(b-\alpha_1.S_1^{(1)}-\alpha_2.S_2^{(1)}\right)$$

$$+(\lambda.X_1).(\lambda_0.X_1).\left(b.(b-1)-\alpha_1.S_1^{(2)}-\alpha_2.S_2^{(2)}\right),$$

$$H_3=(\lambda_0.X_1)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right),$$

$$H_4=(\lambda_0.X_1)\left(\alpha_1.S_1^{(2)}+\alpha_2.S_2^{(2)}\right)+(\lambda_0.X_2)\left(\alpha_1.S_1^{(1)}+\alpha_2.S_2^{(1)}\right)$$

and

$$X_1=E(X),$$

$$S_1^{(1)}=\lambda.X_1.E(S_1),\ S_2^{(1)}=\lambda.X_1.E(S_2),$$

$$V^{(1)}=\lambda_0.X_1.E(V),\ C^{(1)}=\lambda_0.X_1.E(C),$$

$$S_1^{(2)}=\lambda.X_2.E(S_1)+\lambda^2.(E(X))^2.E(S_1^2),$$

$$S_2^{(2)}=\lambda.X_2.E(S_2)+\lambda^2.(E(X))^2.E(S_2^2),$$

$$V^{(2)}=\lambda_0.X_2.E(V)+\lambda_0^2.(E(X))^2.E(V^2),$$

$$C^{(2)}=\lambda_0.X_2.E(C)+\lambda_0^2.(E(X))^2.E(C^2).$$

3.5 Expected Waiting Time

The expected waiting time is obtained by using Little's formula as:

$$E(W)=\frac{E(Q)}{\lambda E(X)},$$

where $E(Q)$ is given in (62).

3.6 Expected length of busy period

Theorem 2:

Let B be the busy period random variable. Then the expected length of busy period is

$$E(B)=\frac{E(T)}{\sum_{n=0}^{a-1} g_n},$$

where $E(T)=\alpha_1 E(S_1)+\alpha_2 E(S_2)$.

Proof:

Let T be the residence time that the server is rendering mode 1 service or mode 2 service.

$$E(T)=\alpha_1 E(S_1)+\alpha_2 E(S_2).$$

Define a random variable J_1 as

$$J_1=\begin{cases}0, & \text{if the server finds less than 'a' customers after the residence time,}\\ 1, & \text{if the server finds atleast 'a' customers after the residence time.}\end{cases}$$

Now the expected length of the busy period is given by

$$E(B)=E(B/J_1=0)P(J_1=0)+E(B/J_1=1)P(J_1=1)$$
$$=E(T)P(J_1=0)+\left[E(T)+E(B)\right]P(J_1=1),$$

where $E(T)$ is the mean service time.

Solving for $E(B)$, we get

$$E(B)=\frac{E(T)}{P(J_1=0)}=\frac{E(T)}{\sum_{n=0}^{a-1} g_n}.$$

3.7 Expected length of idle period

Theorem 3:

Let I be the idle period random variable. Then the expected length of idle period is given by

$$E(I)=E(C)+E(I_1),$$

where $$E(I_1)=\frac{E(V)}{1-\sum_{n=0}^{a-1}\sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\eta_j\beta_{n-i-j}\right]g_i},$$

I_1 is the idle period due to multiple vacation process, $E(C)$ is the expected closedown time.

Proof:

Define a random variable J_2 as

$$J_2=\begin{cases}0, & \text{if the server finds atleast 'a' customers after the first vacation,}\\ 1, & \text{if the server finds less than 'a' customers after the first vacation.}\end{cases}$$

The expected length of idle period due to multiple vacations $E(I_1)$ is given by

$$E(I_1)=E(I_1/J_2=0)P(J_2=0)+E(I_1/J_2=1)P(J_2=1)$$
$$=E(V)P(J_2=0)+\left[E(V)+E(I_1)\right]P(J_2=1).$$

On solving, we get

$$E(I_1)=\frac{E(V)}{P(J_2=0)}=\frac{E(V)}{1-P(J_2=1)}=\frac{E(V)}{1-\sum_{n=0}^{a-1}Q_{1n}(0)}.$$

From equation (45), we get

$$Q_{1n}(0)=\sum_{i=0}^{n}h_{n-i}g_i,$$
$$=\sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\eta_i\beta_{n-i-j}\right]g_i.$$

5. COST MODEL

We derive the expression for finding the total average cost with the following assumptions.

C_s - Start up cost,

C_v - Reward per unit time due to vacation,

C_h - Holding cost per customer,

C_o - Operating cost per unit time,

C_u - Closedown cost per unit time.

The length of cycle is the sum of the idle period and busy period. Now the expected length of the cycle $E(T_c)$ is obtained as

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(J_2 = 0)} + E(C) + \frac{E(T)}{\sum_{n=0}^{a-1} g_n}.$$

$$= \left[C_s + C_u.E(c) - C_v.\frac{E(V)}{P(J_2 = 0)} \right] . \frac{1}{E(T_c)}$$

$$+ C_h.E(Q) + C_o.\rho,$$

where

$$\rho = \lambda E(X)\left[\alpha_1.E(S_1) + \alpha_2.E(S_2)\right]/b.$$

6. NUMERICAL ILLUSTRATION

In this section, various performance measures which are computed in earlier sections are verified numerically. Numerical example is analyzed using MATLAB, the zeros of the function

$z^b - \alpha_1 \tilde{S}_1(\lambda - \lambda X(z)) - \alpha_2 \tilde{S}_2(\lambda - \lambda X(z))$ are obtained and simultaneous equations are solved.

A numerical example is analyzed with the following assumptions:

1. Batch size distribution of the arrival is Geometric with mean two.
2. Service time distribution is Erlang - k for both type of services with $k = 2$.
3. Vacation time and closedown time are exponential with parameter $\eta = 9$ and $\beta = 7$ respectively.
4. The arrival rate $\lambda = 3$, when the server is busy.
5. The arrival rate $\lambda_0 = 2$, when the server is idle.
6. $\alpha_1 = 0.6, \alpha_2 = 0.4, b = 12$.
7. Start-up cost: Rs.3
8. Holding cost per customer: Rs. 0.50
9. Operating cost per unit time: Rs.2
10. Reward per unit time due to vacation: Rs.3
11. Closedown cost per unit time: Rs. 0.25.

In Table 1, for $\alpha_1 = 0.6, \alpha_2 = 0.4, \mu_1 = 7, \mu_2 = 5, b = 12$, we obtained the values of expected queue length, expected busy period, expected idle period and total average cost. From Table 1 and Figure 1, it is clear that the total average cost is minimum when the minimum threshold value is 8.

a	E(Q)	E(B)	E(I)	TAC
1	3.0401	0.9786	1.1964	3.8127
2	3.5342	0.9841	1.1736	3.9486
3	4.2616	0.9894	1.1599	4.2428
4	4.8315	1.0349	1.1323	4.4113
5	5.6489	1.2745	1.118	4.1244
6	6.2205	1.2938	1.0445	4.0088

7	6.7864	1.4651	1.0028	3.5147
8	7.1142	1.5533	1.0002	3.1216
9	7.5364	1.6947	0.9485	3.3459
10	7.9906	1.9219	0.9201	3.7642
11	8.1432	2.5316	0.9116	3.9365

Table 1 Minimum threshold value vs Total average cost and performance measures $\alpha_1 = 0.6, \alpha_2 = 0.4, \mu_1 = 7, \mu_2 = 5, b = 12$

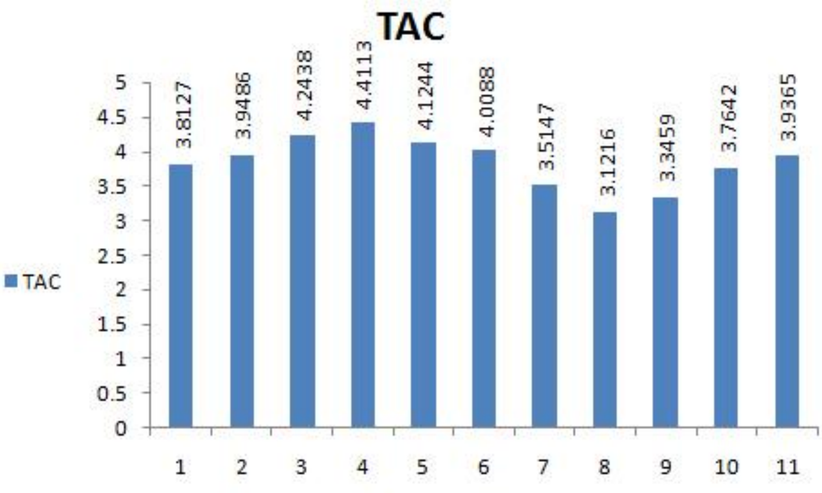

Fig. 1 *TAC for various threshold values*

7. Conclusion and Future Work

In this paper, we have derived the PGF of the queue size for an $M^{[X]} / (G_1(a,b), G_2(a,b)) / 1$ queueing model with multiple vacation, closedown and state dependent arrival rate for the steady-state case. Also we have obtained various performance measures and verified numerically. In future this work may be extended into a queueing model with multiple types of service.

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