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Some Common Fixed Point Theorems for Fuzzy Maps under Non-expansive Type Condition

R. D. Daheriya[1], Umesh Dongre[1*], Manoj Ughade[2]

[1]Department of Mathematics, Government J.H. Post Graduate College, Betul, India
[2]Department of Mathematics, Dr. B. R. A. Government College, Amla, India
[3]Department of Mathematics, Sarvepalli Radhakrishnan University, Bhopal, India
[*]*Correspondingauthor, e-mail: udhelpyou@gmail.com*

Abstract: *In this paper, we prove some common fixed point results for fuzzy mappings satisfying non-expansive type condition.*

Keywords: *fuzzy mapping, common fixed point, linear metric space, non-expansive mapping.*

1. INTRODUCTION

Let (X, d) be a metric space and let T be a self-mappings on X. If T is such that for all x, y in X

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

where $0 < \lambda < 1$, then T is said to be a contraction mapping. If T satisfies (1.1) with $\lambda = 1$, then T is called a non-expansive mapping. If T satisfies any conditions of type

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx) \tag{1.2}$$

where a_i $(i = 1, 2, 3, 4, 5)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T is said to be a contractive type mapping. If T satisfies (1.2) with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, then T is said to be a non-expansive type mapping. Similar terminology is used for multi-valued mappings.

Fixed point theorems for contractive, non-expansive, contractive type and non-expansive type mappings provide techniques for solving a variety of applied problems in mathematical and engineering sciences. It is one of the reason that many authors have studied various classes of contractive type or non-expansive type mappings. For Banach spaces the famous is Gregus's Fixed Point Theorem [10] for non-expansive type single-valued mappings, which satisfy (1.2) with $a_4 = a_5 = 0, a_1 < 1$. The class of mappings T satisfying the following non-expansive type condition:

$$d(Tx, Ty) \leq a(x, y)\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}$$
$$+b(x, y)max\{d(x, Tx), d(y, Ty) + c(x, y)[\, d(x, Ty) + d(y, Tx)] \tag{1.3}$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that $b > 0, c > 0$ and $a + b + 2c = 1$, was introduced and investigated by Ciric [9]. Ciric proved that in a complete metric space such mappings have a unique fixed point. Chandra et al [7] consider the following generalization of (1.3), let $T, f: X \to X$ satisfying:

$$d(Tx, Ty) \leq a(x, y)d(fx, fy) + b(x, y)max\{d(fx, Tx), d(fy, Ty)\}$$
$$+ c(x, y)[\, d(fx, Ty) + d(fy, Tx)] \tag{1.4}$$

where

$$a(x, y) \geq 0, \;\; \beta = inf_{x,y\in X}\, b(x, y) > 0, \;\; \gamma = inf_{x,y\in X}\, c(x, y) > 0$$

with

$$sup_{x,y\in X}\big(a(x, y) + b(x, y) + 2c(x, y)\big) = 1.$$

Jhade et al [12] studied the following non-expansive type condition for two self-maps $T, f: X \to X$;

$$\begin{aligned} d(Tx,Ty) \le\ & a(x,y)d(fx,fy) + b(x,y)max\{d(fx,Tx), d(fy,Ty)\} \\ & + c(x,y)max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\} \\ & + e(x,y)max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\ d(fx,Ty)\} \end{aligned} \quad (1.5)$$

where

$$a(x,y), b(x,y), c(x,y), e(x,y) \ge 0,$$

$$\beta = inf_{x,y\in X}\, e(x,y) > 0$$

$$\gamma = inf_{x,y\in X}\big(1 + b(x,y) + e(x,y)\big) > 0$$

with

$$sup_{x,y\in X}\big(a(x,y) + b(x,y) + c(x,y) + 2e(x,y)\big) = 1.$$

In 1965, Zadeh [25] introduced the concept of a fuzzy set as a new way to represent vagueness in everyday life. The study of fixed point theorems in fuzzy mathematics was investigated by Weiss [24], Butnariu [5], Singh and Talwar [20], Mihet [14], Qiu et al. [16], and Beg and Abbas [2] and many others. Heilpern [11] first used the concept of fuzzy mappings to prove the Banach contraction principle for fuzzy (approximate quantity-valued) mappings on a complete metric linear spaces. The result obtained by Heilpern [11] is a fuzzy analogue of the fixed point theorem for multi-valued mappings of Nadler et al. [15]. Bose and Sahani [4], Vijayaraju and Marudai [21], improved the result of Heilpern. In some earlier work, Watson and Rhoades [22], [23] proved several fixed point theorems involving a very general contractive definition.

In this paper, we establish a common fixed point theorem for fuzzy maps satisfying non-expansive type condition on complete linear metric space. Also, a common fixed point theorem for sequence of fuzzy mappings satisfying non-expansive type condition.

2. PRELIMINARIES

In this paper, we shall generally follow the notations of Heilpern [11].

Definition 2.1 Let (X, d) be a complete linear metric space and $\mathcal{F}(X)$, the collection of all fuzzy sets in X. A fuzzy set in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of A is denoted by

$$A_\alpha = \{x: A(x) \ge \alpha\} \text{ if } \alpha \in (0,1]$$

$$A_0 = \overline{\{x: A(x) > 0\}},$$

where $\bar{B}$ stands for the (non-fuzzy) closure of a set B.

Definition 2.2 A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in (0,1]$ and $\sup_{x\in X} A(x) = 1$, when A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0. From the collection $\mathcal{F}(X)$, a sub-collection of all appropriate quantities is denoted as $\mathcal{W}(X)$.

Definition 2.3 The distance between two appropriate quantities is defined by the following scheme. Let $A, B \in \mathcal{W}(X)$ and $\alpha \in [0,1]$,

$$D_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y);$$

$$H_\alpha(A,B) = dist\, d(A_\alpha, B_\alpha);$$

$$H(A,B) = \sup_\alpha D_\alpha(A,B);$$

wherein the dist is in the sense of Hausdorff distance .The function D_α is called an α-distance (induced by d), H_α a α- distance (induced by dist) and H a distance between A and B. Note that D_α is a non-decreasing function of α.

Definition 2.4 Let $A, B \in \mathcal{W}(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$, iff $A(x) \leq B(x)$ for each $x \in X$. The relation $\subset$ induces a partial ordering on the family $\mathcal{W}(X)$.

Definition 2.5 Let Y be an arbitrary set and X be any metric space. F is called a fuzzy mapping if and only if F is a mapping from the set Y into $\mathcal{W}(X)$. A fuzzy mapping F is a fuzzy subset of $Y \times X$ with membership function $F(y, x)$. The function value $F(y, x)$ is the grade of membership of x in $F(y)$. Note that each fuzzy mapping is a set valued mapping. Let $A \in F(X), B \in F(Y)$. Then he fuzzy set $F(A)$ in $F(X)$ is defined by

$$F(A)(x) = \sup_{y \in X}(F(y, x) \wedge A(y)), \mathrm{x} \in \mathrm{X}$$

and the fuzzy set $F^{-1}(B)$ in $F(Y)$ is defined by

$$F^{-1}(B)\,(y) = \sup_{x \in X} F(y, x) \wedge B(x), \mathrm{y} \in \mathrm{Y}$$

Lee [13] proved the following.

Lemma 2.6 Let (X, d) be a complete linear metric space, F is a fuzzy mapping from X into $\mathcal{W}(X)$ and $x_0 \in X$, then there exists an $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

The following two lemmas are due to Heilpern [11].

Lemma 2.7 Let $x \in X$, $A \in \mathcal{W}(X)$ and $\{x\}$ a fuzzy set with membership function equal to a characteristic function of $\{x\}$. If $\{x\} \subset A$, then $D_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.

Lemma 2.8 Let $A, B \in \mathcal{W}(X)$, $\alpha \in [0,1]$ and $D_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y)$, where $A_\alpha = \{x: A(x) \geq \alpha\}$, then

$$D_\alpha(x, A) \leq d(x, y) + D_\alpha(y, A)$$

for each $x, y \in X$.

Lemma 2.9 Let $H_\alpha(A, B) = \mathrm{dist}\, d(A_\alpha, B_\alpha)$, where 'dist' is the Hausdorff distance. If $\{x_0 \subset A\}$, then $D_\alpha(x_0, B) \leq H_\alpha(A, B)$ for each $B \in \mathcal{W}(X)$.

Rhoades [18] proved the following common fixed point theorem involving a very general contractive condition, for fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.10 Let (X, d) be a complete linear metric space and let F, G be fuzzy mappings from X into $\mathcal{W}(X)$ satisfying

$$H(Fx, Gy) \leq Q(m(x, y)), \text{ for all } x, y \in X, \tag{2.1}$$

where

$$m(x, y) = \max\left\{d(x, y), D_\alpha(x, Fx), D_\alpha(y, Gy), \frac{D_\alpha(x, Gy) + D_\alpha(y, Fx)}{2}\right\}$$

and Q is a real-valued function defined on D, the closure of the range of d, satisfying the following three conditions:

a) $0 < Q(s) < s$ for each $s \in D\backslash\{0\}$ and $Q(0) = 0$,

b) Q is non-decreasing on D, and

c) $g(s) = s/s - Q(s)$ is non-increasing on $D\backslash\{0\}$.

Then there exists a point z in X such that $\{z\} \subset Fz \cap Gz$.

In [17] Rhoades, generalized the result of Theorem 2.10 for sequence of fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.11 Let g be a non-expansive self-mapping of a complete linear metric space (X, d) and $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$. For each pair of fuzzy mappings F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x\}, \{v_y\}) \leq Q(m(x, y)), \text{ for all } x, y \in X, \tag{2.2}$$

where

$$m(x,y) = \max\left\{(g(x), g(y)), d(g(x), g(u_x)), d\left(g(y), g(v_y)\right), \frac{d(g(x), g(v_y)) + d(g(y), g(u_x))}{2}\right\}$$

and Q satisfying the conditions (a)-(c) of Theorem 2.10. Then there exists $\{z\} \subset \bigcap_{i=1}^{\infty} F_i(z)$

3. MAIN RESULTS

Now, we give our first main result.

Theorem 3.1 Let (X, d) be a complete linear metric space. F and G are two fuzzy mappings from X into $\mathcal{W}(X)$ satisfying:

$$\begin{aligned} H(Fx, Gy) \leq\ & a(x,y)d(x,y) + b(x,y)\, max\{D_\alpha(x, Fx), D_\alpha(y, Gy)\} \\ & + c(x,y) max\{d(x,y), D_\alpha(x, Fx), D_\alpha(y, Gy)\} \\ & + e(x,y)\, max\{d(x,y), D_\alpha(x, Fx), D_\alpha(y, Gy), D_\alpha(x, Gy)\} \\ & + h(x,y)\, max\{d(x,y), D_\alpha(x, Fx), D_\alpha(y, Gy), D_\alpha(x, Gy), D_\alpha(y, Fx)\} \end{aligned} \quad (3.1)$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$, $e(x,y)$, $h(x,y)$ are non-negative real functions from $X \times X$ into $[0, +\infty)$ such that

$$\beta = inf_{x,y\in X}\big(e(x,y) + h(x,y)\big) > 0 \quad (3.2)$$

$$\gamma = inf_{x,y\in X}\big(b(x,y) + e(x,y) + h(x,y)\big) > 0 \quad (3.3)$$

with

$$sup_{x,y\in X}\big(a(x,y) + b(x,y) + c(x,y) + 2e(x,y) + 2h(x,y)\big) = 1. \quad (3.4)$$

Then there exists a point z in X, which is a common fixed point of F and G, i.e. $\{z\} \subset Fz \cap Gz$.

Proof. Pick x_0 in X, then by Lemma 2.6, we can choose $x_1 \in X$ such that $\{x_1\} \subset Fx_0$. Choose $x_2 \in X$ such that $\{x_2\} \subset Gx_1$ and $d(x_1, x_2) \leq H(Fx_0, Gx_1)$. Continuing the process, we obtain a sequence $\{x_n\}$ such that $\{x_{2n+1}\} \subset Fx_{2n}$, $\{x_{2n+2}\} \subset Gx_{2n+1}$ such that $d(x_{2n+1}, x_{2n+2}) \leq H(Fx_{2n}, Gx_{2n+1})$, where $n = 0,1,2, \ldots$. Applying (3.1) and using triangle inequality, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) \leq\ & H(Fx_{2n}, Gx_{2n+1}) \\ \leq\ & ad(x_{2n}, x_{2n+1}) + b\, max\{D_\alpha(x_{2n}, Fx_{2n}), D_\alpha(x_{2n+1}, Gx_{2n+1})\} \\ & + c\, max\{d(x_{2n}, x_{2n+1}), D_\alpha(x_{2n}, Fx_{2n}), D_\alpha(x_{2n+1}, Gx_{2n+1})\} \\ & + e\, max\{d(x_{2n}, x_{2n+1}), D_\alpha(x_{2n}, Fx_{2n}), D_\alpha(x_{2n+1}, Gx_{2n+1}) \\ & , D_\alpha(x_{2n}, Gx_{2n+1})\} \\ & + h\, max\{d(x_{2n}, x_{2n+1}), D_\alpha(x_{2n}, Fx_{2n}), D_\alpha(x_{2n+1}, Gx_{2n+1}) \\ & , D_\alpha(x_{2n}, Gx_{2n+1}), D_\alpha(x_{2n+1}, Fx_{2n})\} \\ \leq\ & ad(x_{2n}, x_{2n+1}) + b\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ & + c\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ & + e\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} \\ & + h\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}) \\ & , d(x_{2n+1}, x_{2n+1})\} \\ \leq\ & ad(x_{2n}, x_{2n+1}) + (b + c)\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ & +(e + h)\, max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ & , d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\} \end{aligned}$$

where a, b, c, e and h are evaluated at (x_{2n}, x_{2n+1}).

If for some n, $d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1})$. The last inequality gives

$$d(x_{2n+1}, x_{2n+2}) < (a + b + c + 2e + 2h)d(x_{2n}, x_{2n+1})$$

a contradiction. Therefore, for all n, we have

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$$

Hence, for all positive integers n,

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_0, x_1) \quad (3.5)$$

Again applying (3.1) and triangle inequality, we have

$$\begin{aligned} d(x_2, x_3) &\leq H(Fx_1, Gx_2) \\ &\leq ad(x_1, x_2) + b\, max\{D_\alpha(x_1, Fx_1), D_\alpha(x_2, Gx_2)\} \\ &+ c\, max\{d(x_1, x_2), D_\alpha(x_1, Fx_1), D_\alpha(x_2, Gx_2)\} \\ &+ e\, max\{d(x_1, x_2), D_\alpha(x_1, Fx_1), D_\alpha(x_2, Gx_2), D_\alpha(x_1, Gx_2)\} \\ &+ h\, max\{d(x_1, x_2), D_\alpha(x_1, Fx_1), D_\alpha(x_2, Gx_2), D_\alpha(x_1, Gx_2) \\ &, D_\alpha(x_2, Fx_1)\} \\ &\leq ad(x_1, x_2) + b\, max\{d(x_1, x_2), d(x_2, x_3)\} \\ &+ c\, max\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3)\} \\ &+ e\, max\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\} \\ &+ h\, max\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_3), d(x_2, x_2)\} \end{aligned}$$

where a, b, c, e and h are evaluated at (x_1, x_2). Using (3.5), we have

$$\begin{aligned} d(x_2, x_3) &\leq ad(x_0, x_1) + b\, max\{d(x_0, x_1), d(x_0, x_1)\} \\ &+ c\, max\{d(x_0, x_1), d(x_0, x_1), d(x_0, x_1)\} \\ &+ e\, max\{d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_3)\} \\ &+ h\, max\{d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_3)\} \\ &= (a + b + c)d(x_0, x_1) \\ &+ (e + h)\, max\{d(x_0, x_1), d(x_1, x_3)\} \end{aligned} \quad (3.6)$$

Applying (3.1) again, we have

$$\begin{aligned} d(x_1, x_3) &\leq H(Fx_0, Gx_2) \\ &\leq ad(x_0, x_2) + b\, max\{D_\alpha(x_0, Fx_0), D_\alpha(x_2, Gx_2)\} \\ &+ c\, max\{d(x_0, x_2), D_\alpha(x_0, Fx_0), D_\alpha(x_2, Gx_2)\} \\ &+ e\, max\{d(x_0, x_2), D_\alpha(x_0, Fx_0), D_\alpha(x_2, Gx_2), D_\alpha(x_0, Gx_2)\} \\ &+ h\, max\{d(x_0, x_2), D_\alpha(x_0, Fx_0), D_\alpha(x_2, Gx_2), D_\alpha(x_0, Gx_2), D_\alpha(x_2, Fx_0)\} \\ &\leq ad(x_0, x_2) + b\, max\{d(x_0, x_1), d(x_2, x_3)\} \\ &+ c\, max\{d(x_0, x_2), d(x_0, x_1), d(x_2, x_3)\} \\ &+ e\, max\{d(x_0, x_2), d(x_0, x_1), d(x_2, x_3), d(x_0, x_3)\} \\ &+ h\, max\{d(x_0, x_2), d(x_0, x_1), d(x_2, x_3), d(x_0, x_3), d(x_2, x_1)\} \end{aligned} \quad (3.7)$$

where a, b, c, e and h are evaluated at (x_0, x_2). Since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \leq 2d(x_0, x_1) \\ d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_3) \\ &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \end{aligned}$$

$$\le 3d(x_0, x_1)$$

Using (3.5) and (3.7), we have

$$d(x_1, x_3) \le (2a + b + 2c + 3e + 3h)d(x_0, x_1)$$

Implies that

$$d(x_1, x_3) \le (2 - b - e - h)d(x_0, x_1)$$

Hence, from (3.7)

$$\begin{aligned} d(x_2, x_3) &\le ad(x_0, x_1) + bd(x_0, x_1) + cd(x_0, x_1) \\ &\quad + (e + h)\,(2 - b - e - h)d(x_0, x_1) \\ &= \big(a + b + c + (e + h)\,(2 - b - e - h)\big)d(x_0, x_1) \\ &= \big(1 - (e + h)\,(b + e + h)\big)d(x_0, x_1) \\ &\le (1 - \beta\gamma)d(x_0, x_1) \end{aligned}$$

It is easy to show that

$$d(x_n, x_{n+1}) \le (1 - \beta\gamma)^{\left[\frac{n}{2}\right]} d(x_0, x_1) \tag{3.8}$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Also, since $\beta\gamma > 0$, from (3.8), we have $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there is a point $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z. \tag{3.9}$$

Since $\alpha \in [0,1]$, then using Lemma 2.8 and Lemma 2.9, we have

$$\begin{aligned} D_\alpha(z, Fz) &\le d(z, Gx_n) + D_\alpha(Gx_n, Fz) \\ &\le d(z, Gx_n) + H_\alpha(Fz, Gx_n) \\ &\le d(z, Gx_n) + H(Fz, Gx_n) \end{aligned}$$

Taking limit as $n \to +\infty$, we get

$$D_\alpha(z, Fz) \le \lim_{n\to+\infty} D_\alpha(Fz, Gx_n) \le \lim_{n\to+\infty} H(Fz, Gx_n) \tag{3.10}$$

Again from (3.1), we have

$$\begin{aligned} H(Fz, Gx_n) &\le ad(z, x_n) + b\, max\{D_\alpha(z, Fz), D_\alpha(x_n, Gx_n)\} \\ &\quad + c\, max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n)\} \\ &\quad + e\, max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n), D_\alpha(z, Gx_n)\} \\ &\quad + h\, max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n), D_\alpha(z, Gx_n), D_\alpha(x_n, Fz)\} \\ &\le \sup_{x,y\in X}(a + b + c + e + h) \max\{d(z, x_n), max\{D_\alpha(z, Fz), D_\alpha(x_n, Gx_n)\} \\ &\quad , max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n)\} \\ &\quad , max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n), D_\alpha(z, Gx_n)\} \\ &\quad , max\{d(z, x_n), D_\alpha(z, Fz), D_\alpha(x_n, Gx_n), D_\alpha(z, Gx_n), D_\alpha(x_n, Fz)\} \end{aligned}$$

Letting limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} H(Fz, Gx_n) \le \sup_{x,y\in X}(a + b + c + e + h)\, D_\alpha(z, Fz) = D_\alpha(z, Fz) \tag{3.11}$$

Using (3.10) and (3.11), we have

$$D_\alpha(z, Fz) \le D_\alpha(z, Fz)$$

a contradiction. Hence we must have $D_\alpha(z, Fz) = 0$. Since α is arbitrary number in $[0,1]$. It follows that $D(z, Fz) = 0$, which implies that $\{z\} \subset Fz$. Similarly it can be shown that $\{z\} \subset Gz$. Hence $\{z\} \subset Fz \cap Gz$.

Now, we prove a common fixed point theorem for sequence of fuzzy mappings of non-expansive condition.

Theorem 3.2 Let g be a non-expansive self-mapping of a complete linear metric space (X, d) and $\{F_i\}$ be a sequence of fuzzy mappings from X into $\mathcal{W}(X)$. For each pair of fuzzy mappings F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$\begin{aligned} D(\{u_x\},\{v_y\}) \leq\ & ad(g(x),g(y)) + b\, max\left\{d(g(x),g(u_x)), d\left(g(y),g(v_y)\right)\right\} \\ & + c\, max\left\{(g(x),g(y)), d(g(x),g(u_x)), d\left(g(y),g(v_y)\right)\right\} \\ & +e\, max\left\{(g(x),g(y)), d(g(x),g(u_x)), d\left(g(y),g(v_y)\right), d\left(g(x),g(v_y)\right)\right\} \\ & + h\, max\left\{(g(x),g(y)), d(g(x),g(u_x)), d\left(g(y),g(v_y)\right), d\left(g(x),g(v_y)\right)\right. \\ & \left., d(g(y),g(u_x))\right\} \end{aligned} \tag{3.12}$$

where a, b, c, d, e are non-negative real number such that $\beta = e + h > 0$ and $\gamma = b + e + h > 0$ with $a + b + c + 2e + 2h = 1$.Then there exists a point z in X, which is a common fixed point of sequence of fuzzy mappings, i.e. $\{z\} \subset \bigcap_{i=1}^{\infty} F_i(z)$.

Proof. Choose $x_0 \in X$, then by Lemma 2.6, we can choose $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$. From the hypothesis, there exists an $x_2 \in X$ such that $\{x_2\} \subset F(x_1)$. In general, choose $x_{n+1} \in X$ such that $\{x_{n+1}\} \subset F_{n+1}(x_n)$.

Applying (3.12), we have

$$\begin{aligned} D(\{x_n\},\{x_{n+1}\}) \leq\ & ad(g(x_{n-1}),g(x_n)) \\ & +b\, max\{d(g(x_{n-1}),g(x_n)), d(g(x_n),g(x_{n+1}))\} \\ & + c\, max\{d(g(x_{n-1}),g(x_n)), d(g(x_{n-1}),g(x_n)), d(g(x_n),g(x_{n+1}))\} \\ & +e\, max\{d(g(x_{n-1}),g(x_n)), d(g(x_{n-1}),g(x_n)), d(g(x_n),g(x_{n+1})) \\ & , d(g(x_{n-1}),g(x_{n+1}))\} \\ & +h\, max\{d(g(x_{n-1}),g(x_n)), d(g(x_{n-1}),g(x_n)), d(g(x_n),g(x_{n+1})) \\ & , d(g(x_{n-1}),g(x_{n+1})), d(g(x_n),g(x_n))\} \end{aligned}$$

Since g is a non-expansive self-mapping and $D(\{x_n\},\{x_{n+1}\}) = d(x_n, x_{n+1})$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &= D(\{x_n\},\{x_{n+1}\}) \\ & \leq ad(x_{n-1},x_n) + b\, max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \\ & + c\, max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\} \\ & +e\, max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1})\} \\ & +h\, max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), d(x_n,x_n)\} \end{aligned}$$

If $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some n, then by using triangle inequality, the last inequality gives

$$d(x_n, x_{n+1}) \leq (a + b + c + 2e + 2h)d(x_n, x_{n+1})$$

a contradiction. Thus $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Hence, for all positive integers n,

$$d(x_n, x_{n+1}) \leq d(x_0, x_1) \tag{3.13}$$

Again applying (3.12) and using (3.13), we have

$$\begin{aligned} D(\{x_2\},\{x_3\}) \leq\ & ad(g(x_1),g(x_2)) + b\, max\{d(g(x_1),g(x_2)), d(g(x_2),g(x_3))\} \\ & + c\, max\{d(g(x_1),g(x_2)), d(g(x_1),g(x_2)), d(g(x_2),g(x_3))\} \\ & +e\, max\{d(g(x_1),g(x_2)), d(g(x_1),g(x_2)), d(g(x_2),g(x_3)) \end{aligned}$$

$$, d\big(g(x_1), g(x_3)\big)\}$$
$$+ h\, max\{d\big(g(x_1), g(x_2)\big), d\big(g(x_1), g(x_2)\big), d\big(g(x_2), g(x_3)\big)$$
$$, d\big(g(x_1), g(x_3)\big), d\big(g(x_2), g(x_2)\big)\}$$

Since g is a non-expansive self-mapping and $D(\{x_2\},\{x_3\}) = d(x_2, x_3)$, we get

$$d(x_2, x_3) = D(\{x_2\},\{x_3\})$$
$$\leq ad(x_1, x_2) + (b + c)\, max\{d(x_1, x_2), d(x_2, x_3)\}$$
$$+(e + h)\, max\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3)\} \quad (3.14)$$

Again applying (3.12), we have

$$D(\{x_1\},\{x_3\}) \leq ad\big(g(x_0), g(x_2)\big) + b\, max\{d\big(g(x_0), g(x_2)\big), d\big(g(x_2), g(x_3)\big)\}$$
$$+ c\, max\{d\big(g(x_0), g(x_2)\big), d\big(g(x_0), g(x_2)\big), d\big(g(x_2), g(x_3)\big)\}$$
$$+e\, max\{d\big(g(x_0), g(x_2)\big), d\big(g(x_0), g(x_2)\big), d\big(g(x_2), g(x_3)\big)$$
$$, d\big(g(x_0), g(x_3)\big)\}$$
$$+ h\, max\{d\big(g(x_0), g(x_2)\big), d\big(g(x_0), g(x_2)\big), d\big(g(x_2), g(x_3)\big)$$
$$, d\big(g(x_0), g(x_3)\big), d\big(g(x_2), g(x_2)\big)\}$$

Since g is a non-expansive self-mapping and $D(\{x_1\},\{x_3\}) = d(x_1, x_3)$. By using (3.13) and triangle inequality, we get

$$d(x_2, x_3) = D(\{x_2\},\{x_3\})$$
$$\leq ad(x_0, x_2) + (b + c)\, max\{d(x_0, x_2), d(x_2, x_3)\}$$
$$+(e + h)\, max\{d(x_0, x_2), d(x_2, x_3), d(x_0, x_3)\}$$
$$\leq ad(x_0, x_2) + (b + c)\, max\{d(x_0, x_2), d(x_2, x_3)\}$$
$$+(e + h)\, max\{d(x_0, x_2), d(x_2, x_3), d(x_0, x_3)\}$$
$$\leq (2a + b + 2c + 3e + 3h) d(x_0, x_1)$$
$$= (2 - b - e - h) d(x_0, x_1) \quad (3.15)$$

Hence, from (3.14) and (3.15), we have

$$d(x_2, x_3) \leq ad(x_0, x_1) + bd(x_0, x_1) + cd(x_0, x_1)$$
$$+ (e + h)\,(2 - b - e - h) d(x_0, x_1)$$
$$= \big(a + b + c + (e + h)\,(2 - b - e - h)\big) d(x_0, x_1)$$
$$= \big(1 - (e + h)\,(b + e + h)\big) d(x_0, x_1)$$
$$\leq (1 - \beta\gamma) d(x_0, x_1)$$

It is easy to show that

$$d(x_n, x_{n+1}) \leq (1 - \beta\gamma)^{\left[\frac{n}{2}\right]} d(x_0, x_1) \quad (3.16)$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Also, since $\beta\gamma > 0$, from (3.16), we have $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there is a point $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z.$$

Let F_m be arbitrary member of $\{F_i\}$. Since $\{x_n\} \subset F_m(x_{n-1})$, by Lemma 2.6, there exists a $v_n \in X$ such that $\{v_n\} \subset F_m(z)$ for all n. Applying (3.12), we have

$$D(\{x_n\},\{v_n\}) \leq ad\big(g(x_{n-1}), g(z)\big) + b\, max\{d\big(g(x_{n-1}), g(x_n)\big), d\big(g(z), g(v_n)\big)\}$$

$$+ c\, max\{d(g(x_{n-1}), g(z)), d(g(x_{n-1}), g(x_n)), d(g(z), g(v_n))\}$$
$$+e\, max\{d(g(x_{n-1}), g(z)), d(g(x_{n-1}), g(x_n)), d(g(z), g(v_n))$$
$$, d(g(x_{n-1}), g(v_n))$$
$$+ h\, max\{d(g(x_{n-1}), g(z)), d(g(x_{n-1}), g(x_n)), d(g(z), g(v_n))$$
$$, d(g(x_{n-1}), g(v_n)), d(g(z), g(x_n))\}$$
$$\leq ad(x_{n-1}, z) + b\, max\{d(x_{n-1}, x_n), d(z, v_n)\}$$
$$+ c\, max\{d(x_{n-1}, z), d(x_{n-1}, x_n), d(z, v_n)\}$$
$$+e\, maxd\{d(x_{n-1}, z), d(x_{n-1}, x_n), d(z, v_n), d(x_{n-1}, v_n)\}$$
$$+ h\, max\{d(x_{n-1}, z), d(x_{n-1}, x_n), d(z, v_n),, d(x_{n-1}, v_n), d(z, x_n)\}$$

If $\lim_{n\to+\infty} v_n \neq z$, then letting limit as $n \to +\infty$, we have

$$d(z, v_n) \leq (a + v + c + e + h) max\{d(z, z), max\{d(z, z), d(z, v_n)\}$$
$$, max\{d(z, z), d(z, z), d(z, v_n)\}$$
$$, maxd\{d(z, z), d(z, z), d(z, v_n), d(z, v_n)\}$$
$$, max\{d(z, z), d(z, z), d(z, v_n),, d(z, v_n), d(z, z)\}\}$$
$$< d(z, v_n)$$

a contradiction. Hence

$$\lim_{n\to+\infty} v_n = z.$$

Since F_m be arbitrary, then

$$\{z\} \subset \cap_{i=1}^{\infty} F_i(z).$$

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