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(i, j)-Almost Continuity and (i, j)-Weakly Continuity in Fuzzy Bitopological Spaces

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Abstract: *Fuzzy regular open (closed) sets, fuzzy almost continuous and fuzzy weakly continuous maps on fuzzy topological spaces have been studied in [1]. In the present paper we introduce the concepts of fuzzy (i, j)-regular open (closed) sets, fuzzy (i, j)-almost continuous and fuzzy (i, j)-weakly continuous maps on fuzzy bitopological spaces. Fuzzy (i, j)-semi regular and fuzzy (i, j)-regular spaces have also been studied.*

Keywords: *Fuzzy bitopological spaces, fuzzy regular open sets, fuzzy continuous mappings, fuzzy almost continuous mappings, fuzzy weakly continuous mappings.*

1. Introduction

Fuzzy topological spaces have been developed as an extension of the classical point-set topological spaces. Zadeh in his historical paper [6] has introduced the concept of fuzzy sets. In 1968, Chang [2] introduced the concept of fuzzy topological spaces. Azad [1] in 1981 has introduced and investigated fuzzy regular open (closed) sets, fuzzy almost continuous and fuzzy weakly continuous maps in fuzzy topological spaces.

Fuzzy bitopological spaces have been introduced by Kandil [4] in 1989. Fuzzy (i, j)-semi open (closed) sets and fuzzy (i, j)-semi open (closed) maps in fuzzy bitopological spaces have been studied in [5].

In the present paper we introduce fuzzy (i, j)-regular open (closed) sets in fuzzy bitopological spaces. Significant results have been obtained. Further the concepts of fuzzy (i, j)-almost continuous maps and fuzzy (i, j)-weakly continuous maps, fuzzy (i, j)-semi regular and fuzzy (i, j)-regular spaces have also been introduced and studied in the present paper.

2. Preliminaries

Let X be a nonempty set and let I stand for the closed unit interval [0, 1]. A fuzzy set λ of X is a mapping $\lambda : X \to I$, where for any $x \in X$, λ(x) denotes the degree of membership of element x in fuzzy set λ. The null fuzzy set 0 and the whole fuzzy set 1 are the constant mappings from X to {0} and {1} respectively. The complement, union and intersection of fuzzy sets are defined as follows:

$$\lambda'(x) = 1 - \lambda(x), \qquad x \in X$$

$$\{\cup \lambda_\alpha\}(x) = Sup\{\lambda_\alpha(x) : \alpha \in \wedge\}, \quad x \in X$$

$$\{\cap \lambda_\alpha\}(x) = Inf\{\lambda_\alpha(x) : \alpha \in \wedge\}, \quad x \in X$$

where $\wedge$ is any arbitrary index set.

Let $f : X \to Y$ be a mapping and let λ be a fuzzy set in X, then the image set $f(\lambda)$ is a fuzzy set in Y defined as

$$f(\lambda)(y) = \begin{cases} \underset{x\in\{f^{-1}(y)\}}{Sup} \{\lambda(x)\}, & if\ \{f^{-1}(y)\} \neq \emptyset \\ 0, & otherwise \end{cases}$$

for each $y \in Y$. Further, if μ is a fuzzy set of Y, then $f^{-1}(\mu)$ is a fuzzy set of X defined as

$$f^{-1}(\mu)(x) = \mu(f(x))$$

for each $x \in X$.

A family τ of fuzzy sets of X is called a fuzzy topology (see [2]) on X if it satisfies the following conditions:

i) The null fuzzy set 0 and whole fuzzy set 1 belong to τ.

ii) Any union of members of τ is in τ.

iii) Any finite intersection of members of τ is in τ.

The pair (X,τ) is called a fuzzy topological space. The members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. For a fuzzy set λ of X, the interior ($Int\,\lambda$) and the closure ($Cl\,\lambda$) of λ are defined as

$$Int\,\lambda = Sup\{O : O \leq \lambda \text{ and } O \text{ is a fuzzy open set in } X\}$$

$$Cl\,\lambda = Inf\{C : C \geq \lambda \text{ and } C \text{ is a fuzzy closed set in } X\}$$

If X is a non-empty universal set, then a system (X,τ_1,τ_2) consisting of set X and two fuzzy topologies τ_1 and τ_2 on X is called a fuzzy bitopological space (see [4]).

A fuzzy set λ of (X,τ_1,τ_2) is called fuzzy (i, j)-semi open set if there exists a fuzzy τ_i −open set ν such that $\nu \leq \lambda \leq \tau_j - Cl\,\nu$ and λ is called fuzzy (i, j)-semi closed set if there exists a fuzzy τ_i −closed set μ such that $\tau_j - Int\,\mu \leq \lambda \leq \mu$ (see [5]). In this definition and in the rest of this paper we take i, j = 1, 2 & i ≠j.

A fuzzy set λ is a fuzzy (i, j)-semi open set iff $\lambda \leq \tau_j - Cl\,(\tau_i - Int\,\lambda)$ and fuzzy (i, j)-semi closed set iff $\tau_j - Int\,(\tau_i - Cl\,\lambda) \leq \lambda$ (see [5]).

3. Fuzzy (i, j)-Regular Open (Closed) Sets

Definition 3.1: A fuzzy set λ of fuzzy bitopological space (X,τ_1,τ_2) is called

i) Fuzzy (i, j)-regular open set if $\tau_j - Int\,(\tau_i - Cl\,\lambda) = \lambda$.

ii) Fuzzy (i, j)-regular closed set if $\tau_j - Cl\,(\tau_i - Int\,\lambda) = \lambda$.

Remark 3.1: (a) Every fuzzy (i, j)-regular open set is a fuzzy τ_j-open set, but converse need not be true.

(b) Every fuzzy (i, j)-regular closed set is a fuzzy τ_j-closed set, but converse need not be true.

We exemplify the remarks in the following:

Example 3.1: Let $X = \{a, b\}$ and let A, B, C, D be fuzzy sets on X defined as follows

$$A = \{(a, 0.7), (b, 0.5)\} \qquad B = \{(a, 0.5), (b, 0.4)\}$$

$$C = \{(a, 0.3), (b, 0.4)\} \qquad D = \{(a, 0.8), (b, 0.6)\}$$

Consider $\tau_1 = \{0, A, B, 1\}$ and $\tau_2 = \{0, C, D, 1\}$ as two fuzzy topologies on X. Then we find that fuzzy set C is fuzzy (1, 2)-regular open set and it is a fuzzy τ_2 −open set. Similarly fuzzy set $E = \{(a, 0.7), (b, 0.6)\} = C'$ is a fuzzy (1, 2)-regular closed set and it is a fuzzy τ_2 −closed set.

We observe that fuzzy set D is a fuzzy τ_2-open set, but it is not a fuzzy (1, 2)-regular open set because $\tau_1 - Cl\,D = 1$ and $\tau_2 - Int\,1 = 1$. Thus $\tau_2 - Int\,(\tau_1 - Cl\,D) \neq D$.

Theorem 3.1: A fuzzy set λ of a fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (i, j)-regular open set if and only if λ' is a fuzzy (i, j)-regular closed set.

Proof: Let λ be a fuzzy (i, j)-regular open set of X, so that $\tau_j - Int\,(\tau_i - Cl\,\lambda) = \lambda$. It implies that $\lambda' = \tau_j - Cl\,(\tau_i - Int\,\lambda')$, which show that λ' is a fuzzy (i, j)-regular closed set in X.

Conversely; let λ' be a fuzzy (i, j)-regular closed set of fuzzy space X, so that $\tau_j - Cl\,(\tau_i - Int\,\lambda') = \lambda'$. It implies $\tau_j - Int(\tau_i - Cl\,\lambda) = \lambda$. This proves that λ is a fuzzy (i, j)-regular open set.

Theorem 3.2: (a) The intersection of two fuzzy (i, j)-regular open sets is a fuzzy (i, j)-regular open set.

(b) The union of two fuzzy (i, j)-regular closed sets is a fuzzy (i, j)-regular closed set.

Proof: (a) Let (X, τ_1, τ_2) be a fuzzy bitopological space and let λ and μ be two fuzzy (i, j)- regular open sets in it, so that

$$\tau_j - Int\,(\tau_i - Cl\,\lambda) = \lambda \quad \text{and} \quad \tau_j - Int\,(\tau_i - Cl\,\mu) = \mu \qquad (3.2.1)$$

Thus λ and μ are fuzzy τ_j-open sets, hence $\lambda \cap \mu$ is also fuzzy τ_j-open set. We see that

$$\tau_j - Int\,(\lambda \cap \mu) = \lambda \cap \mu \leq \tau_i - Cl\,(\lambda \cap \mu)$$

Hence $$\lambda \cap \mu \leq \tau_j - Int\,(\tau_i - Cl\,(\lambda \cap \mu)) \qquad (3.2.2)$$

Further $\lambda \cap \mu \leq \lambda,\ \mu$. Therefore $\tau_i - Cl\,(\lambda \cap \mu) \leq \tau_i - Cl\lambda,\ \tau_i - Cl\,\mu$.

Hence $\tau_j - Int\,\left(\tau_i - Cl\,(\lambda \cap \mu)\right) \leq \tau_j - Int\,(\tau_i - Cl\,\lambda),\ \tau_j - Int\,(\tau_i - Cl\,\mu)$

In view of (3.2.1), we have $$\tau_j - Int\,\left(\tau_i - Cl\,(\lambda \cap \mu)\right) \leq \lambda \cap \mu \qquad (3.2.3)$$

Thus in view of (3.2.2) and (3.2.3), we have

$$\tau_j - Int\,\left(\tau_i - Cl\,(\lambda \cap \mu)\right) = \lambda \cap \mu$$

Therefore $\lambda \cap \mu$ is fuzzy (i, j)-regular open set in X. Similarly we can prove (b).

Remark 3.2: Result (a) and (b) of Theorem 3.2 can be generalized to any finite number of fuzzy sets $\lambda_1, \lambda_2, \dots, \lambda_n$.

Theorem 3.3: In a fuzzy bitopological space (X, τ_1, τ_2),

(a) The τ_j-closure of a fuzzy τ_i $-$open set is a fuzzy (i, j)-regular closed set.

(b) The τ_j $-$interior of a fuzzy τ_i $-$closed set is a fuzzy (i, j)-regular open set.

Proof : We prove (a). Part (b) can be proved in a similar manner.

(a) Let λ be a fuzzy τ_i $-$open set of (X, τ_1, τ_2). Consider the set $\theta = \tau_j - Cl\,\lambda$. We show that $\tau_j - Cl(\tau_i - Int\,\theta) = \theta$. Now $\lambda \leq \tau_j - Cl\,\lambda$ and $\tau_i - Int\,\lambda = \lambda$, so that

$$\tau_i - Int\,\left(\tau_j - Cl\,\lambda\right) \leq \tau_j - Cl\,\lambda.$$

Hence $$\tau_j - Cl(\tau_i - Int\left(\tau_j - Cl\,\lambda\right)) \leq \tau_j - Cl(\tau_j - Cl\,\lambda) \equiv \tau_j - Cl\,\lambda$$

Thus $$\tau_j - Cl(\tau_i - Int\left(\tau_j - Cl\,\lambda\right)) \leq \tau_j - Cl\,\lambda \qquad (3.3.1)$$

Now we know $\lambda \leq \tau_j - Cl\,\lambda$.

Then $\tau_i - Int\,\lambda \leq \tau_i - Int\left(\tau_j - Cl\,\lambda\right) \Rightarrow \lambda \leq \tau_i - Int(\tau_j - Cl\,\lambda)$

Therefore $$\tau_j - Cl\,\lambda \leq \tau_j - Cl(\tau_i - Int\left(\tau_j - Cl\,\lambda\right)) \qquad (3.3.2)$$

The inequalities (3.3.1) and (3.3.2) imply that

$$\tau_j - Cl(\tau_i - Int(\tau_j - Cl\,\lambda)) = \tau_j - Cl\,\lambda$$

Thus τ_j −closure of a fuzzy τ_i −open set is a fuzzy (i, j)-regular closed set.

4. Fuzzy (i, j) –Almost Continuous Mapping

Let $(X,\ \tau_1,\ \tau_2)$ and $(X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ are two fuzzy bitopological spaces. We recall that a mapping $f : (X,\ \tau_1,\ \tau_2) \to (X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ is said to be a

(i) fuzzy (i, j)-continuous map if maps $f : (X,\ \tau_1) \to (X^*,\ \tau_1{}^*)$ and $f : (X,\ \tau_2) \to (X^*,\ \tau_2{}^*)$ are fuzzy continuous maps.

(ii) fuzzy (i, j)-semi continuous map if pre-image of every fuzzy $\tau_i{}^*$ −open set in X^* is a fuzzy (i, j)-semi open set in X.

Now we proceed to define a fuzzy (i, j)-almost continuous map from one bitopological space to another bitopological space.

Definition 4.1: A map $f : (X,\ \tau_1,\ \tau_2) \to (X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ is said to be a fuzzy (i, j)-almost continuous map if $f^{-1}(\lambda)$ is a fuzzy τ_j–open set in X for every fuzzy (i, j)-regular open set λ in X^*.

Example 4.1: Let $X = \{x, y\}$ and $X^* = \{a, b\}$ and let $A,\ B,\ C,\ A^*, B^*, C^*$ be the fuzzy sets defined as follows :

$$A = \{(x, 0.6), (y, 0.5)\} \qquad B = \{(x, 0.3), (y, 0.5)\} \qquad C = \{(x, 0.5), (y, 0.4)\}$$

$$A^* = \{(a, 0.6), (b, 0.5)\} \qquad B^* = \{(a, 0.3), (b, 0.5)\} \qquad C^* = \{(a, 0.2), (b, 0.4)\}$$

Consider $\tau_1 = \{0, A, C, 1\}$ and $\tau_2 = \{0, B, 1\}$ as two fuzzy topologies on X and $\tau_1{}^* = \{0, A^*, 1\}$ and $\tau_2{}^* = \{0, B^*, C^*, 1\}$ as two fuzzy topologies on X^*. Let $f : (X,\ \tau_1,\ \tau_2) \to (X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ be a map defined as $f(x) = a$ and $f(y) = b$. Then we see 0, 1 and B^* are fuzzy (1, 2)-regular open sets in X^*. Also $f^{-1}(0) \equiv 0$, $f^{-1}(1) \equiv 1$ and $f^{-1}(B^*) \equiv B$ are fuzzy τ_2-open sets in X. Hence f is a fuzzy (1, 2)-almost continuous map.

Similarly we observe that fuzzy sets 0, 1 and A^* are the only fuzzy (2, 1)-regular open sets in X^* and their pre-images are fuzzy τ_1-open sets in X. Hence f is also a fuzzy (2, 1)-almost continuous map. Thus f is a fuzzy (i, j)-almost continuous map.

Theorem 4.1: Let $(X,\ \tau_1,\ \tau_2)$ and $(X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ be two fuzzy bitopological spaces and let $f : X \to X^*$ be a map, then following statements are equivalent :

(a) f is fuzzy (i, j)-almost continuous map.

(b) $f^{-1}(\mu)$ is fuzzy τ_j-closed set in X for each fuzzy (i, j)-regular closed set μ in X^*.

(c) $f^{-1}(\lambda) \leq \tau_j - \mathrm{Int}[f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}]$ for every fuzzy $\tau_j{}^*$-open set λ of X^*.

(d) For each fuzzy $\tau_j{}^*$-closed set μ of X^*, $\tau_j - \mathrm{Cl}[f^{-1}\{\tau_j{}^* - Cl(\tau_i{}^* - Int\,\lambda)\}] \leq f^{-1}(\mu)$.

Proof : We prove the theorem in following steps:

(I) (a)⇒(b) : Suppose that the map $f : (X,\ \tau_1,\ \tau_2) \to (X^*,\ \tau_1{}^*,\ \tau_2{}^*)$ is a fuzzy (i, j)-almost continuous map. Let μ be a fuzzy (i, j)-regular closed set in X^*, then μ' is fuzzy (i, j)-regular open set in X^*. Therefore $f^{-1}(\mu')$ is a fuzzy τ_j-open set in X. Hence $(f^{-1}(\mu'))' \equiv f^{-1}(\mu')' = f^{-1}(\mu)$ is a fuzzy τ_j-closed set in X.

(II) (b)⇒(a) : Using $(f^{-1}(\lambda'))' \equiv f^{-1}(\lambda')' = f^{-1}(\lambda)$, where λ is a fuzzy (i, j)-regular open set in X^*, it can be easily seen that (b)⇒(a).

(III) (a)⇒(c) : Let $f : X \to X^*$ be a fuzzy (i, j)-almost continuous map and let λ be a fuzzy (i, j)-regular open set in X^*, so that $f^{-1}(\lambda)$ is a fuzzy τ_j–open set in X. Now $\lambda \leq \tau_i{}^* - Cl\,\lambda$. Then $\tau_j{}^* - Int\,\lambda \equiv \lambda \leq \tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)$. Hence

$$f^{-1}(\lambda) \leq f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\} \tag{4.1.1}$$

Since f is fuzzy (i, j)-almost continuous map, therefore $f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}$ is a fuzzy τ_j –open set in X in view of Theorem 3.3(b). So that

$$\tau_j - Int[f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}] = f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}$$

Hence we get $f^{-1}(\lambda) \leq \tau_j - Int[f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}]$ in view of (4.1.1).

(IV) (c)⇒(a) : It can be proved easily.

(V) (b)⇒(d): Let μ be a fuzzy $\tau_j{}^*$-closed set in X^*, so that $\mu = \tau_j{}^* - \mathrm{Cl}\,\mu$. Also $\tau_i{}^* - \mathrm{Int}\,\mu \leq \mu$. Then $\tau_j{}^* - \mathrm{Cl}(\tau_i{}^* - \mathrm{Int}\,\mu) \leq \tau_j{}^* - \mathrm{Cl}\,\mu = \mu$. Hence $f^{-1}\{\tau_j{}^* - \mathrm{Cl}(\tau_i{}^* - \mathrm{Int}\,\mu)\} \leq f^{-1}(\mu)$. Since f satisfies the property that $f^{-1}(\mu)$ is fuzzy τ_j –closed set in X, for each fuzzy (i, j)-regular closed set μ in X^*, we conclude that $f^{-1}\{\tau_j{}^* - \mathrm{Cl}(\tau_i{}^* - \mathrm{Int}\,\mu)\}$ is a fuzzy τ_j -closed set in X. Hence $\tau_j - Cl[f^{-1}\{\tau_j{}^* - \mathrm{Cl}(\tau_i{}^* - \mathrm{Int}\,\mu)\}] = f^{-1}\{\tau_j{}^* - \mathrm{Cl}(\tau_i{}^* - \mathrm{Int}\,\mu)\} \leq f^{-1}(\mu)$.

(VI) (d)⇒(b) : It can be proved easily.

This completes the proof of the Theorem.

Theorem 4.2: If a map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ is a fuzzy (i, j)-continuous map, then it is a fuzzy (i, j)-almost continuous map.

Proof: Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ be a fuzzy (i, j)-continuous map so that maps $f : (X, \tau_1) \to (X^*, \tau_1{}^*)$ and $f : (X, \tau_2) \to (X^*, \tau_2{}^*)$ are continuous. Now if θ is fuzzy (i, j)-regular open set in X^*, then θ is a fuzzy $\tau_j{}^*$ –open set in X^*. Since f is a fuzzy (i, j)-continuous map, therefore $f^{-1}(\theta)$ is τ_j –open set in X. Hence map $f : X \to X^*$ is fuzzy (i, j)-almost continuous map.

Remark 4.1: The converse of Theorem 4.2 may not be true i.e. every fuzzy (i, j)-almost continuous map is not necessarily a fuzzy (i, j)-continuous map.

Example 4.2: Considering Example 4.1, we see that f is a fuzzy (1, 2)-almost continuous map and also a fuzzy (2, 1)-almost continuous map. But we observe that the mapping $f : (X, \tau_2) \to (X^*, \tau_2{}^*)$ is not a continuous map because $f^{-1}(C^*) \notin \tau_2$, where C^* is a fuzzy $\tau_2{}^*$-open set in X^* and thus the map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ is not fuzzy (i, j)-continuous map.

Theorem 4.3: Fuzzy (i, j)-semi continuity and fuzzy (i, j)-almost continuity are independent notions.

Following two examples justify the statement of the theorem.

Example 4.3: Referring to Example 4.1, we see that f is a fuzzy (1, 2)-almost continuous map, but it is not a fuzzy (1, 2)-semi continuous map because $f^{-1}(A^*) \equiv A$ is not a fuzzy (1, 2)-semi open set in X, where A^* is a fuzzy $\tau_1{}^*$ –open set in X^*.

Similarly f is not a fuzzy (2, 1)-semi continuous map because $f^{-1}(B^*) \equiv B$ is not a fuzzy (2, 1)-semi open set in X, B^* being $\tau_2{}^*$ –open set in X^*.

Example 4.4: Let $X = \{x, y\}$ and $X^* = \{a, b\}$ and let A, B, A^*, B^* be the fuzzy sets defined as follows

$$A = \{(x, 0.4), (y, 0.5)\} \qquad B = \{(x, 0.2), (y, 0.4)\}$$

$$A^* = \{(a, 0.6), (b, 0.5)\} \qquad B^* = \{(a, 0.3), (b, 0.5)\}$$

Consider $\tau_1 = \{0, A, 1\}$ and $\tau_2 = \{0, B, 1\}$ as two fuzzy topologies on X and $\tau_1{}^* = \{0, A^*, 1\}$ and $\tau_2{}^* = \{0, B^*, 1\}$ be two fuzzy topologies on X^*. Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ be a map defined as $f(x) = a$ and $f(y) = b$. We observe that map f is a fuzzy (i, j)-semi continuous map, but it is not a fuzzy (i, j)-almost continuous map.

We see that the map f is a fuzzy (1, 2)-semi continuous map because $f^{-1}(0) \equiv 0$, $f^{-1}(1) \equiv 1$ and $f^{-1}(A^*) \equiv \{(x, 0.6), (b, 0.5)\}$ are fuzzy (1, 2)-semi open sets in X for $\tau_1{}^*$ −open sets 0, 1 and A^* in X^*. Similarly we observe that the map f is a fuzzy (2, 1)-semi continuous map.

But we observe that f is not a fuzzy (2, 1)-almost continuous map because $f^{-1}(A^*)$ is not a fuzzy τ_1 −open set in X for fuzzy (2, 1)-regular open set A^* in X^*. Also f is not a fuzzy (1, 2)-almost continuous map because $f^{-1}(B^*)$ is not a fuzzy τ_2 −open set in X for fuzzy (1, 2)-regular open set B^* in X^*. Thus the map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ is not a fuzzy (i, j)-almost continuous map.

5. Fuzzy (i, j)-Semi Regular & Fuzzy (i, j)-Regular Spaces

Definition 5.1: Fuzzy (i, j)-semi regular space : A fuzzy bitopological space (X, τ_1, τ_2) is called a fuzzy (i, j)-semi regular space if and only if the collection of all fuzzy (i, j)-regular open sets of X forms a base for fuzzy topology τ_j of X with i, j = 1, 2.

Example 5.1: Let $X = \{a, b\}$ and let A, B, C, D, E, F, G, H be fuzzy sets on X defined as follows:

$$A = \{(a, 0.4), (b, 0.7)\} \qquad B = \{(a, 0.6), (b, 0.3)\}$$

$$C = A \cup B = \{(a, 0.6), (b, 0.7)\} \qquad D = A \cap B = \{(a, 0.4), (b, 0.3)\}$$

$$E = \{(a, 0.5), (b, 0.2)\} \qquad F = \{(a, 0.3), (b, 0.6)\}$$

$$G = E \cup F = \{(a, 0.5), (b, 0.6)\} \qquad H = E \cap F = \{(a, 0.3), (b, 0.2)\}$$

Consider $\tau_1 = \{0, A, B, C, D, 1\}$ and $\tau_2 = \{0, E, F, G, H, 1\}$ as two fuzzy topologies on X. Since fuzzy sets $0, 1, E, F, G$ and H are the only fuzzy (1, 2)-regular open sets of X and we can write each of fuzzy τ_2 −open sets as union of some of these six fuzzy (1, 2)-regular open sets. Hence fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (1, 2)-semi regular space.

Similarly fuzzy sets $0, 1, A, B, C$ and D are the only fuzzy (2, 1)-regular open sets of X and we can write each of fuzzy τ_1 -open sets as union of some of these six fuzzy (2, 1)-regular open sets, which form a base for fuzzy topology τ_1 of X. Hence fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (2, 1)-semi regular space.

Thus fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (i, j)-semi regular space.

Theorem 5.1: Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ be a map from fuzzy bitopological space (X, τ_1, τ_2) to a fuzzy (i, j)-semi regular space $(X^*, \tau_1^*, \tau_2^*)$. Then the map f is fuzzy (i, j)-almost continuous if and only if f is fuzzy (i, j)-continuous map.

Proof: We know (by Theorem 4.2) that any fuzzy (i, j)-continuous map from one fuzzy bitopological space to another is a fuzzy (i, j)-almost continuous map. Therefore to prove the theorem, it is sufficient to show that if $(X^*, \tau_1^*, \tau_2^*)$ is fuzzy (i, j)-semi regular space and map f is a fuzzy (i, j)-almost continuous map, then it is a fuzzy (i, j)-continuous map. Suppose that $(X^*, \tau_1^*, \tau_2^*)$ is fuzzy (i, j)-semi regular space and the map $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1^*, \tau_2^*)$ is fuzzy (i, j)-almost continuous map. Let λ be a fuzzy τ_j^* −open set in X^*. Then λ is the union of a collection of fuzzy (i, j)-regular open sets $\{\lambda_\alpha\}_{\alpha \in \wedge}$ in X^*, where $\wedge$ is an arbitrary index set. Thus $\lambda = \cup_\alpha \lambda_\alpha$. Since each λ_α is fuzzy (i, j)-regular open set, we have $\lambda_\alpha = \tau_j{}^* - Int(\tau_i{}^* - Cl\, \lambda_\alpha),\ \forall\, \alpha \in \wedge$.

Therefore $f^{-1}(\lambda) = f^{-1}(\cup_\alpha \lambda_\alpha) = \cup_\alpha f^{-1}(\lambda_\alpha)$. Then

$f^{-1}(\lambda) \leq \cup_\alpha \tau_j - Int[f^{-1}\{\tau_j^* - Int(\tau_i{}^* - Cl\,\lambda_\alpha)\}] = \cup_\alpha \tau_j - Int[f^{-1}(\lambda_\alpha)]$ in view of Theorem 4.1(c). This implies $f^{-1}(\lambda) \leq \tau_j - Int[\cup_\alpha f^{-1}(\lambda_\alpha) = \tau_j - Int[f^{-1}(\cup_\alpha \lambda_\alpha)] = \tau_j - Int[f^{-1}(\lambda)]$.

Thus $f^{-1}(\lambda) \leq \tau_j - Int[f^{-1}(\lambda)]$. Therefore, we clearly have

$$\tau_j - Int[f^{-1}(\lambda)] \leq f^{-1}(\lambda) \leq \tau_j - Int[f^{-1}(\lambda)]$$

which shows that $f^{-1}(\lambda)$ is a fuzzy τ_j −open set in X. Thus when X^* is a fuzzy (i, j)-semi regular space and λ is a fuzzy τ_j^* −open set in X^*, then $f^{-1}(\lambda)$ is a fuzzy τ_j −open set in X for j = 1, 2. Therefore f is a fuzzy (i, j)-continuous map from (X, τ_1, τ_2) to $(X^*, \tau_1^*, \tau_2^*)$.

Definition 5.2: Fuzzy (i, j)-regular space : A fuzzy bitopological space (X, τ_1, τ_2) is called a fuzzy (i, j)-regular bitopological space if and only if each fuzzy τ_j −open set λ is a union of fuzzy τ_j −open sets λ_α of X such that $\tau_i - Cl\,\lambda_\alpha \leq \lambda$.

Example 5.2: Let λ and μ be two fuzzy sets of X defined as follows

$$\lambda(x) = x \quad \text{and} \quad \mu(x) = 1 - x, \forall x \in X$$

Consider fuzzy topologies τ_1 = {0, λ, 1} and τ_2 = {0, μ, 1} on X. It is clear that 0, μ and 1 are fuzzy τ_2 −open sets of X. Then we see that

$0 = 0 \cup 0$,	and	$\tau_1 - Cl\,0 = 0 \leq 0$
$\mu = 0 \cup \mu$,	and	(i) $\tau_1 - Cl\,0 = 0 \leq \mu$ and
		(ii) $\tau_1 - Cl\,\mu = \lambda' = \mu$
1 = 0 ∪ μ ∪ 1,	and	(i) $\tau_1 - Cl\,0 = 0 \leq 1$,
		(ii) $\tau_1 - Cl\,\mu = \lambda' \leq 1$, and
		(iii) $\tau_1 - Cl\,1 = 1 \leq 1$

Thus conditions of a fuzzy (1, 2)-regular space are satisfied. Hence fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (1, 2)-regular space.

Similarly we observe that each fuzzy τ_1 −open sets 0, λ and 1 are the union of fuzzy τ_1 −open sets, which satisfy the condition of the definition. Therefore fuzzy bitopological space (X, τ_1, τ_2) is a fuzzy (2, 1)-regular space also. Hence (X, τ_1, τ_2) is a fuzzy (i, j)-regular space.

Theorem 5.2: A fuzzy (i, j)-regular bitopological space is also a fuzzy (i, j)-semi regular bitopological space.

Proof: Let (X, τ_1, τ_2) be a fuzzy (i, j)-regular bitopological space. Let λ be a fuzzy τ_j −open set in X. Suppose λ is the union of family $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ of fuzzy τ_j −open sets λ_α such that $\tau_i - Cl\,\lambda_\alpha \leq \lambda, \forall \alpha \in \Lambda$, Λ being an arbitrary index set. Then $\lambda = \cup_\alpha \lambda_\alpha,\ \alpha \in \Lambda$.

Now for each $\alpha \in \Lambda$, we have $\lambda_\alpha \leq \tau_i - Cl\,\lambda_\alpha \leq \lambda$. Then

$$\tau_j - Int\,\lambda_\alpha \equiv \lambda_\alpha \leq \tau_j - Int(\tau_i - Cl\,\lambda_\alpha) \leq \tau_j - Int\,\lambda \equiv \lambda$$

Thus $\lambda_\alpha \leq \tau_j - Int(\tau_i - Cl\,\lambda_\alpha) \leq \lambda$. Hence $\cup_{\alpha\in\Lambda}\lambda_\alpha \leq \cup_{\alpha\in\Lambda} \tau_j - Int(\tau_i - Cl\,\lambda_\alpha) \leq \lambda$.

It implies $\lambda = \cup_{\alpha\in\Lambda}\lambda_\alpha = \cup_{\alpha\in\Lambda} \tau_j - Int(\tau_i - Cl\,\lambda_\alpha) \leq \lambda$.

Thus $$\lambda = \cup_{\alpha\in\Lambda} \tau_j - Int(\tau_i - Cl\,\lambda_\alpha) \qquad (5.2.1)$$

Since we know that the fuzzy sets $\tau_j - Int(\tau_i - Cl\,\lambda_\alpha)$ is a fuzzy (i, j)-regular open set in X. Hence (5.2.1) indicates that λ is a union of fuzzy (i, j)-regular open sets of X. Therefore X is a fuzzy (i, j)-semi regular bitopological space.

Remark 5.1: A fuzzy (i, j)-semi regular space is not necessarily a fuzzy (i, j)-regular space. This is shown in the following example.

Example 5.3: Consider fuzzy bitopological space (X, τ_1, τ_2) of Example 5.1. It is a fuzzy (1, 2)-semi regular space. We note that τ_2-open fuzzy set G is the union of fuzzy τ_2-open sets E and F, but

$$\tau_1 - \mathrm{Cl}\, E = A' \nleq G \qquad \text{and} \qquad \tau_1 - \mathrm{Cl}\, F = B' \nleq G$$

Hence fuzzy (1, 2)-semi regular bitopological space (X, τ_1, τ_2) is not a fuzzy (1, 2)-regular space

6. Fuzzy (i, j)-Weakly Continuous Map

Definition 6.1: Let (X, τ_1, τ_2) and $(X^*, \tau_1{}^*, \tau_2{}^*)$ be two fuzzy bitopological spaces. A map $f:(X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ is called fuzzy (i, j)-weakly continuous map if for each fuzzy τ_j^*-open set λ in X^*

$$f^{-1}(\lambda) \leq \tau_j - Int\,\{f^{-1}(\tau_i{}^* - Cl\,\lambda)\}$$

Example 6.1: Referring to Example 4.1, we see that fuzzy sets $0, 1, B^*$and $C^*$are fuzzy $\tau_2{}^*$ $-$open sets in X^*. We note that $f^{-1}(B^*) = B$ and $f^{-1}(C^*) = \{(x,\ 0.2), (y,\ 0.4)\} \leq B$. We observe that

$\tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl0) = 0$, so that $f^{-1}(0) \equiv 0 \leq \tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl\,0)\}$ (a)

$\tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl1) = 1$, so that $f^{-1}(1) \equiv 1 \leq \tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl\,1)\}$ (b)

$\tau_2 - Int\,\{f^{-1}(\tau_1{}^* - ClB^*)\} = B$, so that $f^{-1}(B^*) = B \leq \tau_2 - Int\,\{f^{-1}(\tau_1{}^* - ClB^*)\}$ (c)

$\tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl\,C^*)\} = B$, so that $f^{-1}(C^*) \leq \tau_2 - Int\,\{f^{-1}(\tau_1{}^* - Cl\,C^*)\}$ (d)

In view of (a), (b), (c), (d), we conclude that the mapping f is a fuzzy (1, 2)-weakly continuous map.

Similarly we observe that the mapping f is a fuzzy (2, 1)-weakly continuous map.

Theorem 6.1: A fuzzy (i, j)-continuous map from one bitopological space to another is a fuzzy (i, j)-weakly continuous map.

Proof: Let $f:(X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ be a fuzzy (i, j)-continuous map. Let λ be any $\tau_j{}^*$ $-$open set in X^*, then $f^{-1}(\lambda)$ is a fuzzy τ_j $-$open set in X. We know $\lambda \leq \tau_i{}^* - Cl\,\lambda$. Therefore $f^{-1}(\lambda) \leq f^{-1}(\tau_i{}^* - Cl\,\lambda)$. Since $f^{-1}(\lambda)$ is τ_j $-$open set in X, we have

$$\tau_j - Int\{f^{-1}(\lambda)\} = f^{-1}(\lambda) \leq \tau_j - Int\{f^{-1}(\tau_i{}^* - Cl\,\lambda)\}$$

Therefore $\quad f^{-1}(\lambda) \leq \tau_j - \mathrm{Int}\{f^{-1}(\tau_i{}^* - Cl\,\lambda)\}, \qquad \forall\ \lambda \in \tau_j{}^*$

This shows that f is a fuzzy (i, j)-weakly continuous map.

Remark 6.1: The converse of above Theorem 6.1 need not be true i.e. every fuzzy (i, j)-weakly continuous map is not necessarily a fuzzy (i, j)-continuous map. We have the following :

Example 6.2: Considering Example 6.1 we have shown that the map f is a fuzzy (1, 2)-weakly continuous and also a fuzzy (2, 1)-weakly continuous map, whereas in Examples 4.2, we observe that the map f is not a fuzzy (i, j)-continuous map.

Theorem 6.2: A fuzzy (i, j)-almost continuous map is a fuzzy (i, j)-weakly continuous map.

Proof: Let $f:(X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ be a fuzzy (i, j)-almost continuous map. Let λ be any $\tau_j{}^*$ $-$open set in X^*. Then we have

$$f^{-1}(\lambda) \leq \tau_j - Int\,[f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}], \tag{6.2.1}$$

in view of Theorem 4.1(c). Further, we clearly have $\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\} \leq \tau_i{}^* - Cl\,\lambda$. Therefore $f^{-1}[\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}] \leq f^{-1}(\tau_i{}^* - Cl\,\lambda)$. Hence

$$\tau_j - Int[f^{-1}\{\tau_j{}^* - Int(\tau_i{}^* - Cl\,\lambda)\}] \leq \tau_j - Int\{f^{-1}(\tau_i{}^* - Cl\,\lambda)\} \quad (6.2.2)$$

Therefore in view of (6.2.1) & (6.2.2), we conclude that

$$f^{-1}(\lambda) \leq \tau_j - Int\,\{f^{-1}(\tau_i{}^* - Cl\,\lambda)\}, \quad \forall\ \lambda \in \tau_j{}^*$$

Thus f is a fuzzy (i, j)-weakly continuous map.

Remark 6.2: A fuzzy (i, j)-weakly continuous map may not be a fuzzy (i, j)-semi continuous map.

Example 6.3: It is clear from Examples 6.1 that f is a fuzzy (1, 2)-weakly continuous map (and fuzzy (2, 1)-weakly continuous map), but in Example 4.3 we observe that the map f is not a fuzzy (1, 2)-semi continuous map (and fuzzy (2, 1)-semi continuous map).

Remark 6.3: A fuzzy (i, j)-semi continuous map may not be a fuzzy (i, j)-weakly continuous map.

Example 6.4: Referring Example 4.4, we see that map f is a fuzzy (i, j)-semi continuous map. But it is not a fuzzy (1, 2)-weakly continuous map because for any fuzzy $\tau_2{}^*$-open set B^* in X^*, we observe that $\tau_2 - Int\,\{f^{-1}(\tau_i{}^* - Cl\,B^*)\} = B$ and $f^{-1}(B^*) = \{(x, 0.3), (y, 0.5)\}$, so that $f^{-1}(B^*) \geq B$.

Similarly f is not a fuzzy (2, 1)-weakly continuous map because for fuzzy $\tau_1{}^*$-open set A^* in X^*, we see that $\tau_1 - Int\{f^{-1}(\tau_2{}^* - Cl\,A^*)\} = A$ and $f^{-1}(A^*) = \{(x, 0.6), (y, 0.5)\}$ and hence $f^{-1}(A^*) \geq A$.

Thus we have the following Theorem.

Theorem 6.3: In case of fuzzy bitopological spaces the notions of fuzzy (i, j)-weakly continuity and fuzzy (i, j)-semi continuity are independent of each other.

Theorem 6.4: Let $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ be a mapping from a fuzzy bitopological space (X, τ_1, τ_2) to a fuzzy (i, j)-regular bitopological space $(X^*, \tau_1{}^*, \tau_2{}^*)$. Then f is a fuzzy (i, j)-weakly continuous map if and only if f is fuzzy (i, j)-continuous map.

Proof: To prove the Theorem, in view of Theorem 6.1, it is sufficient to show that if mapping $f : (X, \tau_1, \tau_2) \to (X^*, \tau_1{}^*, \tau_2{}^*)$ is a fuzzy (i, j)-weakly continuous and if $(X^*, \tau_1{}^*, \tau_2{}^*)$ is a fuzzy (i, j)-regular bitopological space, then map f is a fuzzy (i, j)-continuous map. Let λ be a fuzzy $\tau_j{}^*$ - open set in X^*. Since X^* is a fuzzy (i, j)-regular space, we have

$\lambda = \cup\,\lambda_\alpha$, $\quad \alpha \in \wedge$, for some index set $\wedge$, (6.4.1)

and for each $\alpha \in \wedge$, λ_α is a fuzzy $\tau_j{}^* -$ open sets in X^* such that $\tau_i{}^* - Cl\,\lambda_\alpha \leq \lambda$. (6.4.2)

Since f is a fuzzy (i, j)-weakly continuous map and λ_α are fuzzy $\tau_j{}^* -$open sets, we have

$$f^{-1}(\lambda_\alpha) \leq \tau_j - Int\{f^{-1}(\tau_i^* - Cl\,\lambda_\alpha)\},\ \forall\ \alpha \in \wedge \quad (6.4.3)$$

Now in view of (6.4.1) we have $f^{-1}(\lambda) = f^{-1}(\cup\,\lambda_\alpha) = \cup\, f^{-1}(\lambda_\alpha)$.

Therefore $f^{-1}(\lambda) \leq \cup\ \tau_j - Int\{f^{-1}(\tau_i^* - Cl\,\lambda_\alpha)\}$, in view of (6.4.3).

This implies that $f^{-1}(\lambda) \leq \cup\ \tau_j - Int\{f^{-1}(\lambda)\} = \tau_j - Int\,\{f^{-1}(\lambda)\}$.

Thus $f^{-1}(\lambda) \leq \tau_j - Int\{f^{-1}(\lambda)\}$. Also we have $\tau_j - Int\,\{f^{-1}(\lambda)\} \leq f^{-1}(\lambda)$.

Hence we conclude that $f^{-1}(\lambda) = \tau_j - Int\{f^{-1}(\lambda)\}$. Thus $f^{-1}(\lambda)$ is a fuzzy $\tau_j -$open set of X for fuzzy $\tau_i^* -$open set λ in $X^*$and this happens for each j = 1, 2. Therefore f is a fuzzy (i, j)-continuous map.

7. CONCLUSION

In [5], fuzzy (i, j)-semi open sets and fuzzy (i, j)-semi continuity in fuzzy bitopological spaces have been studied. In the present paper, we introduced the concepts of fuzzy regular open (closed) sets and studied fuzzy (i, j)-almost continuity and (i, j)-weakly continuity in fuzzy bitopological spaces. We have also studied fuzzy (i, j)-semi regular and fuzzy (i, j)-regular spaces in fuzzy bitopological spaces.

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