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Exponential Matrix and Their Properties

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Abstract: *The matrix exponential is a very important subclass of matrix functions. In this paper, we discuss some of the more common matrix exponential and some methods for computing it. In principle, the matrix exponential could be calculated in different methods some of the methods are preferable to others but none are entirely satisfactory. Due to that, we discussed computations of the matrix exponential using Taylor Series, Scaling and Squaring, Eigenvectors, and the Schur decomposition methods theoretically.*

Keywords: *Matrix Exponential, Commuting Matrix, Non-commuting Matrix.*

1. Introduction

The purpose of this note is matrix functions, The theory of matrix functions was subsequently developed by many mathematicians over the ensuing 100 years. Today, matrices of functions are widely used in science and engineering and are of growing interest, due to the succinct way they allow solutions to be expressed and recent advances in numerical algorithms for computing them [2]. In general is an interesting area in linear algebra, matrix analysis and are used in many areas especially matrix Exponential .The matrix exponential is a very important subclass of functions of matrices that has been studied extensively in the last 50 years [9]. The computation of matrix functions has been one of the most challenging problems in numerical linear algebra. Among the matrix functions one of the most interesting is the matrix exponential. A large number of methods has been proposed for the matrix exponential, many of them of pedagogic interest only or of dubious numerical stability. Some of the more computationally useful methods are surveyed in [5] In principle, the matrix exponential could be computed in many ways and many different methods to calculate matrix exponential [8,9]. In practice, some of the methods are preferable to others, but none are completely satisfactory.

2. Definitions of Exp(A):

The functions of a matrix in which we are interested can be defined in various ways. In mathematics, the matrix exponential is a function on square matrices analogous to the ordinary exponential function [1,4 , 5, 6, 7]. Let A ∊ M_n. The exponential of A, denoted by e^A or exp(A) , is the n × n matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + ... + \frac{A^{n-1}}{(n-1)!} \qquad (`1)$$

Where $A^0 = I$

Note that this is the generalization of the Taylor series expansion of the standard Exponential function. The series (1) converges absolutely for all $A \in C^{n\times n}$ has radius of convergence equal to +1), so the exponential of A is well-defined. To prove the Convergence of the series, we have the following theorem.

Theorem (2.1) for more detail in [6]:

The series (1) converges absolutely for all $A \in M_n$. Furthermore, let $\|\cdot\|$ be a normalized sub multiplicative norm on M_n . Then

$$\left\|e^{A}\right\| \leq e^{\|A\|} \qquad (2)$$

Proof:

The n^{th} partial sum is

$$S_n = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

So

$$\left\|e^{A} - S_n\right\| \equiv \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!} - \sum_{k=0}^{m} \frac{A^k}{k!}\right\| = \left\|\sum_{k=m+1}^{\infty} \frac{A^k}{k!}\right\| \leq \sum_{k=m+1}^{\infty} \frac{\left\|A^k\right\|}{k!} \leq \sum_{k=m+1}^{\infty} \frac{\|A\|^k}{k!}$$

Since $\|A\|$ is a real number and the right-hand side is a part of the convergent series of real numbers

$$e^{\|A\|} = \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$$

then this equation is convergent, if $\varepsilon > 0$ there is an N such that for m > n,

$$e^{\|A\|} = \sum_{k=m+1}^{\infty} \frac{\|A\|^k}{k!} < \varepsilon$$

This is sufficient to prove that S_n is convergent. Furthermore, note that

$$\left\|e^{A}\right\| = \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\right\| \leq \sum_{k=0}^{\infty} \frac{\left\|A^k\right\|}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}$$

In some cases, it is a simple matter to express the matrix exponential of an $n \times n$ complex matrix A shall be denoted by e^{A} and can be defined in a number of equivalent ways [4]:

$$e^{At} = \frac{1}{2\pi i} \oint_{\Gamma} e^{zt} (zI - A)^{-1} dz \qquad (3)$$

Or

$$e^{At} = \lim_{k\to\infty} (1 + \frac{At}{k})^{k} \qquad (4)$$

Or

$$e^{At} \Leftrightarrow \frac{dx}{dt} = AX(t) \ , \quad \mathrm{X}(0) = 1 \qquad (5)$$

For details see [7], and we have other definitions but we leave it to reader to collect them.

3. COMPUTATION OF EXPONENTIAL MATRIX

There are many methods used to compute the exponential of a matrix. Approximation Theory, differential equations, the matrix eigenvalues, and the matrix characteristic Polynomials are some of the various methods used. we will outline various simplistic Methods for finding the exponential of a matrix. The methods examined are given by the type of matrix [2,3,8,9].

3.1- Computing Matrix Exponential for Diagonal Matrix and for Diagonalizable Matrices

if A is a diagonal matrix having diagonal entries then we have

$$e^{A} = \begin{pmatrix} e^{a_1} & & & \\ & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{pmatrix}$$

Now, Let be $A \in R^{n\times n}$ symmetric and has a complete set of linear independent Eigenvectors $v_1, v_2, \ldots\ldots, v_n$ such that

$$Av_k = \lambda_k v_k \qquad k = 1,2,\ldots\ldots n \qquad (6)$$

let us define the matrix T = [$v_1, v_2, \ldots\ldots, v_n$] whose columns are the eigenvector

of A corresponding to the eigenvalues of A, we have

$$AT = \left[Av_1, Av_2, \ldots\ldots, Av_n\right] = \left[\lambda v_1, \lambda v_2, \ldots\ldots \lambda v_3\right] = T\Lambda$$

Since $A = T\Lambda T^{-1}$ where $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

Now using $A = T\Lambda T^{-1}$ to compute e^{A} and we can write it as follow

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum \frac{1}{k!}(T\Lambda T^{-1})^k = T(\sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k)T^{-1} = Te^{\Lambda}T^{-1}$$

And hence

$$e^{A} = T\begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}T^{-1}$$

Example:

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then by using the above formula for diagonal form we get the exponential matrix is

$$e^{A} = \begin{pmatrix} e^{3} & 0 & 0 \\ 0 & e^{5} & 0 \\ 0 & 0 & e^{1} \end{pmatrix}$$

For diagonalizable matrix we give this example

Example:

Let

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}$$

after found the eigenvalues and eigenvectors and construct matrix T we use this formula $A = T\Lambda T^{-1}$ to compute e^A as follow

$$e^A = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^4 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2e^4 - e^3 & e^4 - e^3 \\ 2e^3 - 2e^4 & 2e^3 - e^4 \end{pmatrix}$$

3.2- Computing Matrix Exponential for General Square Matrices

3.2.1- Using Jordan Normal Form

Suppose A is not diagonalizable matrix which it is not possible to find n linearly independent eigenvectors of the matrix A, In this case can use the Jordan form of A. Suppose j is the Jordan form of A, with P the transition matrix. Then

$$e^A = Te^j T^{-1}$$

Where

$$j = diag(j_1\lambda_1, j_2\lambda_2, \ldots\ldots, j_1\lambda_k) = diag(j_1\lambda_1 \oplus j_2\lambda_2 \oplus \ldots\ldots \oplus j_1\lambda)$$

Then

$$e^J = (e^{j_1\lambda_1} \oplus e^{j_2\lambda_2} \oplus \ldots\ldots \oplus e^{j_k\lambda_k})$$

Thus, the problem is to find the matrix exponential of a Jordan block where the Jordan block has the form $J_k(\lambda) = \lambda_k + N_k \in M_k$ and in general N^k as ones on the $k-th$ upper diagonal and is the null matrix if $k \geq n$ the dimension of the matrix. by using the above expression we have

$$e^{J_k(\lambda)} = \sum_{k=0}^{\infty} \frac{1}{k!} J_\lambda^k = \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda I + N)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{J} \lambda^{k-j} N^j$$

This can be written

$$e^{j_\lambda} = e^{\lambda}\left(I + N + \frac{N^2}{2!} + \ldots\ldots + \frac{N^{n-1}}{(n-1)!}\right)$$

Example:

$$A = \begin{pmatrix} 21 & 17 & 6 \\ -5 & -1 & -6 \\ 4 & 4 & 16 \end{pmatrix}$$

Then we calculate the eigenvalues of A which are [6] We have $A = PJP^{-1}$ then we calculate P which is

$$P = \begin{pmatrix} \frac{-1}{4} & 2 & \frac{5}{4} \\ \frac{1}{4} & -2 & \frac{1}{4} \\ 0 & 4 & 0 \end{pmatrix}$$

And

$$J = (4) \oplus \begin{pmatrix} 16 & 1 \\ 0 & 16 \end{pmatrix}$$

Therefore, by using the Jordan canonical form to compute the exponential of matrix A is

$$e^A = \frac{1}{4}\begin{pmatrix} 13e^{16} - e^4 & 13\mathrm{e}^{16} - 5e^4 & 2\mathrm{e}^{16} - 2e^4 \\ -9e^{16} + e^4 & -9\mathrm{e}^{16} + 5e^4 & -2\mathrm{e}^{16} + 2e^4 \\ 16e^{16} & 16\mathrm{e}^{16} & 4\mathrm{e}^{16} \end{pmatrix}$$

3.2.2- Using Hamilton Theorem Cayley

Theorem 3.1 (Cayley Hamilton)

Let A a square matrix and $\Delta(\lambda) = |A - \lambda I|$ its characteristic polynomial then

$\Delta(A) = 0.$

Proof:

Consider a $n \times n$ square matrix A and a polynomial p(x) and $\Delta(x)$ be the characteristic polynomial of A. Then write p(x) in the form

$$p(x) = \Delta(x)q(x) + r(x)$$

by Cayley-Hamilton $\Delta(x) = 0$, then p(A) = r(A) such that we can write polynomial

$$\frac{1}{k!}X^k = r_k(X)$$

Where $r_k(x)$ is the remainder of long division of $\frac{x^k}{k!}$ by $\Delta(x)$, Then the matrix exponential can be written as

$$e^A = \sum_{k=0}^{n-1} \frac{A^k}{k!} = \sum_{k=0}^{\infty} r_k(A)$$

thus e^A is a polynomial of A of degree less than n

$$e^A = \sum_{k=0}^{n-1} a_k A^k$$

Consider now an eigenvector v with the corresponding eigenvalue λ, Then

$$e^A v = \sum_{k=0}^{\infty} \frac{1}{k!} A^k v = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k v = e^{\lambda} v$$

Analogously

$$\sum_{k=0}^{n-1} a_k A^k v = (\sum_{k=0}^{n-1} a_k \lambda^k) v$$

and thus if we have n distinct eigenvalues λ so that the above equation is an interpolation problem which can be used to compute the coefficients a_k. In the case of multiple eigenvalues we use the corresponding generalized eigenvectors.

3.2.3- Using Numerical Integration

Consider the ODE $x_k' = Ax_k$, $\quad x(0) = \mathrm{e}_k = (0,0,...0,1,...0,0)^T$ then when collect the solution from 1 to n we get

$$x_1(t),\dots\dots\dots, x_n(t) = (e^{tA}e_1,\dots\dots, e^{tA}e_2) = e^{tA}(e_1,\dots\dots, e_n) = e^{tA}I = e^{tA}$$

then the general solution for above ODE is

$$X(t) = e^{tA}$$

Now, by using numerical integrator with step $\Delta t = \dfrac{t}{m}$ with $X_0 = I$ we get

$$X_{k+1} = X_k + \Delta t\, \Phi(t_k, X_k), \quad \mathrm{k} = 0,\dots\dots, \mathrm{m}-1$$

That implies

$$e^{tA} \approx X_m$$

3.2.4-The Matrix Exponential Via Interpolation

Here we have two kinds as follow:

3.2.4. A-Lagrange Interpolation Formula

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the distinct eigenvalues of a matrix $A \in M_n$ and f(t) is any function that is well defined at the eigenvalues of A, then the Lagrange formula for e^A is

$$e^{tA} = \sum_{i}^{k} e^{\lambda_i t} \prod_{j=1, i\neq j}^{k} \frac{A - \lambda_i I}{\lambda_i - \lambda_j} \; . \qquad (7)$$

3.2.4.B- Newton's Divided Difference Interpolation

Let $A \in M_n$ be a matrix with eigenvalues $\lambda(A) = \lambda_1, \lambda_2, \dots, \lambda_n$

Now we define f(A) as follows

$$f(A) = \sum_{i=1}^{n} f[\lambda_1, \lambda_2, \dots, \lambda\ ] \prod_{j=1}^{i-1} (A - \lambda_i I) \qquad (8)$$

Where $[\lambda_1, \lambda_2, \dots, \lambda_n]$ is the divided difference at $\lambda_1, \lambda_2, \dots, \lambda_n$ which defined

$$f[\lambda_1, \dots\dots\dots\dots\dots, \lambda_i] = \frac{f[\lambda_1, \dots\dots\dots\dots\dots, \lambda_i] - f[\lambda_2, \dots\dots\dots\dots\dots, \lambda_{n-1}]}{\lambda_n - \lambda_1} \qquad \mathrm{n} \geq 1$$

where the value of divided difference is independent of the order of the arguments.

3,2.5- Using a Limit of Power

From calculus we know that for any numbers a and t the exponential

$$e^{at} = \lim_{k\to\infty} (1 + \frac{A}{k})^k \qquad (9)$$

from equation (4)one can define the matrix exponential as a limit of powers as

$$e^{A} = \lim_{k\to\infty} (1 + \frac{A}{k})^k \qquad (10)$$

This formula is the limit of the first order Taylor expansion of $\dfrac{A}{n}$ raised to the power $n \in Z$.

4. SCALING AND SQUARING

We derive a scaling property from a fundamental nonlinear differential equation whose solution is the so-called q-exponential function. A scaling property has been believed to be given by a power function only, but actually more general expression for the scaling property is found to be a solution of the above fundamental non-linear differential equation. This method will help to control some of the round off error and time or number of terms it would take to find a Taylor approximation The scaling and squaring method is the most widely used method for computing the matrix exponential. The method scales the matrix by a power of 2 to reduce the norm to order 1.The advantage of the scaling methods is that the scaled transition matrix can be made to have a norm less than unity.

5. Taylor Series

Let $A \in M_n$ The exponential of A, denoted by e^A or exp (A), is the $n \times n$ matrix given by the Taylor power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + ... + \frac{A^{n-1}}{(n-1)!} \tag{11}$$

Where $A^0 = I$.

Note that this is the generalization of the Taylor series expansion of the standard exponential function. The above series always converges and well-defined. Now to calculate the matrix exponential we use computers and we cannot calculate the exponential matrix exact only we will be able to approximate it with a truncated Taylor series of k terms. The truncated Taylor series is denoted by $R_k(A)$ the order of the approximation is seen to be the highest power of the truncated Taylor series which represented by k but there are many other factors that can affect the accuracy of a solution technique such that For accuracy and time efficiency, the result for the matrix exponential depends on the matrix norm when the matrix norm is very large then the turn may cause in accuracy due to numerical round off this is problem occurs when the entries of A are large ,otherwise if the norm is small then accuracy and time efficiency as we desired.

6. Eigenvectors and Schur Decomposition Methods

This method based on the similarity transformation of a matrix as follow

$$e^A = Pe^{\Lambda} P^{-1}$$

where A is a real symmetric matrix and P a real unitary matrix and Λ the eigenvalues of A which are real, this is easy to compute when A is non defective (diagonalizable) but, when A may not be diagonalizable and thus is defective such that there is no invertible matrix of eigenvectors P. When P is not invertible then it has ill conditioned so the error will be large. Due to these observations this method relies on diagonalizating the matrix.

7. Properties

In this section of this paper we collect for reference additional important properties of the matrix exponential that are not needed in the development [4,8,9]. Let $A, B \in M_n$ and let t and s be arbitrary complex numbers. We denote the $n \times n$ Zero matrix by 0. The matrix exponential satisfies the following properties

- Property (1) If 0 denotes the zero matrix, then $e^0 = I$ the identity matrix.
- Property (2) If A is invertible, then $e^{ABA^{-1}} = Ae^{B}A^{-1}$.
- Property (3) if $A = diag(A_1, A_2, ..., A_k)$, then $e^A = (e^{A_1}, ..., e^{A_k})$.
- Property (4) $\det(e^A) = e^{trac(A)}$. when A is complex square matrix and trace(A)=0 then $\det(e^A) = 1$.
- Property (5) $e^{(A^T)} = (e^A)^T$. it follows that if A is symmetric, then e^A is also symmetric, and if A is skew symmetric then e^A is orthogonal.

- Property (6) if $AB = BA$ then $Ae^B = e^B A$ and $e^A e^B = e^B e^A$ unfortunately not all familiar properties of the scalar exponential function $y = e^t$ carry over to the matrix exponential. For example, we know from calculus $e^{s+t} = e^s e^t$ when s and t are numbers. However this is often not true for exponentials of matrices. In other words, it is possible to have $n \times n$ matrices A and B such that $e^{A+B} \neq e^A e^B$. Exactly when we have equality $e^{A+B} = e^A e^B$ depends on specific properties of the matrix A and B that discussed in this section.
- Property (7) let A be a complex square $n \times n$ Matrix then $A^m e^A = e^A A^m$ for integer *m*.
- Property (8) Let A be a complex square matrix, and let $s, t \in C$ Then $e^{A(s+t)} = e^{As} e^{At}$.

  Setting s=1 and t=-1 in property (1) we get $e^0 = I$. In other words, regardless of the matrix A, the matrix exponential e^A is always invertible, and has inverse e^{-A}.
- Property (9) $e^{(A^*)} = (e^A)^*$. it follows that if A is Hermitian matrix, then e^A is also Hermitian, and if A is skew-hermitian, then e^A is.
- Property (10) $(e^{At})' = Ae^{tA}$.
- Property (11) let $A, B \in M_n$ then $AB = BA$ if and only if for all t such that $e^{(A+B)t} = e^{At} e^{Bt}$.
- Property (12) let $A, B \in M_n$ be given. If $AB = BA$, then $e^{(A+B)} = e^A e^B = .e^B e^A$.

Proofs and Remarks

in this section we discuss some proofs and some remarks on the previous section and we will give some examples related to them for property (1) it easy to proof it. For property (2) we proof it Recall that, for all integers $s \geq 0$, we have $(ABA^{-1})^m = AB^m A^{-1}$

using the definition (1) to get

$$e^{ABA^{-1}} = I + ABA^{-1} + \frac{(ABA^{-1})^2}{2!} + ...$$

$$= I + ABA^{-1} + A\frac{B^2}{2!} A^{-1} + ...$$

$$= A(I + B + \frac{B^2}{2!} + ...) A^{-1} = Ae^B A^{-1}$$

but If a matrix B is diagonalizable matrix, then there exists an invertible A so that $B = ADA^{-1}$, where D is a diagonal matrix of eigenvalues of B, and A is a matrix having eigenvectors of B as its columns. In this case, $e^B = Pe^D P^{-1}$.For property (3) it easy to proof it. For property (4) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A \in n \times n$,then $(e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_k})$ are the eigenvalues of e^A by the spectral mapping property for diagonalizable matrices such that $f(A) = Pf(A)P^{-1}$ where p is invertible matrix .Then The trace is the sum of the eigenvalues, and the determinant is the product of the eigenvalues of A, so $\det(e^A) = (e^{\lambda_1} e^{\lambda_2} ...... e^{\lambda_n}) = e^{\lambda_1} + e^{\lambda_2} + ... + e^{\lambda_n} = e^{trac(A)}$. For Property (5) it is easy if A is symmetric such that $A = A^T$ then by using the definition of exp(A) that

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^{k}}{k!}$$

Then

$$e^{A^{T}} = \sum_{k=0}^{\infty} \frac{(A^{T})^{k}}{k!} = \sum_{k=0}^{\infty} \frac{(A^{k})^{T}}{k!} = (\sum_{k=0}^{\infty} \frac{A^{k}}{k!})^{T} = (e^{A})^{T} \quad .$$

For property (6)we use the definition of exp(A) then

$$Ae^{B} = A\sum_{k=0}^{\infty} \frac{B^{k}}{k!} = A(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{B^{k}}{k!}) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{AB^{k}}{k!}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{B^{k}A}{k!} = (\sum_{k=0}^{\infty} \frac{\mathrm{B}^{\mathrm{k}}}{\mathrm{k}!})A \quad \text{by } (AB = BA)$$

then we can use induction to proof property(7)from property (6) to get

$$A^{m}e^{A} = e^{A}A^{m}$$

for property(8) we use the definition of exp(A) then we have

$$e^{As}e^{At} = (I + As + \frac{A^{2}s^{2}}{2!} + \frac{A^{3}s^{3}}{3!} + ...)(I + At + \frac{A^{2}t^{2}}{2!} + \frac{A^{3}t^{3}}{3!} + ...)$$

$$= (\sum_{j=0}^{\infty} \frac{A^{j}s^{j}}{j!})\ (\sum_{k=0}^{\infty} \frac{A^{k}t^{k}}{k!}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{A^{j+k}s^{j}t^{k}}{j!k!}$$

Put $m = j + k$, then $j = m - k$ then from the binomial theorem that

$$e^{As}e^{At} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{A^{m}s^{m-k}t^{k}}{(m-k)k!} = \sum_{m=0}^{\infty} \frac{A^{m}}{m!}\sum_{k=0}^{\infty} \frac{m!}{(m-k)!k!}s^{m-k}t^{k} = \sum_{m=0}^{\infty} \frac{A^{m}(s+t)^{m}}{m!} = e^{A(s+t)}$$

property (9) similarly property(5) and property (10) it easy for reader property (11) if

AB = BA and by using the power series expansion of $e^{At}e^{At}$ and $e^{(A+B)t}$ we get it Commute and identical. Conversely, If $e^{(A+B)t} = e^{At}e^{Bt}$ for all t and by differentiating it twice with respect to t and put t = 0 we get AB = BA. For property (12) which is the most important in matrix exponential so we will proof it as follow:

We use the power series for e^{A} to proof this property as follow

$$e^{A}e^{B} = (I + A + \frac{A^{2}}{2!} + \frac{A^{3}}{3!} + ...)(I + B + \frac{B^{2}}{2!} + \frac{B^{3}}{3!} + ...)$$

$$= (\sum_{j=0}^{\infty} \frac{A^{j}}{j!})\ (\sum_{k=0}^{\infty} \frac{B^{k}}{k!}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(A+B)^{j+k}}{j!k!}$$

Put $m = j + k$, then $j = m - k$ then from the binomial theorem that

$$e^{A}e^{B} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{A^{m}B^{m-k}t}{(m-k)k!} = \sum_{m=0}^{\infty} \frac{A^{m}}{m!}\sum_{k=0}^{\infty} \frac{m!}{(m-k)!k!}B^{m-k} = \sum_{m=0}^{\infty} \frac{(A+B)^{m}}{m!} = e^{(A+B)}$$

It then follows that $e^{A}e^{B} = e^{(A+B)} = e^{B+A} = e^{B}e^{A}$. Conclusion, although commuta-

tivity is a sufficient condition for the identities $e^{A}e^{B} = e^{(A+B)} = e^{B+A} = e^{B}e^{A}$ to hold,

but not necessary as the example

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 2\pi i \end{pmatrix}$$

shows ($e^{(A+B)} = e^A = e^B = I$). But if A and B have algebraic entries then their commutativity is necessary for $e^A e^B = e^{(A+B)} = e^{B+A} = e^B e^A$ to hold. An algebraic number is defined by the property that it is a root of a polynomial with rational (or equivalently, integer) coefficients.

8. Conclusion

In summary, we have seen that the commutativity is a sufficient condition for the identities $e^A e^B = e^{(A+B)} = e^{B+A} = e^B e^A$ to hold when A, B have algebraic entries, but not necessary in general as example above that mean the converse in general is not true. And we noticed that when calculated the matrix exponential by the Jordan form this way is very boring for big matrix size. Especially when the matrix is defective which is difficult to determine numerically because any Small changes in a defective matrix will change Jordan form totally. But when we use Taylor's Series numerically which is always converge theoretical we have large cancelation errors due to truncated Taylor's series, so that the convergence can be slow if $\|A\|$ is large. But we can be avoided this problem by careful Scaling and Squaring method. Eigenvalue-Eigenvector method does not work when A is not diagonalizable and we have problem when A anon-diagonalizable matrix. Finally We have seen that there is no uniformly best method for the computation matrix exponential The choice of method depends on the application and the particular matrix

References

[1] Moler. C and Van Loan. C, Nineteen Dubious Ways to Compute the Expo- nential of a Matrix, Twenty-Five Years Later, SIAM Review,2003,3-49 pp

[2] Nicholas J. Higham and Awad H. Al-Mohy, Computing Matrix Func- tions, Manchester Institute for Mathematical Sciences 2010,The University of Manchester, ISSN 1749-9097.pp1-47

[3] Cleve B. Moler and Charles F. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. SIAM Rev., 45(1):349, 2003.

[4] N. J. Higham, Functions of Matrices Theory and Computation, SIAM, Society for Industrial and Applied Mathematics, Philadelphia, A. USA, 2008. ISBN 978-0-89871-646-7. xx+425 pp.

[5] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991,ISBN 978-0-521- 46713-1 , viii+607 pp.

[6] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press,1985, ISBN 0-521-30586-2, xiii+561 pp.

[7] R. Bellman, Introduction to Matrix Analysis, 2nd ed. New York: McGraw- Hill, 1960, reprinted by SIAM, Philadelphia,1995.

[8] Syed Muhammad Ghufran, The Computation of Matrix Functions in Particular, The Matrix Exponential, Master Thesis, University of Birmingham,England,2009,169 pp.

[9] Nathelie Smalls, The exponential of matrices, Master Thesis, University of Georgia, Georgia, 2007,49 pp

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