International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 9, September 2015, PP 26-34
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Fixed Point, Coincidence Point and Common Fixed Point Theorems under Various Expansive Conditions in b-Metric Spaces

Rashmi JAIN[1,♠], R. D. Daheriya[1], Manoj UGHADE[2]

[1]Department of Mathematics, Government J.H. Post Graduate College, Betul, Madhya Pradesh, India

[2]Department of Mathematics, Sarvepalli Radhakrishnan University, Bhopal, Madhya Pradesh, India

♠Corresponding author, e-mail: *krashmi.jain26@gmail.com*

Abstract: *In this article, we establish some fixed point, common fixed point and coincidence point theorems for expansive type mappings in b-metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Also, we introduce some examples the support the validity of our results.*

MATHEMATICS SUBJECT CLASSIFICATION (2010): 47H10, 54H25, 54E50.

Keywords: *Coincidence point, fixed point, common fixed point, b-metric space, weakly compatible maps.*

1. Introduction and Preliminaries

Fixed point theory is one of the most dynamic research subject in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. One of the simplest and most useful results in fixed point theory is the Banach fixed point theorem [2]: Let (X, d) be a complete metric space and T be self mapping of X satisfying

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

where $\lambda \in [0,1)$, then T has a unique fixed point. A mapping satisfying the condition (1.1) is called contraction mapping. As well as, there are a lot of extensions of this famous fixed point theorem in metric space which are obtained generalizing contractive condition, there are a lot of generalizations of it in different space which has metric type structure. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. This celebrated result has been generalized in many abstract spaces for distinct operators. In [3], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [4-8] and the references therein). The study of expansive mappings is a very interesting research area in fixed point theory. A mapping satisfying the condition $d(Tx, Ty) \geq \lambda d(x, y)$ for all $x, y \in X$, where $\lambda > 1$, is called expansive mapping. In 1984, Wang et.al [17] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [16] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Chintaman and Jagannath [18] introduced several meaningful fixed point theorems for one expanding mapping. For more details on expanding mapping and related results we refer the reader to [9-21].

In this paper, we present some new fixed point, coincidence point and common fixed point theorems under various expansive conditions in b-metric spaces. These results improve and generalize some important known results in the literature. Some related results and illustrative some examples to highlight the realized improvements are also furnished.

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ will represents the set of real numbers and nonnegative real numbers, respectively.

The following definitions are required in the sequel which can be found in [3-5].

Definition 1.1 Let X be a nonempty set, $s \geq 1$ be a given real number and $d : X \times X \to \mathbb{R}^+$ be a function. We say d is a b-metric on X if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

A triplet (X, d, s) is called a b-metric space. Obviously, for $s = 1$, b-metric reduces to metric.

Definition 1.2 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a b-metric space (X, d, s).

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for each $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$, $d(x_n, x) < \epsilon$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X, if for each $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n, m \geq n_0$, $d(x_n, x_m) < \epsilon$.
3. (X, d) is said to be complete if every Cauchy sequence is a convergent sequence.

Example 1.3 Let $X = [0,1]$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = (x-y)^2$, $\forall\, x, y \in X$. Then d is a b-metric with constant $s = 2$.

Definition 1.4 Let (X, d, s) be a b-metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

2. Main Results

In this section, we prove some fixed point, common fixed point and coincidence point theorems satisfying expansive condition by considering onto mapping and weakly compatible mappings in the context of b-metric space.

We begin with two simple but a useful Lemmas.

Lemma 2.1 Let $\{x_n\}$ be a sequence in a b-metric space with the coefficient $s \geq 1$ such that

$$d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \tag{2.1}$$

where $h \in \left[0, \frac{1}{s}\right)$ and $n = 1,2, \ldots \ldots \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Proof Let $m > n \geq 1$. It follows that

$$d(x_n, x_m) \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots \ldots \ldots + s^{m-n} d(x_{m-1}, x_m) \tag{2.2}$$

Now (2.1) and $sh < 1$ imply that

$$\begin{aligned} d(x_n, x_m) &\leq s h^n d(x_0, x_1) + s^2 h^{n+1} d(x_0, x_1) + \cdots \ldots \ldots + s^{m-n} h^{m-1} d(x_0, x_1) \\ &= s h^n (1 + (sh) + (sh)^2 + \cdots \ldots \ldots \ldots \ldots + (sh)^{m-n-1}) d(x_0, x_1) \\ &= \frac{sh^n}{1-sh} d(x_0, x_1) \end{aligned} \tag{2.3}$$

Assume that $d(x_0, x_1) > 0$. Letting $n, m \to +\infty$, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Also, if $d(x_0, x_1) = 0$, then $d(x_n, x_m) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. By taking limit $n, m \to +\infty$, we get $d(x_n, x_m) \to 0$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Lemma 2.2 Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y, then $x = y$. That is the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Proof- since $x_n \to x$ and $x_n \to y$ as $n \to +\infty$, that is $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to 0} d(x_n, y) = 0$. By using triangular inequality, we have

$$\begin{aligned} d(x,y) &\leq s[d(x,x_n) + d(x_n,y)] \\ &= s[d(x_n,x) + d(x_n,y)] \end{aligned}$$

By taking limit as $n \to \infty$, we get $d(x,y) = 0$ and so $x = y$.

Theorem 2.3 Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Assume that $T: X \to X$ is surjection and satisfies

$$d(Tx, Ty) \geq \lambda d(x,y) \tag{2.4}$$

$\forall\, x, y \in X$, where $\lambda > s$. Then T has a fixed point.

Proof;- Let $x_0 \in X$, since T is surjection, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

$$x_n = Tx_{n+1}, \ \forall\, n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$,then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n-1}$ for all n. Consider,

$$d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1}) \tag{2.6}$$

Now by (2.4) and definition of the sequence

$$\begin{aligned} d(x_{n-1}, x_n) &= d(Tx_n, Tx_{n+1}) \\ &\geq \lambda\, d(x_n, x_{n+1}) \end{aligned}$$

And so

$$d(x_n, x_{n+1}) \leq \frac{1}{\lambda} d(x_{n-1}, x_n) = h\, d(x_{n-1}, x_n) \tag{2.7}$$

where $h = \frac{1}{\lambda} < \frac{1}{s}$.

Then by Lemma 2.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X, d) is a complete b-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x^\star \in X$. so that

$$\lim_{n\to\infty} x_n = x^\star \tag{2.8}$$

Since T is onto, there exists $p \in X$ such that $x^\star = Tp$. From (2.4), we have

$$\begin{aligned} d(x_n, x^\star) &= d(Tx_{n+1}, Tp) \\ &\geq \lambda\, d(x_{n+1}, p) \end{aligned} \tag{2.9}$$

Taking limit as $n \to +\infty$ in the above inequality, we get

$$0 = \lim_{n\to\infty} d(x_n, x^\star) \geq \lambda \lim_{n\to\infty} d(x_{n+1}, p)$$

This implies that

$$\lim_{n\to\infty} d(x_{n+1}, p) = 0 \tag{2.10}$$

i.e. $x_{n+1} \to p$ as $n \to +\infty$. By Lemma 2.2, we get $x^\star = p$. Hence $x^\star$ is a fixed point of T. Finally, assume $x^\star = y^\star$ is also another fixed point of T. From (2.4), we get

$$\begin{aligned} d(x^\star, y^\star) &= d(Tx^\star, Ty^\star) \\ &\geq \lambda d(x^\star, y^\star) \end{aligned} \tag{2.11}$$

This is true only when $d(x^\star, y^\star) = 0$. so $x^\star = y^\star$. Hence T has a unique fixed point in X.

Example 2.4 Let $X = [0, \infty)$ and let $d(x,y) = |x - y|^2, \forall x, y \in X$. It is obvious that d is a b-metric on X with $s = 2 > 1$ and (X, d) is complete. Also, d is not a metric on X. Define $T: X \to X$ by

$$Tx = \begin{cases} 6x & \text{if } x \in [0,1), \\ 5x+1 & \text{if } x \in [1,2), \\ 4x+3 & \text{if } x \in [2,\infty). \end{cases}$$

Also, clearly T is an onto mapping. Now we consider following cases.

- Let $x, y \in [0,1)$, then

$$\begin{aligned} d(Tx,Ty) &= |6x-6y|^2 = 36|x-y|^2 \\ &\geq 3|x-y|^2 = 3d(x,y) \end{aligned}$$

- Let $x, y \in [1,2)$, then

$$\begin{aligned} d(Tx,Ty) &= |(5x+1)-(5y+1)|^2 \\ &= 25|x-y|^2 \\ &\geq 3|x-y|^2 \\ &= 3d(x,y) \end{aligned}$$

- Let $x, y \in [2,\infty)$, then

$$\begin{aligned} d(Tx,Ty) &= |(4x+3)-(4y+3)|^2 \\ &= |4x-4y|^2 \\ &= 16|x-y|^2 \\ &\geq 3|x-y|^2 \\ &= 3d(x,y) \end{aligned}$$

- Let $x \in [0,1)$and $y \in [1,2)$, then

$$\begin{aligned} d(Tx,Ty) &= |6x-(5y+1)|^2 \\ &\geq |6x-5y|^2 \\ &\geq |5x-5y|^2 \\ &= 25|x-y|^2 \\ &\geq 3|x-y|^2 \\ &= 3d(x,y) \end{aligned}$$

- Let $x \in [0,1)$and $y \in [2,\infty)$, then

$$\begin{aligned} d(Tx,Ty) &= |6x-(4y+3)|^2 \\ &\geq |6x-4y|^2 \\ &\geq |4x-4y|^2 \\ &= 16|x-y|^2 \\ &\geq 3|x-y|^2 \\ &= 3d(x,y) \end{aligned}$$

- Let $x \in [1,2)$and $y \in [2,\infty)$, then

$$\begin{aligned} d(Tx,Ty) &= |(5x+1)-(4y+3)|^2 \\ &= |6x-4y-2|^2 \\ &\geq |5x-4y|^2 \\ &\geq |4x-4y|^2 \\ &= 16|x-y|^2 \\ &\geq 3|x-y|^2 \\ &= 3d(x,y) \end{aligned}$$

That is $d(Tx, Ty) \geq \lambda d(x, y), \forall x, y \in X$ where $\lambda = 3 > 2 = s$. The conditions of Theorem 2.3 are satisfied and T has a unique fixed point $x^\star = 0 \in X$.

Before proving our next result, we prove the following proposition.

Proposition 2.5 Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x) = 0$. Then

$$\frac{1}{s} d(x, y) \leq \lim_{n\to\infty} d(x_n, y) \leq sd(x, y), \ \forall\, y \in X.$$

Proof From triangular inequality,

$$\frac{1}{s} d(x, y) \leq \lim_{n\to\infty} d(x_n, x) \leq \lim_{n\to\infty} d(x_n, y) \leq s(d(x, y) + \lim_{n\to\infty} d(x_n, x))$$

So,

$$\frac{1}{s} d(x, y) \leq \lim_{n\to\infty} d(x_n, y) \leq sd(x, y), \ \forall\, x, y \in X.$$

Now, motivated by the work in [22], we give the following.

Let $\Psi_{\mathcal{B}}^{L}$ denote the class of those function $\mathcal{B}: (0, \infty) \to (L^2, \infty)$ which satisfy the condition $\mathcal{B}(t_n) \to (L^2)^+ \Rightarrow t_n \to 0$, where $L > 0$.

Theorem 2.6 Let (X, d) be a complete b-metric space .Assume that the mapping $T: X \to X$ is surjection and satisfies

$$d(Tx, Ty) \geq \mathcal{B}\big(d(x, y)\big) d(x, y) \tag{2.12}$$

$\forall\, x, y \in X$, where $\mathcal{B} \in \Psi_{\mathcal{B}}^{s}$. Then T has a fixed point.

Proof Let $x_0 \in X$, since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Inductively, we can define a sequence $\{x_n\} \in X$ such that

$$x_n = Tx_{n+1}, \ \forall\, n \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$,then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n-1}$ for all n. Consider,

$$d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1})$$

Now by (2.12) and definition of the sequence

$$\begin{aligned} d(x_{n-1}, x_n) &= d(Tx_n, Tx_{n+1}) \\ &\geq \mathcal{B}\big(d(x_n, x_{n+1})\big) d(x_n, x_{n+1}) \\ &\geq s^2 d(x_n, x_{n+1}) \\ &\geq d(x_n, x_{n+1}) \end{aligned} \tag{2.14}$$

Thus the sequence $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{R}^+$ and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \tag{2.15}$$

Let us prove that $r = 0$. suppose to the contrary that $r > 0$.By (2.14) we can deduce that

$$s^2 \frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1})} \geq \frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1})} \geq \mathcal{B}\big(d(x_n, x_{n+1})\big) \geq s^2 \tag{2.16}$$

By taking limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to\infty} \mathcal{B}\big(d(x_n, x_{n+1})\big) = s^2 \tag{2.17}$$

Hence by definition of $\mathcal{B}$, we have

$$r = \lim_{n\to+\infty} \mathcal{B}\big(d(x_n, x_{n+1})\big) = 0 \tag{2.18}$$

which is a contradiction. That is $r = 0$. we shall show that

$$\lim_{n,m\to\infty}\sup d(x_n,x_m) = 0 \tag{2.19}$$

Suppose to the contrary that $\lim_{n,m\to\infty}\sup d(x_n,x_m) > 0$.

By (2.12), we have

$$\begin{aligned} d(x_n,x_m) &= d(Tx_{n+1},Tx_{m+1}) \\ &\geq \mathcal{B}\big(d(x_{n+1},x_{m+1})\big)d(x_{n+1},x_{m+1}) \end{aligned}$$

That is, $\quad \frac{d(x_n,x_m)}{\mathcal{B}(d(x_{n+1},x_{m+1}))} \geq d(x_{n+1},x_{m+1})$

By Triangular inequality, we have

$$\begin{aligned} d(x_n,x_m) &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{m+1}) + s^2 d(x_{m+1},x_m) \\ &\leq sd(x_n,x_{n+1}) + s^2 \frac{d(x_n,x_m)}{\mathcal{B}(d(x_{n+1},x_{m+1}))} + s^2 d(x_{m+1},x_m) \end{aligned}$$

Therefore,

$$d(x_n,x_m) \leq \left(1 - \frac{s^2}{d(x_{n+1},x_{m+1})}\right)^{-1} \left(\mathcal{B}d(x_n,x_{n+1}) + s^2 d(x_{m+1},x_m)\right) \tag{2.21}$$

By taking limit as $n,m\to+\infty$ in the above inequality, since $\lim_{n,m\to\infty}\sup d(x_n,x_m) > 0$ and $r = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, then we obtain

$$\lim_{n,m\to\infty}\left(1 - \frac{s^2}{d(x_{n+1},x_{m+1})}\right)^{-1} = \infty \tag{2.22}$$

which implies that

$$\lim_{m,n\to+\infty}\sup\ \mathcal{B}\big(d(x_{n+1},x_{m+1})\big) = (s^2)^+ \tag{2.23}$$

And so by definition, we have

$$\lim_{m,n\to\infty}\sup\ d(x_{n+1},x_{m+1}) = 0 \tag{2.24}$$

which is a contradiction. Hence,

$$\lim_{m,n\to\infty}\sup\ d(x_n,x_m) = 0$$

Hence $\lim_{m,n\to\infty} d(x_n,x_m) = 0$. So, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X,d) is a complete b-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x^\star \in X$. so that

$$\lim_{n\to\infty} d(x_n,x^\star) = 0 \tag{2.25}$$

As T is surjective, so there exists $p \in X$ such that $x^\star = Tp$. Let us prove that $x^\star = p$. Suppose to the contrary that $x^\star \neq p$. Then by (2.12), we have

$$\begin{aligned} d(x_n,x^\star) &= d(Tx_{n+1},Tp) \\ &\geq \mathcal{B}\left(d(x_{n+1},p)\right) d(x_{n+1},p) \end{aligned} \tag{2.26}$$

By Taking limit as $n\to+\infty$ in the above inequality and applying proposition 2.5, we obtain

$$\begin{aligned} 0 = \lim_{n\to\infty} d(x_n,x^\star) &\geq \lim_{n\to\infty}\mathcal{B}\left(d(x_{n+1},p)\right).\lim_{n\to\infty} d(x_{n+1},p) \\ &\geq \frac{1}{s}\lim_{n\to\infty}\mathcal{B}\left(d(x_{n+1},x^\star)\right) d(x^\star,p) \end{aligned} \tag{2.27}$$

And hence,

$$\lim_{n\to\infty}\mathcal{B}\left(d(x_{n+1},x^\star)\right) = 0 \tag{2.28}$$

which is a contradiction. Indeed, $\lim_{n\to\infty}\mathcal{B}\big(d(x_{n+1},x_n)\big) \geq s^2$. since $\mathcal{B}(t) > s^2, \forall\, t \in [0,\infty)$, therefore $x^\star = p$. Hence $x^\star = Tp = Tx^\star$.

Now, we prove the following common fixed point theorem, which is generalization of Theorem 2.2 of W. Shatanwi and F. Awawdeh [23] in the setting of b-metric space.

Theorem 2.7 Let $T, S: X \to X$ be two surjective mappings of a complete b-metric space (X, d) with the coefficient $s \geq 1$. Suppose that T and S satisfying inequalities

$$d(T(Sx), Sx) + kd(T(Sx), x) \geq ad(Sx, x) \tag{2.29}$$

$$d(S(Tx), Tx) + kd(S(Tx), x) \geq bd(Tx, x) \tag{2.30}$$

for $x \in X$ and some nonnegative real numbers a, b and k with $a > s(1 + k) + s^2k$ and $b > s(1 + k) + s^2k$.. If T or S is continuous, then T and S have a common fixed point in X.

Proof Let x_0 be an arbitrary point in X. Since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Also, since S is surjective, there exists $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we construct a sequence $\{x_n\}$ in X such that

$$x_{2n} = Tx_{2n+1} \text{ and } x_{2n+1} = Sx_{2n+2} \tag{2.31}$$

for all $n \in N \cup \{0\}$, Now for $n \in \mathbb{N} \cup \{0\}$, by (2.29) we have

$$d(T(Sx_{2n+2}), Sx_{2n+2}) + kd(T(Sx_{2n+2}), x_{2n+2}) \geq ad(Sx_{2n+2}, x_{2n+2})$$

Thus, we have

$$d(x_{2n}, x_{2n+1}) + kd(x_{2n}, x_{2n+2}) \geq ad(x_{2n+1}, x_{2n+2})$$

which implies that

$$d(x_{2n}, x_{2n+1}) + sk[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \geq ad(x_{2n+1}, x_{2n+2})$$

Hence

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1+sk}{a-sk}\, d(x_{2n}, x_{2n+1}) \tag{2.32}$$

On other hand, we have (from (2.30))

$$d(S(Tx_{2n+1}), Tx_{2n+1}) + kd(S(Tx_{2n+1}), x_{2n+1}) \geq bd(Tx_{2n+1}, x_{2n+1})$$

Thus we have

$$d(x_{2n-1}, x_{2n}) + kd(x_{2n-1}, x_{2n+1}) \geq bd(x_{2n}, x_{2n+1})$$

which implies that

$$d(x_{2n-1}, x_{2n}) + sk[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n1})] \geq bd(x_{2n}, x_{2n+1})$$

Hence

$$d(x_{2n-1}, x_{2n}) \leq \frac{1+sk}{b-sk}\, d(x_{2n-1}, x_{2n}) \tag{2.33}$$

Let $h = \max\left\{\frac{1+sk}{a-sk}, \frac{1+sk}{b-sk}\right\} < \frac{1}{s}$

Then by combining (2.32) and (2.33), we have

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n) \tag{2.34}$$

where $h \in \left[0, \frac{1}{s}\right)$, $\forall\, n \in \mathbb{N} \cup \{0\}$.

Then by Lemma 2.1, $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in the complete b-metric space. Then there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to +\infty$. Therefore $x_{2n+1} \to x^\star$ and $x_{2n+2} \to v$ as $n \to +\infty$. Without loss of generality, we may assume that T is continuous, then $Tx_{2n+1} \to Tx^\star$ as $n \to +\infty$. But $Tx_{2n+1} = x_{2n} \to x^\star$ as $n \to +\infty$.Thus, we have $Tx^\star = x^\star$. since S is surjective, there exists $p \in X$ such that $Sp = x^\star$. Now

$$d(T(Sp) + kd(T(Sp), p) \geq ad(Sp, p)$$

implies that

$$kd(x^\star, p) \geq ad(x^\star, p)$$

Then $d(x^\star, p) \leq \frac{k}{a} d(v, w)$. Since $a > k$, we conclude that $d(x^\star, p) = 0$. so $x^\star = p$. Hence $Tx^\star = Sx^\star = x^\star$. Therefore $x^\star$ is a common fixed point of T and S.

Bt taking $b = a$ in theorem 2.7, we have the following result.

Corollary 2.8 Let $T, S: X \to X$ be two surjective mappings of a complete b-metric space (X, d) with the coefficient $s \geq 1$. Suppose that T and S satisfying inequalities

$$d(T(Sx), Sx) + kd(T(Sx), x) \geq ad(Sx, x) \quad (2.29)$$

$$d(S(Tx), Tx) + kd(S(Tx), x) \geq ad(Tx, x) \quad (2.30)$$

for $x \in X$ and some nonnegative real numbers a, b and k with $a > s(1 + k) + s^2 k$. If T or S is continuous, then T and S have a common fixed point in X.

By taking $S = T$ in Corollary 2.8, we have the following Corollary.

Corollary 2.9 Let $T: X \to X$ be a surjective mappings of a complete b-metric space (X, d) with the coefficient $s \geq 1$. Suppose that T satisfying inequality

$$d(T(Tx), Tx) + kd(T(Tx), x) \geq ad(Tx, x) \quad (2.29)$$

for $x \in X$ and some nonnegative real numbers a, b and k with $a > s(1 + k) + s^2 k$. If T is continuous, then T has a fixed point in X.

Now, we present an example to illustrate the usability of Corollary 2.9.

Example 2.10 Let $X = [0, \infty)$ and define $d: X \times X \to R^+$ by $d(x, y) = |x - y|^2, \forall\, x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Define $T: X \to X$ by $T(x) = 2x$. Then T has a fixed point.

Proof Note that

$$\begin{aligned} d(T(Tx), Tx) &+ d(T(Tx), x) \\ &= d(4x, 2x) + d(4x, x) \\ &= |4x - 2x|^2 + |4x - x|^2 \\ &= 4x^2 + 9x^2 \\ &= 13x^2 \\ &\geq 12x^2 \\ &= 12|2x - x|^2 \\ &= 12d(Tx, x) \end{aligned}$$

for all $x \in X$. Here $k = 1$ and $a = 12$. Clearly $12 = a > s(1 + k) + sk^2 = 2(1 + 1) + 2(1)^2 = 6$.

Also T is surjection on X. Thus T satisfies all the hypotheses of Corollary 2.9 and hence T has a fixed point. Here $0 \in X$ is the fixed point of T.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1]. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985).

[2]. Banach, S: Sur les operations dans les ensembles abstrait et leur application aux equations, integrals. Fundam. Math. 3, 133-181 (1922).

[3]. Bakhtin, IA: The contraction mapping principle in almost metric spaces. Funct. Anal., Gos. Ped. Inst. Unianowsk 30, 26-37 (1989).

[4]. S. Czerwik, “Contraction mappings in b-metric spaces,” Acta Mathematica et Informatica Universitatis Ostraviensis, vol. 1, pp. 5–11, 1993.

[5]. S. Czerwik, “Nonlinear set-valued contraction mappings in b-metric spaces,” Atti del Seminario matematico e fisico dell’Universit`a di Modena, vol. 46, no. 2, pp. 263–276, 1998.

[6]. Boriceanu, M, Bota, M, Petrusel, A: Mutivalued fractals in b-metric spaces. Cent. Eur. J. Math. 8(2), 367-377 (2010).

[7]. Bota, M, Molnar, A, Csaba, V: On Ekeland's variational principle in b-metric spaces. Fixed Point Theory 12, 21-28 (2011).

[8]. Marta Demma and Pasquale Vetro, Picard Sequence and Fixed Point Results on b-Metric Spaces, Journal of Function Spaces,Volume 2015, Article ID 189861, 6 pages http://dx.doi.org /10.1155/2015/189861.

[9]. A. S. Saluja, A. K. Dhakda and D. Magarde, Some fixed point theorems for expansive type mapping in dislocated metric space, Mathematical Theorey and Modeling, 3(2013).

[10]. R. D. Daheriya, Rashmi Jain, and Manoj Ughade "Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space", ISRN Mathematical Analysis, Volume 2012, Article ID 376832, 5 pages, doi:10.5402/2012/376832.

[11]. Yan Han and Shaoyuan Xu, Some new theorems of expanding mappings without continuity in cone metric spaces, Fixed Point Theory and Applications, 2013, 2013:3.

[12]. Wasfi Shatanawi and Fadi Awawdeh, Some fixed and coincidence point theorems for expansive maps in cone metric spaces, Fixed Point Theory and Applications 2012, 2012:19.

[13]. Xianjiu Huang, Chuanxi Zhu, Xi Wen, Fixed point theorems for expanding mappings in partial metric spaces, An. St. Univ. Ovidius Constant_a Vol. 20(1), 2012, 213-224.

[14]. Z. Mustafa, F. Awawdeh and W. Shatanawi, Fixed Point Theorem for Expansive Mappings in G-Metric Spaces, Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 50, 2463-2472.

[15]. A. Muraliraj and R. Jahir Hussain, Coincidence and Fixed Point Theorems for Expansive Maps in d−Metric Spaces, Int. Journal of Math. Analysis, Vol. 7, 2013, no. 44, 2171 – 2179.

[16]. P. Z. Daffer, H. Kaneko, On expansive mappings, Math. Japonica. 37 (1992), 733-735.

[17]. S. Z. Wang, B. Y. Li, Z. M. Gao, K. Iseki, Some fixed point theorems for expansion mappings, Math. Japonica. 29 (1984), 631-636.

[18]. Aage, CT, Salunke, JN: Some fixed point theorems for expansion onto mappings on cone metric spaces. Acta Math. Sin. Engl. Ser. 27(6), 1101-1106 (2011).

[19]. G. Jungck, "Commuting mappings and fixed points," The American Mathematical Monthly, vol. 83, no.4, pp. 261-263, 1976.

[20]. M. A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, Journal of Inequalities and Applications, 2013, 2013:402.

[21]. R. D. Daheriya, Rashmi Jain, and Manoj Ughade, Unified fixed point theorems for non-surjective expansion mappings by using class Ψ on 2- metric spaces, Bessel J. Math., 3(3), (2013), 271-291.

[22]. M. A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, Journal of Inequalities and Applications, 2013, 2013:402.

[23]. W. Shatanwi and F. Awawdeh, Some fixed and coincidence point theorems for expansive maps in cone metric spaces, Fixed point theory and apllicaions, 2012, 2012;19, doi:10.1186/1687-1812-2012-19.