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# Some results in co-Noetherian BL-algebras

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**Abstract:** *In this paper, we investigate further properties of co-Noetherian $BL$-algebras. We introduce a special case of $BL$-algebras and give some characterizations for $pBL$-algebras. Finally, we study co-Noetherian $BL$-algebras by prime ideals.*

**Keywords:** *$BL$-algebra, co-Noetherian $BL$-algebra, ideal, prime ideal*

## 1. Introduction

The notion of $BL$-algebra was initiated by Hάjek ([2]) in order to provide an algebraic proof of the completeness theorem of Basic Logic (**BL**, in short). Soon after, Cignoli et al.([1]) proved that Hάjek's logic really is the logic of continuous $t$-norms as conjectured by Hάjek. At the same time started a systematic study of $BL$-algebras, and in particular, filters theory (see [3], [4], [7], [10], [11]). Using the model of rings, Motamed ([6]) introduced the notion of Northerian $BL$-algebras and gave some of its equivalent definitions. Since filters and ideals are not dual concepts in $BL$-algebras, Meng ([5]) systematically studied properties of ideals and introduced co-Noetherian $BL$-algebras based on ideals theory in $BL$-algebras.

The structure of the paper is as follows: In section 2, we recall some definitions and facts about $BL$-algebras that we use in the sequel. In the section 3, we investigate further properties of co-Noetherian $BL$-algebras. This part of paper contains providing the sufficient conditions for a quotient $BL$-algebra to be co-Noetherian. We introduce a special kinds of $BL$-algebras named $pBL$-algebra , and give some characterizations for $pBL$-algebra. After that, we characterize co-Noetherian $BL$-algebras by prime ideals.

## 2. Preliminaries

Let us recall some definitions and results on $BL$-algebras.

**Definition 2.1**([2]). An algebra $(A;\wedge,\vee,*,\rightarrow,0,1)$ of type $(2,2,2,2,0,0)$ is called a $BL$-algebra if it satisfies the following conditions:

(BL1) $(A;\wedge,\vee,0,1)$ is a bounded lattice,

(BL2) $(A;*,1)$ is a commutative monoid,

(BL3) $x*y\le z$ if and only if $x\le y\rightarrow z$ (residuation),

(BL4) $x\wedge y = x*(x\rightarrow y)$, thus $x*(x\rightarrow y)=y*(y\rightarrow x)$ (divisibility),

(BL5) $(x\rightarrow y)\vee(y\rightarrow x)=1$ (prelinearity).

**Definition 2.2**([9])**.** Let $A, B$ be two $BL$-algebras. A map $h: A \to B$, defined on $A$, is called a $BL$-homomorphism if, for all $x, y \in A$,

$$h(x * y) = h(x) * h(y),\ h(x \to y) = h(x) \to h(y) \text{ and } h(0_A) = 0_B.$$

If $h: A \to B$ is a $BL$-homomorphism then, for all $x, y \in A$,

$$h(x \wedge y) = h(x) \wedge h(y),\ h(x \vee y) = h(x) \vee h(y),$$

$$h(x^-) = h(x)^-,\ h(1_A) = 1_B \text{ and if } x \le y \text{ then } h(x) \le h(y).$$

**Definition 2.3**([2])**.** Let $A$ be a $BL$-algebra. A nonempty subset $I \subseteq A$ is called an ideal of $A$, if the following conditions are satisfied:

(1) $0 \in I$,

(2) if $x, (x^- \to y^-)^- \in I$ then $y \in I$.

A proper ideal $P$ of $A$ is called a prime ideal if, for all $x, y \in A$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

Through the paper, the set of all ideals of a $BL$-algebra $A$ is denoted by $I(A)$ and all prime ideals of $A$ by $PI(A)$.

**Proposition 2.4**([5])**.** Let $I$ be an ideal of a $BL$-algebra $A$ and $a \in A$, then

$$(I \cup \{a\}] = \{x \in A : ((a^-)^n \to x^-)^- \in I, \exists n \in N\}.$$

An ideal of $A$ is called a finitely generated ideal, if there exist $a_1, a_2, ..., a_k \in A$ such that

$$I = (a_1, a_2, ..., a_k].$$

Obviously, if $I, J \in I(A)$ are finitely generated ideals, $I \cap J$ is a finitely generated ideal.

**Proposition 2.5**([5])**.** Let $A$ be a $BL$-algebra. Then for any $a_1, a_2, ..., a_k \in A (k \in N)$

$$((a_1] \cup (a_2] \cup \dots \cup (a_k]] = ((a_1^- * a_2^- * \dots * a_k^-)^-].$$

**Definition 2.6**([5])**.** A $BL$-algebra $A$ is said to be co-Noetherian with respect to ideals if every ideal of $A$ is finitely generated;

We say that $A$ satisfying the ascending chain condition with respect to ideals if for every ascending sequence $I_1 \subseteq I_2 \subseteq \dots$ of ideals of $A$, there is $n \in N$ such that $I_n = I_k$ for $k \ge n$;

$A$ is said to satisfy the maximal condition with respect to ideals if every nonempty set of $I(A)$ has a maximal element.

**Theorem 2.7**([5])**.** Let $A$ be a $BL$-algebra. Then the following conditions are equivalent:

$(i)$ $A$ is co-Noetherian with respect to ideals,

$(ii)$ $A$ satisfies the ascending chain condition with respect to ideals,

$(iii)$ $A$ satisfies the maximal condition with respect to ideals.

## 3. Some Further Properties of CO-Notherian BL-algebras

In this section, we investigate further properties of co-Noetherian $BL$-algebras and give some of its characterizations.

**Proposition 3.1.** Let $h: A \to B$ be a $BL$-homomorphism. Then the following assertions are true:

$(i)$. For any (proper, prime) ideal $J$ of $B$, the set

$$h^{\leftarrow}(J)=\{a\in A: h(a)\in J\}$$

is an (proper, prime) ideal of $A$. In particular,

$$\ker(h)=\{a\in A: h(a)=0\}$$

is a proper ideal of $A$,

$(ii)$. If $M$ is a maximal ideal of $B$, then $h^{\leftarrow}(M)$ is a maximal ideal of $A$,

$(iii)$. If $h$ is surjective, $I$ is an ideal of $A$ and $\ker(h)\subseteq I$, then

$$h^{\rightarrow}(I)=\{h(x): x\in I\}$$

is an ideal of $B$,

$(iv)$. If $h$ is surjective, $M$ is a maximal ideal of $A$ and $\ker(h)\subseteq I$, then $h^{\rightarrow}(I)$ is a maximal ideal of $B$,

$(v)$. $h$ is injective if and only if $\ker(h)=\{0\}$.

**Proof:** $(i)$. Let $J$ be an ideal of $B$. Clearly, $0\in h^{\leftarrow}(J)$. If $(x^{-}\rightarrow y^{-})^{-}, x\in h^{\leftarrow}(J)$, then

$$(h(x)^{-}\rightarrow h(y)^{-})^{-}=h((x^{-}\rightarrow y^{-})^{-})\in J \text{ and } h(x)\in J.$$

Since $J$ is an ideal, we have $h(y)\in J$, i.e., $y\in h^{\leftarrow}(J)$. Therefore $h^{\leftarrow}(J)$ is an ideal of $A$.

$(ii)$. First we show that $M$ is a maximal ideal of $A$ if and only if $x\notin M$, then $(x^{-})^{n}\in M$ for some $n\in N$.

Let $M$ be a maximal ideal of $A$ and $x\notin M$, then $(M\cup\{x\}]=A$. Since $1\in A$, by Proposition 2.4 we get

$$(x^{-})^{n}=((x^{-})^{n})^{--}=((x^{-})^{n}\rightarrow 1^{-})^{-}\in M$$

for some $n\in N$.

Conversely, let $M\subset I$ where $I$ is a proper ideal of $A$. Take $x\in I\setminus M$, then $(x^{-})^{n}\in M$ for some $n\in N$ by hypothesis. Since

$$\begin{aligned}(x^{-})^{n}&=((x^{-})^{n})^{--}=((x^{-})^{n}\rightarrow 1^{-})^{-}\\&=(x^{-}\rightarrow((x^{-})^{n-1}\rightarrow 1^{-}))^{-}\\&=(x^{-}\rightarrow((x^{-})^{n-1}\rightarrow 1^{-})^{--})^{-}\in M\subset I,\end{aligned}$$

then we obtain $((x^{-})^{n-1}\rightarrow 1^{-})^{-}\in I$ by $x\in I$ and $I$ is an ideal. By repeating the above proceeding $n$ times, we have $1\in I$, thus $I=A$. Therefore $M$ is a maximal ideal of $A$. Now let $M$ be a maximal ideal of $B$, then $h^{\leftarrow}(M)$ is a proper ideal of $A$ by $(i)$. If $x\notin h^{\leftarrow}(M)$, then $h(x)\notin M$. Since $M$ is maximal, by the above conclusion we have

$$h((x^{-})^{n})=h(x^{-})^{n}\in M$$

for some $n\in N$, i.e., $(x^{-})^{n}\in h^{\leftarrow}(M)$, hence $h^{\leftarrow}(M)$ is maximal.

$(iii)$. Let $h$ be surjective and $I$ an ideal of $A$. Trivially $0\in h^{\rightarrow}(I)$. Now let

$$(x^{-}\rightarrow y^{-})^{-}, x\in h^{\rightarrow}(I),$$

then there exist $a,b \in I$ such that

$$h(a) = (x^- \to y^-)^-, h(b) = x .$$

Since $h$ is homomorphic, so we have

$$h(a^-) = x^- \to y^-, h(b^-) = x^-,$$

then

$$h(a^- * b^-) = x^- * (x^- \to y^-) \le y^- .$$

By $h$ is surjective, there exist $c \in A$ such that $h(c) = y$, hence

$$h(a^- * b^-) \le h(c^-),$$

i.e.,

$$h(a^- * b^- \to c^-) = 1,$$

by hypothesis we have

$$(a^- \to (b^- \to c^-))^- \in \ker(h) \subseteq I \ .$$

Since $I$ is an ideal and $a,b \in I$ then $c \in I$, hence $y \in h^{\to}(I)$. This shows that $h^{\to}(I)$ is an ideal of $B$.

$(iv)$. By $(iii)$ we know that $h^{\to}(M)$ is an ideal of $B$. Now let $h^{\to}(M) \subset J$ where $J$ is a proper ideal of $B$, then

$$M \subseteq h^{\leftarrow}(h^{\to}(M)) \subseteq h^{\leftarrow}(J).$$

Since $J$ is a proper ideal, by $(i)$ we have $h^{\leftarrow}(J)$ is also proper ideal, thus $M = h^{\leftarrow}(J)$ because $M$ is maximal. Since $h$ is surjective, we get

$$h^{\to}(M) = h^{\to}(h^{\leftarrow}(J)) = J \ .$$

This completed the proof.

$(v)$. It is clear.

**Proposition 3.2.** Let $h: A \to B$ be a surjective $BL$-homomorphism. If $A$ is co-Noetherian, then $B$ is co-Noetherian.

**Proof:** Let $J_1 \subseteq J_2 \subseteq \cdots$ be a chain of ideals of $B$, then by proposition 3.1 $(i)$ we know $h^{\leftarrow}(J_i)(i \in N)$ are ideals of $A$, hence

$$h^{\leftarrow}(J_1) \subseteq h^{\leftarrow}(J_2) \subseteq \cdots$$

a chain of ideal of $A$. Since $A$ is co-Noetherian, by Theorem 2.3, then there exists $k \in N$ such that $h^{\leftarrow}(J_n) = h^{\leftarrow}(J_k)$ for any $n \ge k$. By $h$ is surjective, we get

$$h^{\to}(h^{\leftarrow}(J_i)) = J_i (\forall i \in N),$$

hence $J_k = J_n$ for any $n \ge k$, $B$ is co-Noetherian.

**Proposition 3.3.** Let $A$ be a $BL$-algebra and $I$ an ideal of $A$. Then $K$ is an ideal of $A/I$ if and only if there exists an ideal $J$ of $A$ such that $I \subseteq J$ and $K = J/I$.

**Proof:** $\Rightarrow$. Denote $J = \cup\{[x]:[x]\in K\}$. Since $I = [0] \in K$, then $I \subseteq J$ and $0 \in J$. Now let

$$(x^- \to y^-)^-, x \in J,$$

it follows that

$$([x]^- \to [y]^-)^-, [x] \in K.$$

By $K$ being an ideal we have $[y] \in K$, hence $y \in J$, $J$ is an ideal of $A$.

$\Leftarrow$. Let $([x]^- \to [y]^-)^-, [x] \in K$, then there exist $a, b \in J$ such that

$$[(x^- \to y^-)^-] = ([x]^- \to [y]^-)^- = [a],\ [x] = [b].$$

Since

$$(a^- \to (x^- \to y^-)^{--})^- \leq ((x^- \to y^-)^- \to a)^- \in I \subseteq J,$$

by $a \in J$ and $J$ being an ideal of $A$, we have $(x^- \to y^-)^- \in J$. Similarly, we can prove $x \in J$, hence $y \in J$, $[y] \in K$. Therefore $K$ is an ideal of $A/I$.

The above proposition shows that for any ideal of $A/I$, it must be the form of $J/I$ where $J \supseteq I$ is an ideal of $A$.

By Proposition 3.2 and 3.3 the following conclusion holds.

**Corollary 3.4.** Let $A$ be a co-Noetherian $BL$-algebra and $I$ an ideal of $A$. Then $A/I$ is co-Noetherian.

**Proposition 3.5.** Let $A$ be co-Noetherian and $h$ a surjective $BL$-homomorphism on $A$. Then $h$ is one-to-one homomorphism.

**Proof:** Trivially $h^n (n \in N)$ is surjective. Consider

$$\ker(h) \subseteq \ker(h^2) \subseteq \ldots$$

a chain of ideals of $A$. Since $A$ is co-Noetherian, then there exists $k \in N$ such that

$$\ker(h^n) = \ker(h^k)$$

for any $n \geq k$. Let

$$a \in \ker(h) \subseteq \ker(h^n)(n \geq k).$$

By $h^n$ being surjective we have $h^n(b) = a$ for some $b \in A$. Hence

$$h^{n+1}(b) = h(a) = 0,\ b \in \ker(h^{n+1}),$$

thus $a = h^n(b) = 0$, i.e., $\ker(h) = \{0\}$, by proposition 3.1$(v)$ we get $h$ is one-to-one.

**Definition 3.6.** Let $A$ be a $BL$-algebra. If for any $I \in I(A)$, $I$ is a principal ideal, i.e., $I = (a]$ where $a \in A$, then $A$ is called a principal ideal $BL$-algebra, simply $pBL$-algebra.

Obviously, every $pBL$-algebra must be co-Noetherian.

**Proposition 3.7.** Let $A$ be co-Noetherian. If for any ideal $I$ which is generated with two elements is principal, then $A$ is a $pBL$-algebra.

**Proof:** Let $I$ be an ideal of $A$. Since $A$ is co-Noetherian, then

$$I = (a_1, a_2, \ldots, a_n]$$

for some $a_1, a_2, \dots, a_n \in A$. We demonstrate, by induction on $n$, that $I$ is principal. If $n = 2$, then the claim is true by hypothesis. Now assume it is true for $n = k$ and set

$$(a_1, a_2, \dots, a_k] = (b].$$

Let $n = k+1$, by Proposition 2.5 we have

$$(a_1, a_2, \dots, a_k, a_{k+1}] \subseteq ((b] \cup (a_{k+1}]] = ((b^- * a_{k+1}{}^-)^-].$$

Conversely, since $(b], (a_{k+1}] \subseteq (a_1, a_2, \dots, a_k, a_{k+1}]$, then

$$((b^- * a_{k+1}{}^-)^-] = ((b] \cup (a_{k+1}]] \subseteq (a_1, a_2, \dots, a_k, a_{k+1}],$$

hence $(a_1, a_2, \dots, a_k, a_{k+1}] = ((b^- * a_{k+1}{}^-)^-]$, a principal ideal of $A$. Thus, the claim holds for all natural numbers $n$ and the proof is completed.

By Proposition 2.5 we have the following conclusion.

**Proposition 3.9.** Let $A$ be a $BL$-algebra. If for any ideal $I$ of $A$, which is generated by union of finite principal ideals, then $A$ is a $pBL$-algebra.

**Corollary 3.10.** Let $A$ be a $pBL$-algebra and $I$ an ideal of $A$. Then $A/I$ is co-Noetherian.

**Proposition 3.11.** Let $A$ be a $BL$-algebra. If for any nontrivial ideal $I$ of $A$, $A/I$ is co-Noetherian, then $A$ is co-Noetherian.

**Proof:** Let $J_1 \subseteq J_2 \subseteq \dots$ be a chain of nontrivial ideals of $A$, then we have

$$J_1/I \subseteq J_2/I \subseteq \dots$$

a chain of ideals of $A/I$. Since $A/I$ is co-Noetherian, then there exists $k \in N$ such that

$$J_n/I = J_k/I$$

for any $n \geq k$, hence $J_n = J_k$. Therefore, $A$ is co-Noetherian.

**Theorem 3.12.** Let $A$ be a $BL$-algebra. Then $A$ is co-Noetherian if and only if every prime ideal of $A$ is finitely generated.

**Proof:** The necessity is obvious, so we just prove the sufficiency. Denote by $K$ the set of all ideals $I$ of $A$ where $I$ is not finitely generated. Assume that $K \neq \emptyset$ and

$$I_1 \subseteq I_2 \subseteq \dots$$

be a chain where $I_i \in K$ for any $i \in N$. Clearly, $I = \cup I_i$ is an ideal of $A$ and $I \in K$. By Zorn's lemma, $K$ has a maximal element $M$. Now we show that $M$ is prime. Supposed that $a \wedge b \in M$ but $a, b \notin M$, then

$$(M \cup \{a\}] \cap (M \cup \{b\}] = M.$$

Since $M$ is maximal in $K$ and

$$M \subset (M \cup \{a\}], (M \cup \{b\}],$$

hence

$$(M \cup \{a\}], (M \cup \{b\}] \notin K.$$

Thus

$$(M \cup \{a\}], (M \cup \{b\}]$$

are finitely generated, it follows that $M$ is finitely generated, a contradiction. Therefore $K=\varnothing$, the conclusion is true.

**Theorem 3.13.** Let $A$ be a $BL$-algebra satisfying the ascending chain condition with respect to finitely generated ideals, then $A$ is co-Noetherian.

**Proof:** Let $A$ be not co-Noetherian, then there exists an ideal $I$ not finitely generated. Obviously, $\{0\}\subset I$, $a_1 \in I\setminus\{0\}$ such that $(a_1]\subset I$. Take $a_2 \in I\setminus(a_1]$ then $(a_1,a_2]\subset I$ .Continuing this procedure, we will get an increasing proper ideals chain

$$(a_1]\subset(a_1,a_2]\subset\dots,$$

which every element is finitely generated, a contradiction.

By the prime ideal extended theorem (see [5]) we know that, if $I\subseteq J$ then $J$ is prime, where $I$ is an prime ideal and $J$ is proper ideal of $A$.

Let $A$ be a $BL$-algebra and $I$ a proper ideal of $A$. We denote

$$K=\{J\in PI(A): I\subseteq J\},$$

then by prime ideal theorem (see [5]) we get $K\neq\varnothing$ and by dual Zorn's lemma $K$ has a minimal element, which is called a minimal prime ideal associated with $I$. Denote the set of all minimal prime ideals associated with $I$ by $m(I)$.

**Theorem 3.14.** Let $A$ be a $BL$-algebra and $I$ a proper ideal of $A$. If every element of $m(I)$ is finitely generated, then $m(I)$ is a finite set.

**Proof:** We denote the set of all finite intersections of $m(I)$ by $F(I)$. It is Obvious that $F(I)$ is nonempty. Now supposed that $I\subset P$ for any $P\in F(I)$. Consider the set

$$S=\{J\in I(A): I\subseteq J, P\not\subset J, \forall P\in F(I)\}.$$

We have $I\in S$. Take $J_1\subseteq J_2\subseteq\dots$ a chain of $S$. Obviously, $J^*=\cup_i J_i$ is an ideal of $A$ and $I\subseteq J^*$. If there exists $P\in F(I)$ such that $P\subseteq J^*$. Let

$$P=P_1\cap P_2\cap\dots\cap P_n.$$

It is clear $P$ is finitely generated since each $P_i$ is finitely generated, then

$$P=(x_1,x_2,\dots,x_n]\subseteq J^*$$

for some $x_1,x_2,\dots,x_n\in A$. Since $\{J_i\}$ is a chain, for some $k\in N$, all $x_i\in J_k$, thus $P\subseteq J_k$, a contradiction. Therefore, $J^*\in S$ and a upper bound of the chain in $S$. By Zorn's lemma, $S$ has a maximal element $M$. We show that $M$ is prime. If not, there is $a\wedge b\in M$ but $a,b\notin M$. By Corollary 4.9([5]) we have

$$(M\cup\{a\}]\cap(M\cup\{b\}]=M.$$

It is obvious that $M\subset(M\cup\{a\}],(M\cup\{b\}]$, and so $(M\cup\{a\}],(M\cup\{b\}]\notin S$. Thus there exist $Q_1,Q_2\in F(I)$ such that

$$Q_1\subseteq(M\cup\{a\}], Q_2\subseteq(M\cup\{b\}].$$

It follows that $Q_1\cap Q_2\subseteq M$, a contradiction because $Q_1\cap Q_2\subseteq F(I)$, hence $M$ is prime. By Proposition 5.19 ([5]) there exists $P\in m(I)$ such that $I\subseteq P\subseteq M$, a contradiction with $M\in S$, that is to say, there exists $P'\in F(I)$ such that $I=P'$. Let

$$P'=P_1\cap P_2\cap\dots\cap P_k\in F(I).$$

Then for any $P \in m(I)$, $P_1 \cap P_2 \cap \ldots \cap P_k = I \subseteq P$. By Theorem 2.39 ([5]), there exists $1 \le i \le k$ such that $P = P_i$, hence $m(I) = \{P_1, P_2, ..., P_k\}$ a finite set.

We denote all maximal ideals of a $BL$-algebra $A$ by $M(A)$. Obviously, $M(A) \subseteq PI(A)$.

**Corollary 3.15.** Let $A$ be a co-Noetherian $BL$-algebra. If $PI(A) = M(A)$, then $M(A)$ is a finite set.

**Proof:** Since $A$ be co-Noetherian, for any $P \in PI(A)$, $P$ is finitely generated, hence every element of $M(A)$ is finitely generated. By hypothesis, we obtain that every element of $M(A)$ is also a minimal ideal of $A$ containing the ideal $\{0\}$, thus $M(A) = m(\{0\})$, by Theorem 3.14 we have $M(A)$ is a finite set.

## 4. CONCLUSION

In the paper, we investigate further properties of co-Noetherian $BL$-algebras. We also provide the sufficient conditions for a quotient $BL$-algebra to be co-Noetherian. A special $BL$-algebra is introduced and some characterizations for $pBL$-algebra are given. Finally, we characterize co-Noetherian $BL$-algebras by prime ideals.

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