International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 9, September 2015, PP 8-17
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

On Common Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Menger Space

V. H. Badshah

School of Studies in Mathematics, Vikram University, Ujjain (M.P.), India

Suman Jain

Department of Mathematics, Govt. College, Kalapipal (M.P.) India

Arihant Jain

Department of Applied Mathematics,
Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.), India

Nitin Jauhari

Department of Applied Mathematics, Alpine Institute of Technology, Ujjain (M.P.)

Abstract: *In this paper, the concept of occasionally weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps. Our result generalizes and extends the result of Pathak and Verma [1].*

Keywords: *Probabilistic metric space, Menger space, common fixed point, compatible maps, occasionally weak compatibility.*

AMS Subject Classification: *Primary 47H10, Secondary 54H25.*

1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [2]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [3] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [4] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [5] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [6] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [7] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [8]. In the sequel, Pathak and Verma [1] proved a common fixed point theorem in Menger space using compatibility and weak compatibility. Using the concept of compatible mappings of type (A), Jain et. al. [9, 10] proved some interesting fixed point theorems in Menger space. Afterwards, Jain et. al. [11] proved the fixed point theorem using the concept of weak compatible maps in Menger space.

In this paper a fixed point theorem for six self maps has been proved using the concept of occasionally weak compatibility which turns out to be a material generalization of the result of Pathak and Verma [1].

2. Preliminaries

Definition 2.1. A mapping $\mathcal{F} : \mathrm{R} \to \mathrm{R}^+$ is called a *distribution* if it is non-decreasing left continuous with

$\inf\{F(t) \mid t \in R\} = 0$ and $\sup\{F(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$.

Definition 2.2. [8] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [8] A *probabilistic metric space (PM-space)* is an ordered pair $(X, \mathcal{F})$ consisting of a non-empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of F at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [8] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \ge t\{F_{u,v}(x), F_{v,w}(y)\}$, for all $u, v, w \in X$, $x, y \ge 0$.

Definition 2.5. [8] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \text{ for all } n \ge M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1 - \lambda \qquad \text{for all } m, n \ge M(\varepsilon, \lambda).$$

A Menger PM-space (X, F, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way:

Proposition 2.1. [8] If (X, d) is a metric space then the metric d induces mappings $\mathcal{F} : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$$H(k) = 0, \quad \text{for } k \le 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a, b) = \min\{a, b\}$. Then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.

The space (X, F, t) so obtained is called the *induced Menger space.*

Definition 2.6. [1] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [1] Self mappings A and S of a Menger space (X, $\mathcal{F}$, t) are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all x>0, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [12] Self maps A and S of a Menger PM-space (X, $\mathcal{F}$, t) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Example 2.1. Let (X, $\mathcal{F}$, t) be the Menger PM-space, where X = [0, 4]. Define F by

$$F_{x,y}(t) = \begin{cases} \dfrac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}.$$

Define A, S : X $\to$ X by

$Ax = 9x$ and $Sx = x^3$ for all $x \in X$ then $Ax = Sx$ for $x = 0$ and 3.

But AS(0) = SA(0) and AS(9) $\neq$ SA(9).

Thus, S and T are occasionally weakly compatible mappings but not weakly compatible.

Remark 2.1. In view of above example, it follows that the concept of occasionally weakly compatible is more general than that of weak compatibility.

Lemma 2.1. [1] Let (X, $\mathcal{F}$, *) be a Menger space with t-norm * such that the family $\{*_n(x)\}_{n\in N}$ is equicontinuous at x = 1 and let E denote the family of all functions $\phi : R^+ \to R^+$ such that ϕ is non-decreasing with $\lim_{n\to\infty} \phi^n(t) = +\infty$, $\forall\, t > 0$. If $\{y_n\}_{n\in N}$ is a sequence in X satisfying the condition

$$F_{y_n, y_{n+1}}(t) \geq F_{y_{n-1}, y_n}(\phi(t)),$$

for all t > 0 and $\alpha \in [-1, 0]$, then $\{y_n\}_{n\in N}$ is a Cauchy sequence in X.

Proposition 2.2. Let $\{x_n\}$ be a Cauchy sequence in a Menger space (X, $\mathcal{F}$, t) with continuous t-norm t. If the subsequence $\{x_{2n}\}$ converges to x in X, then $\{x_n\}$ also converges to x.

Proof. As $\{x_{2n}\}$ converges to x, we have

$$F_{x_n,x}(\varepsilon) \geq t\left(F_{x_n,x_{2n}}\left(\frac{\varepsilon}{2}\right), F_{x_{2n},x}\left(\frac{\varepsilon}{2}\right)\right).$$

Then

$\lim_{n\to\infty} F_{x_n,x}(\varepsilon) \geq t(1,1)$, which gives $\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1$, $\forall\, \varepsilon > 0$ and the result follows.

3. Main Result

Theorem 3.1. Let A, B, S, T, P and Q be self mappings on a Menger space (X, $\mathcal{F}$, *) with continuous t-norm * satisfying :

(3.1.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(3.1.2) AB = BA, ST = TS, PB = BP, QT = TQ;

(3.1.3) One of ST(X), Q(X), AB(X) or P(X) is complete;

(3.1.4) The pairs (P, AB) and (Q, ST) are occasionally weak compatible;

(3.1.5) $[1 + \alpha F_{ABx, STy}(t)] * F_{Px, Qy}(t)$

$$\geq \alpha \min\{F_{Px, ABx}(t) * F_{Qy, STy}(t),\ F_{Px, STy}(2t) * F_{Qy, ABx}(2t)\}$$
$$+ F_{ABx, STy}(\phi(t)) * F_{Px,ABx}(\phi(t)) * F_{Qy, STy}(\phi(t)) * F_{Px, STy}(2\phi(t)) * F_{Qy, ABx}(2\phi(t))$$

for all x, y $\in$ X, t > 0 and $\phi \in$ E.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$. From condition (3.1.1) $\exists\ x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \quad \text{and} \quad Qx_1 = ABx_2.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Px_{2n} = STx_{2n+1} \quad \text{and} \quad y_{2n+1} = Qx_{2n+1} = ABx_{2n+2} \qquad \text{for } n = 0, 1, 2, \ldots .$$

Step I. Let us show that $F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t))$.

For, putting x_{2n+2} for x and x_{2n+1} for y in (3.1.5) and then on simplification, we have

$$[1 + \alpha F_{ABx_{2n+2}, STx_{2n+1}}(t)] * F_{Px_{2n+2}, Qx_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{Px_{2n+2}, ABx_{2n+2}}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Px_{2n+2}, STx_{2n+1}}(2t)\ F_{Qx_{2n+1}, ABx_{2n+2}}(2t)\}$$

$$+ F_{ABx_{2n+2}, STx_{2n+1}}(\phi(t)) * F_{Px_{2n+2}, ABx_{2n+2}}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t))$$

$$* F_{Px_{2n+2}, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABx_{2n+2}}(2\phi(t))$$

$$[1 + \alpha F_{y_{2n+1}, y_{2n}}(t)] * F_{y_{2n+2}, y_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{y_{2n+2}, y_{2n+1}}(t) * F_{y_{2n+1}, y_{2n}}(t), F_{y_{2n+2}, y_{2n}}(2t) * F_{y_{2n+1}, y_{2n+1}}(2t)\} + F_{y_{2n+1}, y_{2n}}(\phi(t))$$

$$* F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n}}(2\phi(t)) * F_{y_{2n+1}, y_{2n+1}}(2\phi(t))$$

$$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{y_{2n+2}, y_{2n}}(2t), F_{y_{2n+2}, y_{2n}}(2t)\} + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$$

$$* F_{y_{2n+2}, y_{2n}}(2\phi(t)) * 1$$

$$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$$

$$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(2\phi(t))$$

$$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+2}, y_{2n}}(2t)$$

$$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y2n}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t))$$

$$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$$

or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n+2}}(\phi(t)) * F_{y_{2n}, y_{2n+1}}(\phi(t))$

or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq \min\{F_{y_{2n+1}, y_{2n+2}}(\phi(t)), F_{y_{2n}, y_{2n+1}}(\phi(t))\}$.

If $F_{y_{2n+1}, y_{2n+2}}(\phi(t))$ is chosen 'min' then we obtain

$$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \ \forall\ t > 0$$

a contradiction as $\phi(t)$ is non-decreasing function.

Thus,

$$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)), \ \forall \ t > 0.$$

Similarly, by putting x_{2n+2} for x and x_{2n+3} for y in (3.1.5), we have

$$F_{y_{2n+3}, y_{2n+2}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \ \forall \ t > 0.$$

Using these two, we obtain

$$F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t)), \ \forall \ n = 0, 1, 2, \dots , t > 0.$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X.

Case I. ST(X) is complete. In this case $\{y_{2n}\} = \{STx_{2n+1}\}$ is a Cauchy sequence in ST(X), which is complete. Thus $\{y_{2n+1}\}$ converges to some $z \in ST(X)$. By proposition 2.2, we have

$$\{Qx_{2n+1}\} \to z \quad \text{and} \quad \{STx_{2n+1}\} \to z, \tag{3.1.6}$$

$$\{Px_{2n}\} \to z \quad \text{and} \quad \{ABx_{2n}\} \to z. \tag{3.1.7}$$

As $z \in ST(X)$ there exists $u \in X$ such that $z = STu$.

Step I. Put $x = x_{2n}$ and $y = u$ in (3.1.5), we get

$$[1 + \alpha F_{ABx_{2n}, STu}(t)] * F_{Px_{2n}, Qu}(t)$$
$$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{Qu, STu}(t), F_{Px_{2n}, STu}(2t) * F_{Qu, ABx_{2n}}(2t)\}$$
$$+ F_{ABx_{2n}, STu}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{Qu, STu}(\phi(t)) * F_{Px_{2n}, STu}(2\phi(t))$$
$$* F_{Qu, ABx_{2n}}(2\phi(t)).$$

Letting $n \to \infty$ and using (3.1.6), (3.1.7), we get

$$[1 + \alpha F_{z, z}(t)] * F_{z, Qu}(t)$$
$$\geq \alpha \min\{F_{z, z}(t) * F_{Qu, z}(t), F_{z, z}(2t) * F_{Qu, z}(2t)\}$$
$$+ F_{z, z}(\phi(t)) * F_{z, z}(\phi(t))$$
$$* F_{Qu, z}(\phi(t)) * F_{z, z}(2\phi(t)) * F_{Qu, z}(2\phi(t))$$

$$F_{z, Qu}(t) + \alpha F_{z, Qu}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(2t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(2\phi(t))$$

$$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(t) * F_{z, z}(t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha F_{Qu, z}(t) + F_{Qu, z}(\phi(t))$$

$$F_{Qu, z}(t) \geq F_{Qu, z}(\phi(t))$$

which is a contradiction and we get

$Qu = z$ and so $Qu = z = STu$.

Since (Q, ST) is occasionally weakly compatible, we have

$STz = Qz$.

Step III. Put $x = x_{2n}$ and $y = Tz$ in (3.1.5), we have

$$[1 + \alpha F_{ABx_{2n}, STTz}(t)] * F_{Px_{2n}, QTz}(t)$$

$$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{QTz, STTz}(t), F_{Px_{2n}, STTz}(2t) * F_{QTz, ABx_{2n}}(2t)\}$$

$$+ F_{ABx_{2n}, STTz}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{QTz, STTz}(\phi(t))$$

$$* F_{Px_{2n}, STTz}(2\phi(t)) * F_{QTz, ABx_{2n}}(2\phi(t)).$$

As QT = TQ and ST = TS, we have

$$QTz = TQz = Tz \quad \text{and} \quad ST(Tz) = T(STz) = Tz.$$

Letting $n \to \infty$, we get

$$[1 + \alpha F_{z, Tz}(t)] * F_{z, Tz}(t) \geq \alpha \min\{F_{z, z}(t) * F_{Tz, Tz}(t), F_{z, Tz}(2t) * F_{Tz, z}(2t)\}$$

$$+ F_{z, Tz}(\phi(t)) * F_{z, z}(\phi(t))$$

$$* F_{Tz, Tz}(\phi(t)) * F_{z, Tz}(2\phi(t)) * F_{Tz, z}(2\phi(t))$$

$$F_{z, Tz}(t) + \alpha\{F_{z, Tz}(t) * F_{z, Tz}(t)\} \geq \alpha \min\{1 * F_{Tz, z}(2t)\} + F_{z, Tz}(\phi(t)) * 1 * 1 * F_{Tz, z}(2\phi(t))$$

$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(2t) + F_{Tz, z}(\phi(t)) * F_{Tz, z}(2\phi(t))$$

$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha \{F_{Tz, z}(t) * F_{z, z}(t)\} + F_{Tz, z}(\phi(t))*F_{Tz, z}(\phi(t))*F_{z, z}(\phi(t))$$

$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(t) + F_{Tz, z}(\phi(t))$$

$$F_{Tz, z}(t) \geq F_{Tz, z}(\phi(t))$$

which is a contradiction and we get Tz = z.

Now, STz = Tz = z implies Sz = z.

Hence, Sz = Tz = Qz = z.

Step IV. As $Q(X) \subseteq AB(X)$, there exists $w \in X$ such that

$$z = Qz = ABw.$$

Put x = w and y = x_{2n+1} in (3.1.5), we get

$$[1 + \alpha F_{ABw, STx_{2n+1}}(t)] * F_{Pw, Qx_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{Pw, ABw}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Pw, STx_{2n+1}}(2t)$$

$$* F_{Qx_{2n+1}, ABw}(2t)\} + F_{ABw, STx_{2n+1}}(\phi(t)) * F_{Pw, ABw}(\phi(t))$$

$$* F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{Pw, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABw}(2\phi(t)).$$

Letting $n \to \infty$, we get

$$[1 + \alpha F_{z, z}(t)] * F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t) * F_{z, z}(t), F_{Pw, z}(2t) * F_{z, z}(2t)\}$$

$$+ F_{z, z}(\phi(t)) * F_{Pw, z}(\phi(t))$$

$$* F_{z, z}(\phi(t)) * F_{Pw, z}(2\phi(t)) * F_{z, z}(2\phi(t))$$

$$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t), F_{Pw, z}(2t)\} + F_{Pw, z}(\phi(t)) * F_{Pw, z}(2\phi(t))$$

$$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t), F_{Pw, z}(t) * F_{z, z}(t)\} + F_{Pw, z}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{Pw,\,z}(t) + \alpha F_{Pw,\,z}(t) \geq \alpha \min\{F_{Pw,\,z}(t),\, F_{Pw,\,z}(t)\} + F_{Pw,\,z}(\phi(t))$$

$$F_{Pw,\,z}(t) + \alpha F_{Pw,\,z}(t) \geq \alpha F_{Pw,\,z}(t)\} + F_{Pw,\,z}(\phi(t))$$

$$F_{Pw,\,z}(t) \geq F_{Pw,\,z}(\phi(t))$$

which is a contradiction and hence, we get $Pw = z$.

Hence, $Pz = z = ABz$.

Step V. Put $x = z$ and $y = x_{2n+1}$ in (3.1.5), we have

$$[1 + \alpha F_{ABz,\, STx_{2n+1}}(t)] * F_{Pz,\, Qx_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{Pz,\, ABz}(t) * F_{Qx_{2n+1},\, STx_{2n+1}}(t),\, F_{Pz,\, STx_{2n+1}}(2t) * F_{Qx_{2n+1},\, ABz}(2t)\}$$

$$+ F_{ABz,\, STx_{2n+1}}(\phi(t)) * F_{Pz,\, ABz}(\phi(t)) * F_{Qx_{2n+1},\, STx_{2n+1}}(\phi(t)) * F_{Pz,\, STx_{2n+1}}(2\phi(t))$$

$$* F_{Qx_{2n+1},\, ABz}(2\phi(t)).$$

Letting $n \to \infty$, we get

$$[1 + \alpha F_{Pz,\, z}(t)] * F_{Pz,\, z}(t)$$

$$\geq \alpha \min\{F_{Pz,\, Pz}(t) * F_{z,\, z}(t),\, F_{Pz,\, z}(2t) * F_{z,\, Pz}(2t)\} + F_{Pz,\, z}(\phi(t)) * F_{Pz,\, Pz}(\phi(t))$$

$$* F_{z,\, z}(\phi(t)) * F_{Pz,\, z}(2\phi(t)) * F_{z,\, Pz}(2\phi(t))$$

$$F_{Pz,\, z}(t) + \alpha\{F_{Pz,\, z}(t) * F_{Pz,\, z}(t)\}$$

$$\geq \alpha \min\{1 * 1,\, F_{Pz,\, z}(2t) * F_{Pz,\, z}(2t)\} + F_{Pz,\, z}(\phi(t)) * 1 * 1 * F_{Pz,\, z}(2\phi(t)) * F_{z,\, Pz}(2\phi(t))$$

$$F_{Pz,\, z}(t) + \alpha F_{Pz,\, z}(t) \geq \alpha \min\{1,\, F_{Pz,\, z}(2t)\} + F_{Pz,\, z}(\phi(t)) * F_{Pz,\, z}(2\phi(t))$$

$$F_{Pz,\, z}(t) + \alpha F_{Pz,\, z}(t) \geq \alpha F_{Pz,\, z}(2t) + F_{Pz,\, z}(\phi(t)) * F_{Pz,\, z}(2\phi(t))$$

$$F_{Pz,\, z}(t) + \alpha F_{Pz,\, z}(t) \geq \alpha\{F_{Pz,\, z}(t) * F_{z,\, z}(t)\} + F_{Pz,\, z}(\phi(t)) * F_{Pz,\, z}(\phi(t)) * F_{z,\, z}(\phi(t))$$

$$F_{Pz,\, z}(t) + \alpha F_{Pz,\, z}(t) \geq \alpha\{F_{Pz,\, z}(t) * 1\} + F_{Pz,\, z}(\phi(t)) * 1$$

$$F_{Pz,\, z}(t) + \alpha F_{Pz,\, z}(t) \geq \alpha F_{Pz,\, z}(t) + F_{Pz,\, z}(\phi(t))$$

$$F_{Pz,\, z}(t) \geq F_{Pz,\, z}(\phi(t))$$

which is a contradiction and hence, $Pz = z$

and so $z = Pz = ABz$.

Step VI. Put $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$[1 + \alpha F_{ABBz,\, STx_{2n+1}}(t)] * F_{PBz,\, Qx_{2n+1}}(t)$$

$$\geq \alpha \min\{F_{PBz,\, ABBz}(t) * F_{Qx_{2n+1},\, STx_{2n+1}}(t),\, F_{PBz,\, STx_{2n+1}}(2t)$$

$$* F_{Qx_{2n+1},\, ABBz}(2t)\} + F_{ABBz,\, STx_{2n+1}}(\phi(t)) * F_{PBz,\, ABBz}(\phi(t))$$

$$* F_{Qx_{2n+1},\, STx_{2n+1}}(\phi(t)) * F_{PBz,\, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1},\, ABBz}(2\phi(t)).$$

As $BP = PB$, $AB = BA$ so we have

$$P(Bz) = B(Pz) = Bz \text{ and } AB(Bz) = B(AB)z = Bz.$$

Letting $n \to \infty$ and using (3.1.6), we get

$$[1 + \alpha F_{Bz, z}(t)] * F_{Bz, z}(t)$$
$$\geq \alpha \min\{F_{Bz, Bz}(t) * F_{z, z}(t), F_{Bz, z}(2t) * F_{z, Bz}(2t)\}$$
$$+ F_{Bz, z}(\phi(t)) * F_{Bz, Bz}(\phi(t)) * F_{z, z}(\phi(t)) * F_{Bz, z}(2\phi(t)) * F_{z, Bz}(2\phi(t))$$

$$F_{Bz, z}(t) + \alpha\{F_{Bz, z}(t) * F_{Bz, z}(t)\}$$
$$\geq \alpha \min\{1 * 1, F_{Bz, z}(2t)\} + F_{Bz, z}(\phi(t)) * 1 * 1 * F_{Bz, z}(2\phi(t))$$

$$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha F_{Bz, z}(2t) + F_{Bz, z}(\phi(t)) * F_{Bz, z}(2\phi(t))$$

$$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha \{F_{Bz, z}(t) * F_{z, z}(t)\} + F_{Bz, z}(\phi(t)) * F_{Bz, z}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha \{F_{Bz, z}(t) * 1\} + F_{Bz, z}(\phi(t)) * 1$$

$$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha F_{Bz, z}(t) + F_{Bz, z}(\phi(t))$$

$$F_{Bz, z}(t) \geq F_{Bz, z}(\phi(t))$$

which is a contradiction and we get $Bz = z$ and so

$$z = ABz = Az.$$

Therefore, $Pz = Az = Bz = z$.

Combining the results from different steps, we get

$$Az = Bz = Pz = Qz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case when $P(X)$ is complete follows from above case as $P(X) \subseteq ST(X)$.

Case II. AB(X) is complete. This case follows by symmetry. As $Q(X) \subseteq AB(X)$, therefore the result also holds when $Q(X)$ is complete.

Uniqueness:

Let z_1 be another common fixed point of A, B, P, Q, S and T. Then

$Az_1 = Bz_1 = Pz_1 = Sz_1 = Tz_1 = Qz_1 = z_1$, assuming $z \neq z_1$.

Put $x = z$ and $y = z_1$ in (3.1.5), we get

$$[1 + \alpha F_{ABz, STz_1}(t)] * F_{Pz, Qz_1}(t)$$
$$\geq \alpha \min\{F_{Pz, ABz}(t) * F_{Qz_1, STz_1}(t), F_{Pz, STz_1}(2t) * F_{Qz_1, ABz}(2t)\}$$
$$+ F_{ABz, STz_1}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qz_1, STz_1}(\phi(t)) * F_{Pz, STz_1}(2\phi(t)) * F_{Qz_1, ABz}(2\phi(t))$$

$$[1 + \alpha F_{z, z_1}(t)] * F_{z, z_1}(t)$$
$$\geq \alpha \min\{F_{z, z}(t) * F_{z_1, z_1}(t), F_{z, z_1}(2t) * F_{z_1, z}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z}(\phi(t))$$
$$* F_{z_1, z_1}(\phi(t)) * F_{z, z_1}(2\phi(t)) * F_{z_1, z}(2\phi(t))$$

$$F_{z, z_1}(t) + \alpha\{F_{z, z_1}(t) * F_{z, z_1}(t)\} \geq \alpha \min\{1, F_{z, z_1}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z_1}(2\phi(t))$$

$$F_{z, z_1}(t) + \alpha F_{z, z_1}(t) \geq \alpha F_{z, z_1}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z_1}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{z_1, z}(t) + \alpha F_{z_1, z}(t) \geq \alpha\{F_{z_1, z}(t) * F_{z, z}(t)\} + F_{z_1, z}(\phi(t)) * 1$$

$$F_{z_1, z}(t) + \alpha F_{z_1, z}(t) \geq \alpha F_{z_1, z}(t) + F_{z_1, z}(\phi(t))$$

$$F_{z_1, z}(t) \geq F_{z_1, z}(\phi(t))$$

which is a contradiction.

Hence $z = z_1$ and so z is the unique common fixed point of A, B, S, T, P and Q.

This completes the proof.

Remark 3.1. If we take B = T = I, the identity map on X in theorem 3.1, then condition (3.1.2) is satisfied trivially and we get

Corollary 3.1. Let A, S, P and Q be self mappings on a Menger space $(X, \mathcal{F}, *)$ with continuous t-norm * satisfying :

(i) $P(X) \subseteq T(X)$, $Q(X) \subseteq A(X)$;

(ii) One of S(X), Q(X), A(X) or P(X) is complete;

(iii) The pairs (P, A) and (Q, S) are occasionally weak compatible;

(iv) $$[1 + \alpha F_{Ax, Sy}(t)] * F_{Px, Qy}(t) \geq \alpha \min\{F_{Px, Ax}(t) * F_{Qy, Sy}(t), F_{Px, Sy}(2t) * F_{Qy, Ax}(2t)\} + F_{Ax, Sy}(\phi(t)) * F_{Px,Ax}(\phi(t)) * F_{Qy, Sy}(\phi(t)) * F_{Px, Sy}(2\phi(t)) * F_{Qy, Ax}(2\phi(t))$$

for all x, y $\in$ X, t > 0 and $\phi \in$ E.

Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Pathak and Verma [1] in the sense that both the pair of self maps has been restricted to occasionally weak compatibility and we have dropped the condition of continuity in a Menger space with continuous t-norm.

REFERENCES

[1]. Pathak, H.K. and Verma, R.K., Common fixed point theorems for weakly compatible mappings in Menger space and application, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 24, 1199-1206.

[2]. Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.

[3]. Schweizer, B. and Sklar, A., Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.

[4]. Sehgal, V.M. and Bharucha-Reid, A.T., Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.

[5]. Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.

[6]. Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.

[7]. Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.

[8]. Mishra, S.N., Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.

[9]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger space through compatible maps of type (A), Chh. J. Sci. Tech. 2 (2005), 1-12.

[10]. Jain, Arihant and Singh, Bijendra, A fixed point theorem in Menger space through compatible maps of type (A), V.J.M.S. 5(2), (2005), 555-568.

[11]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger Spaces, The Aligarh Bull. of Math. 25 (1), (2006), 23-31.

[12]. Jain, A. and Chaudhary, B., On common fixed point theorems for semi-compatible and occasionally weakly compatible mappings in Menger space, International Journal of Research and Reviews in Applied Sciences, Vol. 14 (3), (2013), 662-670.

AUTHORS' BIOGRAPHY

Dr. V. H. Badshah, presently working on the post of Professor and Head, School of Studies in Mathematics, Vikram University, Ujjain. He has 25 years of teaching experience. He selected the post of Scientific Assistance in the Indian Institute of Geo-Magnetism Colaba, Mumbai. His area of interest includes Functional Analysis, Fixed Point Theory, Mathematical Modeling and Operations Research. He has authored many books. Twenty students have got Ph.D. under his supervision.

Dr. Suman Jain, presently working of the post of Professor and Head, Department of Mathematics, Govt. College Kalapipal (M.P.). She has 25 years of teaching experience. Her areas of interest includes fixed point theory and Mathematical Modeling. Eight students have got Ph.D. under her supervision. She has published about 30 research papers in national and international journals.

Dr. Arihant Jain, did his Post Graduation in Mathematics in the year 2000 from School of Studies in Mathematics, Vikram University, Ujjain. He has been awarded doctorate degree in Mathematics in the year 2007 from School of Studies in Mathematics, Vikram University, Ujjain on Fixed Point Theory. He has published 100 research papers in national and international journals of repute. He has a post graduate teaching experience of 7 years and graduate teaching experience of 6 years. Two students have got Ph.D. under his guidance. Presently, he is working on the post of Academic Dean, Professor and Head, Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain and actively engaged with his researchers.

Mr. Nitin Jauhari, presently working on the post of Assistant Professor in Alpine Institute of Technology, Ujjain. He has 16 years of teaching experience. His area of interest is fixed point theorem. He has published three papers in national journal.