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About New Statement and About New Method of Cauchy Problem for Singular Perturbed Differential Equation of the Type of Lighthill

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Abstract: *Here the initial problem of Cauchy for singularly perturbed ordinary differential equation of the type of Lighthill will set at the regularly singularly point of the corresponding unperturbed equation. In practice, it is usually the initial problem will set at nonsingular point of differential equations. At first the asymptotic solution of this problem will construct by the method of uniformization, after we will suppose new method.*

Keywords: *problem of Caushy, singular point, perturbed and unperturbed equation, asymptotic solution, method of uniformization,new asymptotical method,derivative Freshe.*

1. Introduction

English mathematician and mechanic M. Lighthill at 1948 y. [1] investigated next problem:

$$\left(x+\varepsilon u(x)\right)\frac{du(x)}{dx}=-q(x)u(x)+r(x), \tag{1}$$

$$u(1)=a.$$

here $0<\varepsilon$ is small parameter, $0\le x\le 1$ - nondependent variable, a - given constant, $q(x),r(x)\in C^{\omega}[0,1]$ - analytical functions, $u(x)$ - unknown function.

We will suppose that $q(0)=q_0>0$. The point $x=0$ is singular point for unperturbed equation (1) ($\varepsilon=0$):

$$L_0 u_0(x):=x\frac{du(x)}{dx}+q(x)u_0(x)=r(x),\ \ u_0(1)=a.$$

(2)

The solution of the problem (2) is:

$$u_0(x)=x^{-q_0}w(x), \tag{3}$$

here

$$p(x)=\exp\{\int_1^x\left(q_0-q(s)\right)s^{-1}ds\}\in C^{\omega}[0,1], w(x)=p(x)[w_0+\int_0^x s^{q_0-1}\,p^{-1}(s)\,r(s)ds]\in C^{\omega}[0,1],$$

$$w_0=(a-A)\,p(0), A=\int_0^1 p^{-1}(s)s^{q_0-1}r(s)ds.$$

Let $w_0\ne 0$, then the solution (3) is the unbounded function on the interval [0,1] and the point $x=0$ is the pole of (3). If we will seek the solution of the problem (1) by classical method of the small parameter in the form:

$$u(x)=u_0(x)+\varepsilon u_1(x)+\varepsilon^2 u_2(x)+\dots\ . \tag{4}$$

then, for determination of the unknown functions $u_k(x)$ we will have got next equations:

$$L_0 u_1(x) = -u_0(x)\frac{du_0(x)}{dx}, \quad u_1(1) = 0, \tag{5.1}$$

$$L_0 u_2(x) = -u_0(x)\frac{du_1(x)}{dx} - \frac{du_0(x)}{dx}u_1(x), \quad u_2(1) = 0, \tag{5.2}$$

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$$L_0 u_n(x) = -\sum_{\substack{i+j=n-1\\ i,j\ge 0}} u_i(x)\frac{du_j(x)}{dx}, \quad u_n(1) = 0. \tag{5.n}$$

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By using (3) the problem (5.1) we can rewrite in the form:

$$L_0 u_1(x) \sim w_0^2 q_0 x^{-2q_0-1},\ x \to 0.$$

From here we have

$$u_1(x) \sim -q_0(1+q_0)^{-1} x^{-1-2q_0},\ x \to 0.$$

Analogously we have

$$u_k(x) \sim O\left(x^{-q_0-(q_0+1)k}\right),\ x \to 0, \quad \forall k \in N.$$

Therefore the solution (4) we can represent in the form

$$u(x) \sim x^{-q_0}\left[w_0 + \alpha_1\left(\varepsilon x^{-(q_0+1)}\right) + \alpha_2\left(\varepsilon x^{-(q_0+1)}\right)^2 + \dots + \alpha_n\left(\varepsilon x^{-(q_0+1)}\right)^n + \dots\right],\ x \to 0, \tag{6}$$

here $\alpha_k = const$. It is seen from (6) that this is asymptotical approximation of the solution only on the interval $(\varepsilon^{1/(q_0+1)}, 1]$.

Therefore Lighthill supposed to seek the solution of the problem (1) in the form:

$$u(\xi) = u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \dots, \tag{7}$$

$$x = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \dots .$$

Functions $u_k(\xi),\ x_k(\xi)$ will determine after substituting (7) to the equation (1) and we will have one equation for to ones. This approach was named after him as the method of Lighthill. The method of Lighthill developed by Career G.F., Wasov W.A., Sibuya Y., Takahasi K.I., Temple G., Pritulo M.F., Tsien H.S., Comstok C., Hapets P., Alymkulov K and others. It is possible to read these historical reviews in [4, 8, 10, 11]. We must note, that C.Comstok [8] on an example and K. Alymkulov [10, 11] in common case showed that in the application of the method of Lighthill to this problem will appears the unnecessary condition

$$x\frac{du_0(x)}{dx} \ne 0,\ \ x \in (0,1). \tag{8}$$

J.Temple [5] on an example suppose the method of uniformization, K.Alymkulov [10, 11] in the general case proved, that the problem [1] is equivalent to next uniformized problem:

$$\xi\frac{du(\xi)}{d\xi} = -q\left(x(\xi)\right)u(\xi) + r\left(x(\xi)\right),\ u(1) = a, \tag{9}$$

$$\xi\frac{dx(\xi)}{d\xi} = x(\xi) + \varepsilon u(\xi),\ x(1) = 1.$$

here $\xi \in [\xi_0(\varepsilon),1] = \mathrm{J}(\varepsilon),\ 0 < \xi_0(\varepsilon),\ \xi_0(0) = 0$. This equation determined by unique way functions $u(\xi), x(\xi)$.. Of course, in order that equations (1) and (9) were equivalent must be:

$x(\xi) + \varepsilon u(\xi) \neq 0$.

We must note in the method of uniformization will removed of the condition Wasov [8]. For example in [10] proved next

Theorem 1. Let

1) $q(x), r(x) \in C^\infty[0,1], q(0) = m \in N, w_0 > 0$. Then the solution of the problem (9) is represented in the form (7), that will convergent uniformly on $\mathrm{J}(\varepsilon)$;

2) it is correspondent to point $x = 0$ the unique point

$$0 < \xi_0 < \xi_0(\varepsilon) \sim \left(\frac{\varepsilon w_0}{q_0 + 1}\right)^{\frac{1}{q_0+1}},\ \varepsilon \to 0;$$

3) $x(\xi) + \varepsilon u(\xi) \neq 0$. It means, that problems (1) and (9) are equivalent.

Remark1. If $w_0 < 0$, then the equation $x(\xi) + \varepsilon u(\xi) = 0$ will have the solution $0 < \eta \sim \left(\frac{\varepsilon w_0 q_0}{q_0 + 1}\right)^{\frac{1}{q_0+1}},\ \varepsilon \to 0.$ Therefore the problem (1) will have a solution only on the interval $[\eta, 1]$. (Please compare this theorem with theorem 2)

2. Statement of the Problem

We will study next problem

$$\left(x + \varepsilon u(x)\right)\frac{du(x)}{dx} = -q(x)u(x) + r(x),\ u(0) = b, \tag{10}$$

here b is given real number. We will impose to function $q(x), r(x)$ satisfied the same conditions as earlier.

3. Construction of Asymptotic of the Solution of the Problem by Method of Uniformization

We will uniformize of the problem (10) by (9), where the number a undefined as yet. After substitution series (7) to the equation (9) we will have next equations for functions $u_k(\xi),\ x_k(\xi)$:

$$Lu_0(\xi) = r(\xi),\quad u_0(1) = a, \tag{11.0}$$

$$Lu_1(\xi) = -q_1(\xi)x_1(\xi)u_0(\xi) + r_1(\xi)x_1(\xi),\quad u_1(1) = 0, \tag{11.1.1}$$

$$M\,x_1(\xi) := \xi\frac{dx_1(\xi)}{d\xi} - x_1(\xi) = u_0(\xi),\quad x_1(1) = 0, \tag{11.1.2}$$

$$Lu_2(\xi) = -q_1x_1u_1 - \left(q_2x_1^2 + q_1x_2\right)u_0 + r_1\,x_2 + r_2\,x_1^2,\quad u_2(1) = 0, \tag{11.2.1}$$

$$M\,x_2(\xi) := u_1(\xi),\quad x_2(1) = 0, \tag{11.2.2}$$

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$$Lu_n(\xi) = -q_1 x_1 u_{n-1} - \left(q_2 x_1^2 + q_1 x_2\right) u_{n-1} + \ldots + q_n x_1^n u_0 + \ldots + r_1 x_n + \ldots + r_n x_1^n, u_n(1) = 0, \quad (11.n.1)$$

$$M x_n(\xi) := u_{n-1}(\xi), \quad x_n(1) = 0, \quad (11.n.2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution of the problem (11.0) will represented in the form (3), where $x = \xi$, that is

$$u_0(\xi) = \xi^{-q_0} w(\xi) \sim \xi^{-q_0} w_0, \ \xi \to 0. \quad (12.0)$$

Therefore the equation (11.1.2) will have a form:

$$M x_1(\xi) \sim w_0 \xi^{-q_0}.$$

From here we have got

$$x_1(\xi) \sim -(1+q_0)^{-1} w_0 \xi^{-q_0}, \ \xi \to 0. \quad (12.1)$$

Therefore the equation (11.1) will have a form:

$$Lu_1(\xi) \sim \alpha_1 (w_0 \xi^{-q_0})^2, \ \xi \to 0,$$

here and further by α_k, β_k $(k \in N)$ we will denote some real numbers. From here,

$$u_1(\xi) \sim \beta_1 (w_0 \xi^{-q_0})^2, \ \xi \to 0.$$

Analogously, we can have next functions

$$x_n(\xi) \sim \gamma_n (w_0 \xi^{-q_0})^n, \ u_n(\xi) \sim \beta_n (w_0 \xi^{-q_0})^{n+1}, \ \xi \to 0, \quad (13.n)$$

here $\alpha_1 = -w_0 (1+q_0)^{-1}$.

So, the main asymptotical term of the solution (7) we can rewrite in the next form:

$$u(\xi) \sim \xi^{-q_0} \left[w_0 + \alpha_1 \left(\varepsilon w_0 \xi^{-q_0}\right) + \alpha_2 \left(\varepsilon w_0 \xi^{-q_0}\right)^2 + \ldots + \alpha_n \left(\varepsilon w_0 \xi^{-q_0}\right)^n + \ldots \right] \quad (14.1)$$

$$x(\xi) \sim \xi - \gamma_1 \varepsilon \frac{w_0}{1+q_0} \xi^{-q_0} + \gamma_2 \left(\varepsilon w_0 \xi^{-q_0}\right)^2 + \ldots + \gamma_n \left(\varepsilon w_0 \xi^{-q_0}\right)^n + \ldots \ . \quad (14.2)$$

Let's the point $x = 0$ will correspond the point $\xi = \xi_0$, then

$$\xi_0 + \varepsilon (1+q_0)^{-1} w_0 \xi_0^{-q_0} + \gamma_2 \left(\varepsilon w_0 \xi_0^{-q_0}\right)^2 + \ldots + \gamma_n \left(\varepsilon w_0 \xi_0^{-q_0}\right)^n + \ldots = 0, \ \xi_0 \to 0.$$

From here we have got the main asymptotic of ξ_0:

$$\xi_0 \sim \left(\frac{\varepsilon w_0}{1+q_0}\right)^{\frac{1}{1+q_0}}, \ \varepsilon \to 0. \quad (15)$$

It is evidently, that series (7) or (14) are asymptotical on the segment $[\xi_0, 1]$. If we will substitute (15) to the first equality (14), then

$$u(0) = u(x)\big|_{x=0} = b = \sim w_0 \left(\varepsilon (1+q_0)^{-1} w_0\right)^{-\frac{q_0}{1+q_0}} = w_0^{\frac{1}{1+q_0}} \left((1+q_0)^{-1} \varepsilon\right)^{-\frac{q_0}{1+q_0}}, \varepsilon \to 0.$$

From here we have got

$$w_0 = \varepsilon^{q_0} b^{1+q_0} (1+q_0)^{-q_0}. \quad (16)$$

From (16) it is seen $w_0 > 0$ when $b^{1+q_0} > 0$. We proved next

Theorem 2. Let's $q(x), r(x) \in C^{\omega}[0,1]$, $q(0) > 0$. Then the solution of the problem (10) exist for the all $b^{1+q_0} > 0$.

For example, if q_0 is an odd number then it condition is true. Strictly this theorem we can prove by method of the majorant.

4. New Approach for this Problem

Here we will suppose that $q(x), r(x) \in C^{(\infty)}[0,1]$ and $q(0) = q_0 = m \in N$ for the simplicity. We will make next substitution

$$x = \mu t,\ u(x) = \mu^{-m} z(t),\ \mu^{1+m} = \varepsilon, \tilde{\mu} = \mu^{-1}, \tag{17}$$

in the (1), then

$$(t + z(t))\frac{dz(t)}{dt} + q(\mu t) z(t) = \mu^m r(\mu t),\ t \in [0, \tilde{\mu}], \tag{18}$$

$$z(0) = \mu^m b. \tag{19}$$

The solution of the problem (18)-(19) we will seek in the form

$$z(t,\varepsilon) = z_0(t) + z_1(t)\mu + z_2(t)\mu^2 + ..., \tag{20}$$

here functions $z_k(t)$ will depend also from μ, but this dependent we do not will write for simplicity. After substitution (20) to the equation we will have next equation for the functions $z_k(t)$.

$$(t + z_0(t)) z_0'(t) = q(\mu t) z_0(t),\ z_0(0) = \mu^m b, \tag{21.0}$$

$$Lz_1(t) := (t + z_0(t)) z_1'(t) + (z_0'(t) + q(\mu t)) z_1(t) = 0,\ u_1(0) = 0 \tag{21.1}$$

$$Lz_2(t) = -z_1(t) z_1'(t) +,\ z_2(0) = 0 \tag{21.2}$$

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$$Lz_m(t) = -\sum_{\substack{i+k=m \\ i,k\geq 1}} z_i(t) z_k'(t) + r(\mu t),\ z_m(0) = 0 \tag{21.m}$$

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$$Lz_\nu(t) = -\sum_{\substack{i+k=\nu \\ i,k\geq 1}} z_i(t) z_k'(t),\ z_\nu(0) = 0\ (\nu \neq m) \tag{26. ν}$$

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In order solve (21.0) we will rewrite this one in next equivalent form

$$(t + z_0(t)) z_0'(t) + m z_0(t) = \tilde{q}(\mu t) z_0(t),\ z_0(0) = \mu^m b \tag{21.0*}$$

here $\tilde{q}(\mu t) = m - q(\mu t)$.

At first we will solve next «unperturbed» equation:

$$\left(t+y(t)\right)\frac{dy(t)}{dt}+my(t)=0,\ \ y(0)=\mu^{m}b$$

The solution of this problem is:

$$t=c_0\,y^{-1/m}-\frac{1}{1+m}y:=\psi\left(c_0,y\right),c_0=\alpha\,\mu^{1+m},\ \ \alpha=\frac{1}{1+m}b^{\frac{1+m}{m}}.$$

The function t(y) is strictly decreasing functions on the segment

$$J\left(\mu\right)=\left[\tilde{y}\left(\mu\right),a_0\,\mu^{1+m}\right],here\ \tilde{y}\left(\mu\right)=\{c_0\mu-\frac{1}{1+m}\mu\,\tilde{y}^{1+\frac{1}{m}}(\mu)\}\sim c_0^{m}\,\mu^{m},\mu\to 0.$$

Therefore $t=\psi\left(c_0,y\right)$ will have the inverse function $y=\varphi\left(t,c_0\right),\ t\in\left[0,\mu\right]$.

The solution of the problem (21.0) or (21.0*) we will seek by method of Lagrange, that is in the form

$$z_0\left(t\right)=\varphi\left(t,c\left(t\right)\right)$$

here $c=c\left(t\right)$ is function from t. Then we will have next equation for the function $c\left(t\right)$

$$\frac{dc(t)}{dt}=\frac{\tilde{q}\left(\mu t\right)\cdot\varphi\left(t,c\right)}{\left(t+\varphi\left(t,c\right)\right)\varphi_c\left(t,c\right)},\ \ c\left(0\right)=c_0=\alpha\,\mu^{1+m}. \tag{23}$$

But from next equation

$$t=c\varphi^{-1/m}\left(t,c\right)-\frac{1}{1+m}\varphi\left(t,c\right)$$

we will have

$$\left(1+m\right)t\,\varphi^{1/m}\left(t,c\right)=\left(1+m\right)c-\varphi^{1+\frac{1}{m}}\left(t,c\right). \tag{24}$$

After differentiate this by c we have

$$\varphi_{\tilde{n}}\left(t,c\right)=m\frac{\varphi\left(t,c\right)}{\left(1+\varphi\left(t,c\right)\right)\varphi^{1/m}\left(t,c\right)} \tag{25}$$

Substitute this to (24) we have

$$\frac{dc(t)}{dt}=\frac{1}{m}\tilde{q}\left(\mu t\right)\cdot\varphi^{1/m}\left(t,c\left(t\right)\right).$$

By using (24) this equation we can rewrite in the next form:

$$\frac{dc(t)}{dt}=\frac{1+m}{m}\cdot\frac{c\left(\mu t\right)}{\left(1+m\right)t+\varphi\left(t,c\right)},\ \ c\left(0\right)=\alpha\,\mu^{1+m}.$$

By integrating this equation we have

$$\tilde{n}\left(t\right)=\alpha\,\mu^{1+m}\exp\left\{\frac{1+m}{m}\int_0^t\frac{\tilde{q}\left(\mu s\right)}{\left(1+m\right)s+\varphi\left(t,c\left(s\right)\right)}ds\right\}=T\left[c\right]. \tag{26}$$

Let's S is set of functions $c\left(t\right)$ satisfying next inequality

$$\alpha\,\mu^{1+m}\,e^{-\frac{l}{m}+1} \le c(t) \le \alpha\,\mu^{1+m}\,e^{\frac{l}{m}}, \tag{27}$$

here l is satisfied inequality

$$-l\mu t \le \tilde{q}(\mu t) \le l\mu t,\ \ \forall t \in [0,\tilde{\mu}].$$

We can easily proof that the operator T transforms S to S. Now we will proof, that the operator T from (26) is the contracting in S. Really the derivative Freshe of the (26) is

$$\frac{dT}{dc} = T\cdot\left\{\frac{1+m}{m}\int_0^t \frac{\tilde{q}(\mu s)}{\left[(1+m)s+\varphi(s,c(s))\right]^2}\varphi_c(s,c(s))\,ds\right\}.$$

Using (25) after estimating we will have

$$\left|\frac{dT}{dc}\right| \le a\mu^{2+m}e^{l/m}m\int_0^{1/\mu}\frac{s}{\left[(1+m)s+\varphi(s,c(s))\right]^2}\cdot\frac{\varphi(s,c(s))\,ds}{\left(s+\varphi(s,c(s))\right)\varphi^{1/m}(s,c(s))} \le$$

$$\overset{(25)}{\le} a\mu^{2+m}e^{l/m}m\int_0^{1/\mu}\frac{s}{(1+m)^2 s^2\varphi^{-2/m}(s,c(s))\left(s+\varphi(s,c(s))\right)}\cdot\frac{\varphi(s,c(s))\,ds}{\varphi^{1/m}(s,c(s))} =$$

$$= \frac{ae^{l/m}m}{(1+m)^2}\mu^{1+m}\int_0^{1/\mu}\frac{s\,\varphi^{1+1/m}(s,c(s))\,ds}{\left(s+\varphi(s,c(s))\right)c^2(s)} \le$$

$$\overset{(27)}{\le} \frac{ae^{2l/m}m}{(1+m)^2}\mu^{1+m}\int_0^{1/\mu}\frac{\left(a\mu^{1+m}\right)\left(a\mu^{1+m}\right)^{1/m}ds}{a^2\mu^{2(1+m)}} = \frac{a^{1/m}e^{2l/m}}{(1+m)^2}\mu^{1+1/m} = \delta(\mu)$$

If μ sufficiently small, then $\delta(\mu)<1$ and $\left|\frac{dT}{dc}\right| < \delta(\mu) < 1$. Therefor the operator $T[c]$ is contracting [[12] pp.390-392] and the problem (23) will have unique solution and $c(t)\in S$. It is true next

Theorem 3. The problem (21.0) will have unique solution $z_0(t)=z_0(t,\mu)$ and

$$a_1\mu^{1+m} \le z_0(t,\mu) \le a_2\mu^{2m+m^2}\ \ (0<a_1,a_2=const)$$

Lemma 1. Fundamental solution of the equation $L\eta(t)=0$ we can represent in the form

$$\hat{O}(t) = \frac{u_0(0)}{t+u_0(t)}\cdot X(t)\exp\left\{-\int_0^t\frac{m-1}{s+u_0(s)}ds\right\},\qquad X(t)=\exp\left\{\int_0^t\frac{m-q(\mu s)}{s+u_0(s)}ds\right\} \tag{28}$$

Really from (21.1) we will have

$$\eta(t) = \exp\left\{-\int_0^t\frac{u_0'(s)+q(\mu s)}{s+u_0(s)}ds\right\} = \exp\left\{-\int_0^t\frac{u_0'(s)+1+q(\mu s)-1}{s+u_0(s)}ds\right\} =$$

$$= \frac{u_0(0)}{t+u_0(t)}\exp\left\{\int_0^t\frac{1-q(\mu s)}{s+u_0(s)}ds\right\} =$$

$$= \frac{u_0(0)}{t+u_0(t)}\exp\left\{\int_0^t\frac{m-q(\mu s)}{s+u_0(s)}ds - \int_0^t\frac{m-1}{s+u_0(s)}ds\right\}$$

It is follow from (28), that $\left|X(t)\right|\le l,\ \left|X^{-1}(t)\right|\le l,\ \forall t\in\left[0,\tilde{\mu}\right]$. Here and further l will denote some constant, that not depend from μ.

Lemma 2. The inhomogeneous problem

$$L\eta(t)=g(t),\ \eta(0)=0 \tag{29}$$

here $g(t)\in C[0,\tilde{\mu}]$ and $\left|g(t)\right|\le l, t\in[0,\tilde{\mu}]$ will unique solution and $\left|\eta(t)\right|\le l,\ \forall t\in[0,\ \tilde{\mu}]$.

Proof. The solution of the problem (29) will presented in the form

$$\eta(t)=\frac{X(t)}{t+u_0(t)}\int_0^t X^{-1}(s)\cdot\exp\left\{-(m-1)\int_s^t\frac{d\tau}{\tau+u_0(\tau)}\right\}g(s)\,ds$$

Since

$$\frac{1}{t+u_0(t)}>0,\ \left|g(t)\right|\le l$$

we have got

$$\exp\left\{-(m-1)\int_s^t\frac{d\tau}{\tau+u_0(\tau)}\right\}\le 1.$$

Therefore

$$\eta(t)\le\frac{l\,t}{t+u_0(t)}\le l.$$

Lemma 2 is proved. From (29) we have also

$$\left|\eta'(t)\right|\le {l}/{(t+u_0(t))}.$$

It is follow from Lemma 2 next

Lemma 3. The problem (21.0) will have unique uniformly bounded solution in the segment $[0,\ \tilde{\mu}]$,

$$\left|u_n(t)\right|\le l,\ \forall n\in N.$$

From this Lemma we have got

Theorem 4. Series (20) is asymptotical on the $[0,\tilde{\mu}]$ that is

$$z(t,\varepsilon)=z_0(t)+z_1(t)\mu+z_2(t)\mu^2+\ldots+z_n(t)\mu^n+R_{n+1}(t,\mu)\mu^{n+1},$$

here

$$R_{n+1}(t,\mu)=O(1),\forall t\in\left[0,\tilde{\mu}\right].$$

and

$$z'(t,\varepsilon)=z_0'(t)+z_1'(t)\mu+z_2'(t)\mu^2+\ldots+z_n'(t)\mu^n+\frac{R_{n+1}(t,\mu)}{t+u_0(t)}\mu^{n+1},\ \left|z_n'(t)\right|\le\frac{l}{t+u_0(t)}.$$

Turning to the function $u(x)$ we have got:

$$u(x)=\frac{1}{\mu^m}\left[u_0\left(\frac{x}{\mu}\right)+\mu u_1\left(\frac{x}{\mu}\right)+...+\mu^n u_n\left(\frac{x}{\mu}\right)+\mu^{n+1}R_n\left(\frac{x}{\mu}\right)\right], \quad (30)$$

therefore

Theorem 5. Let's 1) $q(x),r(x)\in C^{(\infty)}[0,1]$; 2) $q_0=m$; 3) $b\neq 0$ then the solution of the problem (10) will represent in the form (30).

Example.

$$\left(x+\varepsilon u(x)\right)u(x)+u(x)=1,\ u(0)=b>0\ . \quad (31)$$

Here $q(x)=q_0=1,\ r(x)=1, u(x)=\mu^{-1}z(t),\ \varepsilon=\mu^2.$

Equations (21.0),(21.1), (21.2) will have next solutions:

$$z_0(t)=-t+\sqrt{t^2+b^2\mu^2},\ z_1(t)=\frac{t}{\sqrt{t^2+b^2\mu^2}}, z_2(t)=-\frac{t^2}{2\sqrt{(t^2+b^2\mu^2)^3}}.$$

Therefore

$$u(x,\varepsilon)=\frac{1}{\mu}\left[-t+\sqrt{t^2+\mu^2b^2}+\frac{\mu t}{\sqrt{t^2+\mu^2 b^2}}-\frac{\mu^2 t^2}{2\sqrt{\left(t^2+\mu^2 b^2\right)^3}}+O\left(\mu^3\right)\right](t=x/\mu). \quad (32)$$

We note that the problem (31) will have explicit solution

$$u(x)=\varepsilon^{-1}\left[-x+\sqrt{x^2+\varepsilon^2b^2+2\varepsilon x}\right].$$

After the substitution $x=\mu t$ we have

$$u(\mu t)=\frac{1}{\mu}\left[-t+\sqrt{t^2+\mu^2b^2+2\mu t}\right] \quad (33)$$

From (33) we can receive the expression (32).
Further we can solve the problem (31) by method of uniformization, then

$$\begin{cases}\xi\dfrac{du}{d\xi}=-u(\xi)+1,\quad u(1)=b,\\ \xi\dfrac{dx}{d\xi}=x(\xi)+\varepsilon u(\xi).\end{cases}$$

Solving first equation for $u(\xi)$ we have got

$$u(\xi)=\frac{\alpha}{\xi}+1,\ \alpha=b-1..$$

After substitution this in equation for $x(\xi)$ then we will have

$$x(\xi)=\left(1+\frac{\varepsilon\alpha}{2}+\varepsilon\right)\xi-\frac{\varepsilon\alpha}{2}\xi^{-1}-\varepsilon.$$

So, the parametric solution of the problem (31) will have the form:

$$u(\xi)=\frac{\alpha}{\xi}+1,\ x(\xi)=\left(1+\frac{\varepsilon\alpha}{2}+\varepsilon\right)\xi-\frac{\varepsilon\alpha}{2}\xi^{-1}-\varepsilon. \quad (34)$$

If the point $x=0$ corresponds to the point η , then

$$\eta \approx \sqrt{\frac{\varepsilon\alpha}{2}}=\sqrt{\frac{\varepsilon(b-1)}{2}}.$$

After substituting on the first expression of (34), we will have

$$u(0)\approx \frac{\alpha}{\sqrt{\frac{\varepsilon\alpha}{2}}}=\sqrt{\frac{2\alpha}{\varepsilon}}=a\Rightarrow b=1+\frac{a^2\varepsilon}{2}.$$

By exception from the second equation (34) a parameter ξ and putting in the first, we will get an exact solution (33).

5. CONCLUSION

Here new statement for the initial problem Cauchy for singularly perturbed ordinary differential equation of the type of Lighthill with singularly regularly point at first was solved by method of uniformization, after by new method. It is considered an example. The problem (1) we can solve by new method also. We note, that earlier asymptotic of the problem (1) was constructed by method of the structural matching in [13] (but, here estimation of the Remainder term less exact) and by boundary function method in [14].

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