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On the Direct Transformation of a Matrix Spectrum

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Abstract: *A method is presented for calculating a matrix spectrum with a given set of eigenvalues. It can be used to build systems with different spectrums with the aim of choosing desired alternative. It enables a practical implementation of control algorithms without resort to transformation of variables.*

Keywords: *Matrix spectrum, Frobenius matrix, Frobenius transformation, Spectral equation.*

1. Introduction

The problem of target transforming a spectrum is the subject of control theory. It is called as the method of characteristic equation setting, arrangement of eigenvalues, spectrum control, and modal control [1-4]. To change a spectrum the relationships between coefficients of characteristic polynomial and its roots are used. They are known as Vieta's formulas. A given set of numbers defines these coefficients. Because the matrix has alternate spectrum and coefficients, the elements of the matrix need to be changed. However, this procedure is made not with the parent matrix, but its transformed form called Frobenius. A Frobenius matrix have a row of elements representing coefficients of characteristic polynomial up to a sign. They are changed by summing with elements called feedback coefficients. As a result, we obtain a given spectrum.

To apply the method in a real-time control a transformation of variables is required to obtain a Frobenius matrix. The matter is that variables in technical system are physical parameters characterizing energy stores such as a speed of moving mass, a solenoid current, a capacitor voltage, and so on that are measured by sensors. Implementing the transformation by hardware requires additional schematic expenditures and software implementation needs an extra time that may cause a delay in the feedback loop and deteriorate dynamical properties of the system.

The method for calculating a desired spectrum, for which the authors found possible to use the definition in the headline, does not based on a Frobenius matrix.

It can be used to calculate the feedback coefficients of a control system with the aim to obtain a desired spectrum of closed-loop system without resort to transformation of variables. This allows practical problems of control to be solved at the design phase of the system. By simulating the system behavior with different spectrums it is possible to find a suitable alternative, which can be further implemented as a direct digital control algorithm. The paper is an outgrowth of the work [5].

2. Aim of the Work

Suppose $A=(a_{i,j})$, $i,\ j\in[1,k]$ is a given $k\times k$ real matrix, $\sigma(A)$ is its spectrum, and $\Lambda=\{\lambda_i\}$ is a set of real numbers. By A_x denote the matrix A with k replaced elements by unknowns.

The objective is to consider a range of issues related to evaluation of the unknowns, which are substituted into the matrix A_x, such that the condition $\sigma(A_x)=\Lambda$ is satisfied.

3. Definitions

Definition 1. Replacement is a replacing the elements of the matrix A (replaced elements) by other elements (replacing elements). Replacing matrix A_x (matrix with replacement) is a matrix with replacing elements.

Definition 2. Spectral equations of matrix A (replacement system) are k equations that was formed by replacing the coefficients of Vieta's formulae by the sums of main minors of the matrix A_x and by replacing the roots by the elements from a given set Λ.

Definition 3. Replacement of the i-th order is a replacement leading to spectral equations of the i-th order. Linear replacement is a replacement of the first order. Non-linear replacement is a replacement of the second order or higher.

Definition 4. Spectral transformation of the matrix A is a replacing the elements of the matrix A_x by the solution of spectral equations.

4. Frobenius Transformation of a Spectrum and Its Alternative

For a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}, \tag{1}$$

it is known Vieta's formalas

$$\begin{aligned} a_1 &= \sigma_1 + \sigma_2 + \dots + \sigma_k, \\ a_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \dots + \sigma_{k-1}\sigma_k, \\ &\dots \\ a_k &= \sigma_1\sigma_2 \dots \sigma_k, \end{aligned} \tag{2}$$

where σ_i and a_i are the i-th root of the characteristic polynomial and the result of summation in the i-th row, which are the coefficients of the characteristic polynomial considering the sign.

Frobenius transformation of a spectrum is based on obtaining the elements on the left-hand side (taking into account the sign) by non-singular transformation of the matrix A and by supplementing them to the values that satisfy a given set. This corresponds to the fact that the sum on the right-hand side (2) are replaced by the same relationships between the numbers of a given set $\Lambda = \{\lambda_i\}$, and the elements on the left-hand side are supplemented by unknowns x_i. This leads the system to the equations

$$\begin{aligned} a_1 + x_1 &= \lambda_1 + \lambda_2 + \dots + \lambda_k = d_1, \\ a_2 + x_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{k-1}\lambda_k = d_2, \\ &\dots \\ a_k + x_k &= \lambda_1\lambda_2 \dots \lambda_k = d_k \end{aligned} \tag{3}$$

with an obvious solution $x_1 = d_1 - a_1$, where d_i is a sum in the i-th row. Substituting the solution into Frobenius matrix one forms its spectrum with the values from a given set Λ.

The possibility to change a matrix spectrum by supplementing the matrix elements to the values that satisfy a given set provides an alternative to Frobenius transformation of a matrix.

To perform this procedure, we use the system (2) in the form of sums of main minors on the left-hand side. The example of such system for a matrix of the 3-rd order is given by

$$\begin{aligned} &a_{11} + a_{22} + a_{33} = a_1, \\ &a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32} = a_2, \\ &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} = a_3. \end{aligned}$$

Now, we supplement arbitrary elements of A, for example, the main diagonal elements by unknowns x_1, x_2, and x_3. As a result, the matrix A takes the form

$$A_x = \begin{pmatrix} a_{11}+x_1 & a_{12} & a_{13} \\ a_{21} & a_{22}+x_2 & a_{23} \\ a_{31} & a_{32} & a_{33}+x_3 \end{pmatrix},$$

and we obtain the system of equations for supplements the same as (3):

$$\begin{aligned} & a_{11}+x_1+a_{22}+x_2+a_{33}+x_3=d_1; \\ & (a_{11}+x_1)(a_{22}+x_2)-a_{12}a_{21}+(a_{11}+x_1)(a_{33}+x_3)-a_{13}a_{31}+(a_{22}+x_2)(a_{33}+x_3)-a_{23}a_{32}=d_2; \\ & (a_{11}+x_1)(a_{22}+x_2)(a_{33}+x_3)+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{21}(a_{33}+x_3)-(a_{11}+x_1)a_{23}a_{32}- \\ & -a_{13}(a_{22}+x_2)a_{31}=d_3. \end{aligned} \tag{4}$$

By solving (4), we consider the goal has been achieved. Indeed, substituting the solutions into the matrix A_x one makes it equal to a given set without resort to transforming the matrix.

5. Spectral Equations and Their Types

The above computational difficulties can be significantly reduced by choosing as the unknowns the elements instead of just the supplements. For this purpose, the k arbitrary elements of A are replaced by unknowns, which are denoted for presentation by the capital letter X with the same indexes. For example, instead of the matrix

$$A_x = \begin{pmatrix} a_{11} & a_{12}+x_1 & a_{13} \\ a_{21}+x_2 & a_{22}+x_3 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with unknown supplements it is assumed the replacing matrix of the third order:

$$A_x = \begin{pmatrix} a_{11} & X_{12} & a_{13} \\ X_{21} & X_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{5}$$

The result is the system of equations for X_{12}, X_{21}, X_{22} of the form

$$\begin{aligned} & a_{11}+X_{22}+a_{33}=d_1, \\ & a_{11}X_{22}-X_{12}X_{21}+a_{11}a_{33}-a_{13}a_{31}+X_{22}a_{33}-a_{23}a_{32}=d_2, \\ & a_{11}X_{22}a_{33}+X_{12}a_{23}a_{31}+a_{13}X_{21}a_{32}-X_{12}X_{21}a_{33}-a_{11}a_{23}a_{32}-a_{13}X_{22}a_{31}=d_3. \end{aligned} \tag{6}$$

In general case, replacement of k elements of A with combining the replacing elements $X_{i,j}$ into the vector $X_{i,j}$ and building A_x gives the system of equations

$$F(X) = 0, \tag{7}$$

where F is the non-linear vector function with size of k called by the spectral equation.

In a similar way, we can choose

$$N = C_{k^2}^{k} \tag{8}$$

different replacing sets of elements and obtain replacing matrices in the form of (5) and equations in the form of (7). The number N very rapidly increases with the size of A. For small values of k, it is given in Table 1.

Table 1.

k	2	3	4	5	6	7
N	6	84	1820	53130	1947772	85900584
n	1	20	495	15504	776475	26978328
M	5	64	1325	37626	1171297	58922256

The type of the system (7) depends on the arrangement of replacing elements in A_x. If we allocate the replacing elements in different rows and columns, as it is shown for the matrix (5), the system can takes the linear or non-linear form of degree from 2 to k. However, not all of the systems have a solution. Using a particular matrix, we can at once determine a group of systems that do not have a solution.

Further, for the sake of simplicity, we will denote the replacing and non-replaced elements of matrices by the numbers that equal to the indexes and dots, respectively.

The right-hand side of the first equations of the system (6)

$$a_{11} + X_{22} + a_{33} = d_1$$

is the fixed sum, and the left-hand side has the unknown, therefore, the equation is consistent with arbitrary values of a_{11}, a_{33}, and d_1. But, for the other matrix

$$A_x = \begin{pmatrix} . & 12 & . \\ . & . & 23 \\ 31 & . & . \end{pmatrix}, \tag{9}$$

there are no unknowns on the left-hand side of

$$a_{11} + a_{22} + a_{33} = d_1,$$

so, the last expression is inconsistent.

It is straightforward to make the following generalization. The matrix (9) belongs to the family of matrices, which is formed by replacing k elements of A that lie outside of the main diagonal in the two triangle areas containing $k^2 - k$ elements. This means that a necessary condition to solve (7) is that at least a one replacing element must be located on the main diagonal. It follows that the number of inconsistent equations (7) is equal to the number of combinations

$$n = C_{k^2-k}^{k}. \tag{10}$$

The dependence (10) is also given in Table 1.

Subtracting (10) from (8), we obtain

$$M = N - N_k \tag{11}$$

(given in Table 1) that is the number of solvable systems (7). Under appropriate conditions, this number is the sum

$$M = \sum_{i=1}^{k} M_i , \tag{12}$$

where M_i is the number of the i-th order equations.

Determining the terms in (12) for a general case as functions of k is the problem that needs to be solved. Even calculating M_1, i.e. determining the number of the linear systems (7), is unobvious procedure that requires an analysis of equations of the form (6). We can say definitely (or, rather, we can suggest, since there is no rigorous proof) about only the single term M_i for $i = k$. It is equal to 1. In other words, there is only one way to replace k elements of matrix that allows a spectral equation of order k to be obtained by replacing the elements on the main diagonal.

We consider next a particular case for a matrix of the third order. By analyzing 64 consistent equations (7), we establish 18, 45 and 1 variants to replace 3 elements according to (11). Replacements are described by two types of the first order equations, six types of the second order equations, and one type of third order equation. Equations of all types are resulted.

6. Spectral Transformation of the Third Order Matrix

At first, we discuss the variants with evident solving the problem of choosing elements for linear replacement associated with a replacement of rows and columns of A.

There is only one element of replacing rows and columns in the summands of minors. Each of summand contains one unknown, and the multipliers obtained from the remaining elements give the coefficient at the summand. Assembly of these coefficients forms a matrix denoted by R. These equations belong to the type 1.1.

Some summands on the left-hand side, as it can be seen from the system (6), do not have replacing elements. We combine these elements in the row i into the element b_i. Then, after combining the elements b_i and d_i into the vectors b and d respectively, we can represent the equation (7) in the linear form

$$RX = d - b \tag{13}$$

by replacing rows and columns.

The solution to (13) exists under condition

$$\det R \neq 0 . \tag{14}$$

6.1 Linear Spectral Equations

Replacing rows and columns one gives the matrices

$$\begin{gathered} 1)\begin{pmatrix} 11 & 12 & 13 \\ . & . & . \\ . & . & . \end{pmatrix},\ 2)\begin{pmatrix} . & . & . \\ 21 & 22 & 23 \\ . & . & . \end{pmatrix},\ 3)\begin{pmatrix} . & . & . \\ . & . & . \\ 31 & 32 & 33 \end{pmatrix}, \\ 4)\begin{pmatrix} 11 & . & . \\ 21 & . & . \\ 31 & . & . \end{pmatrix},\ 5)\begin{pmatrix} . & 12 & . \\ . & 22 & . \\ . & 32 & . \end{pmatrix},\ 6)\begin{pmatrix} . & . & 13 \\ . & . & 23 \\ . & . & 33 \end{pmatrix} \end{gathered} \tag{15}$$

with appropriate equations. For example, for the matrix 1), we obtain the following equations

$$\begin{aligned} & X_{11} + a_{22} + a_{33} = d_1; \\ & X_{11}a_{22} - X_{12}a_{21} + X_{11}a_{33} - X_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32} = d_2; \\ & X_{11}a_{22}a_{33} + X_{12}a_{23}a_{31} + X_{13}a_{21}a_{32} - X_{12}a_{21}a_{33} - X_{11}a_{23}a_{32} - X_{13}a_{22}a_{31} = d_3. \end{aligned} \tag{16}$$

They can be presented in the form (13) as

$$R_i X_i = d - b_i , \tag{17}$$

where $X_1 = (X_{11}, X_{12}, X_{13})^T$, $X_2 = (X_{21}, X_{22}, X_{23})^T$, $X_3 = (X_{31}, X_{32}, X_{33})^T$, $X_4 = (X_{11}, X_{21}, X_{31})^T$, $X_5 = (X_{12}, X_{22}, X_{32})^T$, $X_6 = (X_{13}, X_{23}, X_{33})^T$, $d = (d_1, d_2, d_3)^T$, $f_1 = a_{22} + a_{33}$, $f_2 = a_{11} + a_{33}$, $f_3 = a_{11} + a_{22}$, $g_1 = a_{22}a_{33} - a_{23}a_{32}$, $g_2 = a_{23}a_{31} - a_{21}a_{33}$, $g_3 = a_{21}a_{32} - a_{22}a_{31}$, $g_4 = a_{13}a_{32} - a_{12}a_{33}$, $g_5 = a_{11}a_{33} - a_{13}a_{31}$, $g_6 = a_{12}a_{31} - a_{11}a_{32}$, $g_7 = a_{12}a_{23} - a_{13}a_{22}$, $g_8 = a_{13}a_{21} - a_{11}a_{23}$, $g_9 = a_{11}a_{22} - a_{12}a_{21}$, $b_1 = b_4 = (f_1, g_1, 0)^T$,

$b_2 = b_5 = (f_2, g_5, 0)^T$, $b_3 = b_6 = (f_3, g_9, 0)^T$, $R_1 = \begin{pmatrix} 1 & 0 & 0 \\ f_1 & -a_{21} & -a_{31} \\ g_1 & g_2 & g_3 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -a_{12} & f_2 & -a_{32} \\ g_4 & g_5 & g_6 \end{pmatrix}$,

$R_3 = \begin{pmatrix} 0 & 0 & 1 \\ -a_{13} & -a_{23} & f_3 \\ g_7 & g_8 & g_9 \end{pmatrix}$, $R_4 = \begin{pmatrix} 1 & 0 & 0 \\ f_1 & -a_{12} & -a_{13} \\ g_1 & g_2 & g_3 \end{pmatrix}$, $R_5 = \begin{pmatrix} 0 & 1 & 0 \\ -a_{21} & f_2 & -a_{23} \\ g_4 & g_5 & g_6 \end{pmatrix}$,

$R_6 = \begin{pmatrix} 0 & 0 & 1 \\ -a_{31} & -a_{32} & f_3 \\ g_7 & g_8 & g_9 \end{pmatrix}$.

Equations (17) do not describe all of possible linear systems but determine only obvious ones. If replacing elements are not rows or columns, we can also get the linear system (7). In this case, the summands of minors can contain a product of replacing elements. Indeed, for example, for the matrix

$$A_x = \begin{pmatrix} 11 & 12 & . \\ . & . & . \\ . & 32 & . \end{pmatrix}, \tag{18}$$

the replacement system is

$$\begin{aligned} &X_{11} + a_{22} + a_{33} = d_1; \\ &X_{11}a_{22} - a_{12}a_{21} + X_{11}a_{33} - a_{13}X_{31} + a_{22}a_{33} - a_{23}X_{32} = d_2; \\ &X_{11}a_{22}a_{33} + X_{12}a_{23}a_{31} + a_{13}a_{21}X_{32} - X_{12}a_{21}a_{33} - X_{11}a_{23}X_{32} - a_{13}a_{22}a_{31} = d_3. \end{aligned} \tag{19}$$

In the third row, we obtain the summand with a product of unknowns X_{11} and X_{32}. However, we can find X_{11} from the first equation (i.e. X_{11} is known), and the system becomes linear. These equations belong to the type 1.2. The given types of equations exhaust linear replacements. The number of equations with types 1.1 and 1.2 is equal to 6 and 12 respectively.

7. The Second Order Replacements

7.1 Replacing a Single Diagonal Element

Replacements with a single diagonal elements lead to different types of the second order equations. Consider the matrix

$$A_x = \begin{pmatrix} 11 & 12 & . \\ . & . & . \\ 31 & . & . \end{pmatrix}, \tag{20}$$

which differs from (16) in a single element. The second and third equations for (18) is given by

$$\begin{aligned} &X_{11}a_{22} - X_{12}a_{21} + X_{11}a_{33} - a_{13}X_{31} + a_{22}a_{33} - a_{23}a_{32} = d_2; \\ &X_{11}a_{22}a_{33} + X_{12}a_{23}X_{31} + a_{13}a_{21}a_{32} - X_{12}a_{21}a_{33} - X_{11}a_{23}a_{32} - a_{13}a_{22}X_{31} = d_3. \end{aligned} \tag{21}$$

The last equation of (21) is like (19) only externally. In the product, there is no variable expressed from the first equation. This type of replacement is denoted as 2.1.

The matrix

$$A_x = \begin{pmatrix} 11 & 12 & . \\ 21 & . & . \\ . & . & . \end{pmatrix} \tag{22}$$

differs from (20) in a single element and contains by a single product of elements in two equations:

$$\begin{aligned} &X_{11}a_{22} - X_{12}X_{21} + X_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32} = d_2; \\ &X_{11}a_{22}a_{33} + X_{12}a_{23}a_{31} + a_{13}X_{21}a_{32} - X_{12}X_{21}a_{33} - X_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} = d_3. \end{aligned} \tag{23}$$

This type of replacement is denoted as 2.2.

The matrix

$$A_x = \begin{pmatrix} 11 & . & . \\ . & . & 23 \\ . & 32 & . \end{pmatrix} \tag{24}$$

differs from (16) in a single element and also contains the product of elements in two equations

$$\begin{aligned} &X_{11}a_{22} - a_{12}a_{21} + X_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - X_{23}X_{32} = d_2; \\ &X_{11}a_{22}a_{33} + a_{12}X_{23}a_{31} + a_{13}a_{21}X_{32} - a_{12}a_{21}a_{33} - X_{11}X_{23}X_{32} - a_{13}a_{22}a_{31} = d_3, \end{aligned} \tag{25}$$

but the third equations has the product of three elements.

This type is denoted as 2.3.

The matrix

$$A_x = \begin{pmatrix} 11 & 12 & . \\ . & . & 23 \\ . & . & . \end{pmatrix} \tag{26}$$

is characterized by the equations

$$\begin{gathered} X_{11}a_{22} - X_{12}a_{21} + X_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - X_{23}a_{32} = d_2; \\ X_{11}a_{22}a_{33} + X_{12}X_{23}a_{31} + a_{13}a_{21}a_{32} - X_{12}a_{21}a_{33} - X_{11}X_{23}a_{32} - a_{13}a_{22}a_{31} = d_3 \end{gathered} \tag{27}$$

with two products of two unknowns in the third row. This type is denoted as 2.4.

7.2 Replacement of Two Diagonal Elements

When two diagonal elements are replaced the type of equations depends on a choosing the third element. Consider two matrices

$$\text{a)} \begin{pmatrix} 11 & 12 & . \\ . & 22 & . \\ . & . & . \end{pmatrix}; \quad \text{b)} \begin{pmatrix} 11 & . & 13 \\ . & 22 & . \\ . & . & . \end{pmatrix} \tag{28}$$

with identical replaced diagonal elements and common first equation $X_{11} + X_{22} + a_{33} = d_1$. For the matrix a), the equations

$$\begin{gathered} X_{11}X_{22} - X_{12}a_{21} + X_{11}a_{33} - a_{13}a_{31} + X_{22}a_{33} - a_{23}a_{32} = d_2; \\ X_{11}X_{22}a_{33} + X_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - X_{12}a_{21}a_{33} - X_{11}a_{23}a_{32} - a_{13}X_{22}a_{31} = d_3 \end{gathered} \tag{29}$$

differ from 2.2 in the first equation. They are denoted as 2.5. For the matrix b), the equations

$$\begin{gathered} X_{11}X_{22} - a_{12}a_{21} + X_{11}a_{33} - X_{13}a_{31} + X_{22}a_{33} - a_{23}a_{32} = d_2; \\ X_{11}X_{22}a_{33} + a_{12}a_{23}a_{31} + X_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - X_{11}a_{23}a_{32} - X_{13}X_{22}a_{31} = d_3 \end{gathered} \tag{30}$$

with a single product in the second row and two products in the third row are denoted as 2.6.

The number of equations with a single replaced diagonal element of the types 2.1 and 2.2 is equal to 6. The number of equations of types 2.3 and 2.4 is equal to 3 and 12 accordingly. The number of equations with two replaced diagonal elements of the types 2.5 and 2.6 is equal to 6 and 12 accordingly.

7.3 Replacement of *k*-order

In general case, the last equation of (7) contains the $k!$ summands with products of k elements while the single summand has all unknown multipliers. Replacement of k elements using variants of (10) gives spectral equations of the k order only for unique case when the main diagonal of a matrix is replaced. Other variants of replacement lead to equations of the lower order. This conclusion is done without proving due to analysis of all spectral equations of the third order matrix.

For the third order matrix, 63 of 64 variants for choosing three elements lead to equations of the first and second order. The remaining matrix

$$A_x = \begin{pmatrix} 11 & . & . \\ . & 22 & . \\ . & . & 33 \end{pmatrix} \tag{31}$$

is characterized by the third order equation

$$
\begin{aligned}
&X_{11}+X_{22}+X_{33}=d_1;\\
&X_{11}X_{22}-a_{12}a_{21}+X_{11}X_{33}-a_{13}a_{31}+X_{22}X_{33}-a_{23}a_{32}=d_2;\\
&X_{11}X_{22}X_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-X_{11}a_{23}a_{32}-a_{13}X_{22}a_{31}-a_{12}a_{21}X_{33}=d_3.
\end{aligned}
\tag{32}
$$

The result obtained can be generalize for an arbitrary order matrix.

Example 1. With a set $\Lambda=\{1, 0, -1\}$, consider variants of transforming a spectrum by replacing rows and columns in the matrices

$$
1)\ \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix},\quad 2)\ \begin{pmatrix}0&1&0\\2&2&2\\3&5&3\end{pmatrix},\quad 3)\ \begin{pmatrix}0&1&0\\1&2&2\\0&1&0\end{pmatrix}.
$$

The matrix 1). Let's calculate $d_1=0$, $d_2=-1$, $d_3=0$ and determine matrices and vectors (17):

$d=(0,-1,0)^T$, $f_1=14$, $f_2=10$, $f_3=6$, $g_1=-3$, $g_2=6$, $g_3=-3$, $g_4=6$, $g_5=-12$, $g_6=6$, $g_7=-3$, $g_8=6$, $g_9=-3$, $b_1=b_4=(14,-3,0)^T$, $b_2=b_5=(10,-12,0)^T$, $b_3=b_6=(6,-3,0)^T$,

$$
R_1=\begin{pmatrix}1&0&0\\14&-4&-7\\-3&6&-3\end{pmatrix},\ R_2=\begin{pmatrix}0&1&0\\-2&10&-8\\6&-12&6\end{pmatrix},\ R_3=\begin{pmatrix}0&0&1\\-3&-6&6\\-3&6&-3\end{pmatrix},
$$

$$
R_4=\begin{pmatrix}1&0&0\\14&-2&-3\\-3&6&-3\end{pmatrix},\ R_5=\begin{pmatrix}0&1&0\\-4&10&-6\\6&-12&6\end{pmatrix},\ R_6=\begin{pmatrix}0&0&1\\-7&-8&6\\-3&6&-3\end{pmatrix}.
$$

The matrices are non-singular, hence, there are all the solutions:

$$
\begin{aligned}
X_1&=R_1^{-1}(d-b_1)=(X_{11},X_{12},X_{13})^T=(-14,-16.444,-18.889)^T,\\
X_2&=R_2^{-1}(d-b_2)=(X_{21},X_{22},X_{23})^T=(-8.167,-10,-11.833)^T,\\
X_3&=R_3^{-1}(d-b_3)=(X_{31},X_{32},X_{33})^T=(-3.333,-4.667,-6)^T,\\
X_4&=R_4^{-1}(d-b_4)=(X_{11},X_{21},X_{31})^T=(-14,-30,-46)^T,\\
X_5&=R_5^{-1}(d-b_5)=(X_{12},X_{22},X_{32})^T=(-4.5,-10,-3.636)^T,\\
X_6&=R_6^{-1}(d-b_6)=(X_{13},X_{23},X_{33})^T=(-1.273,-15.5,-6)^T.
\end{aligned}
$$

With these solutions, replacing matrices (15)

$$
A_{x_1}=\begin{pmatrix}-14&-16.444&-18.889\\4&5&6\\7&8&9\end{pmatrix},\ A_{x_2}=\begin{pmatrix}1&2&3\\-8.167&-10&-11.833\\7&8&9\end{pmatrix},
$$

$$
A_{x_3}=\begin{pmatrix}1&2&3\\4&5&6\\-3.333&-4.667&-6\end{pmatrix},\ A_{x_4}=\begin{pmatrix}-14&2&3\\-30&5&6\\-46&8&9\end{pmatrix},
$$

$$
A_{x_5}=\begin{pmatrix}1&-4.5&3\\4&5&6\\7&-3.636&-6\end{pmatrix},\ A_{x_6}=\begin{pmatrix}1&-4.5&-1.273\\4&5&-15.5\\7&-3.636&-6\end{pmatrix}
$$

take the spectrum

$$
\sigma(A_{x_1})=\{1,-1,1.516\cdot10^{-14}\},\ \sigma(A_{x_2})=\{1,-8.847\cdot10^{-15},-1\},\ \sigma(A_{x_3})=\{1,2.044\cdot10^{-15},-1\},
$$

$$
\sigma(A_{x_4})=\{1,-1,5.437\cdot10^{-14}\},\ \sigma(A_{x_5})=\{1,-1,2.421\cdot10^{-14}\},\ \sigma(A_{x_6})=\{-1,-2.876\cdot10^{-15},1\}
$$

that is equal to a given set with calculating accuracy.

The matrix 2). Omitting intermediate calculations, here and further, we find matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & -2 & -3 \\ -4 & 0 & 4 \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & -5 \\ -3 & 0 & 3 \end{pmatrix},\ R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 2 \\ 2 & 6 & -2 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & -1 & 0 \\ -4 & -3 & 2 \end{pmatrix},\ R_5 = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix},\ R_6 = \begin{pmatrix} 0 & 0 & 1 \\ -3 & -5 & 2 \\ 4 & 3 & -2 \end{pmatrix}.$$

Among all the matrices only R_5 is singular and hence the solution X_5 does not exist.

With the remaining matrices, the solutions to the systems (17) are

$$X_1 = (-6, -8, -6)^T,\ X_2 = (-2, -4, -2)^T,\ X_3 = (-3, -4, -3)^T,$$
$$X_4 = (-6, -34, -63)^T,\ X_6 = (-0.545, -1.273, -3)^T.$$

Substituting them into the matrices (15)

$$A_{x_1} = \begin{pmatrix} -6 & -8 & -6 \\ 2 & 2 & 2 \\ 3 & 5 & 3 \end{pmatrix},\ A_{x_2} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -4 & -2 \\ 3 & 5 & 3 \end{pmatrix},\ A_{x_3} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ -3 & -4 & -3 \end{pmatrix},$$

$$A_{x_4} = \begin{pmatrix} -6 & 1 & 0 \\ -34 & 2 & 2 \\ -63 & 5 & 3 \end{pmatrix},\ A_{x_6} = \begin{pmatrix} 0 & 1 & -0.545 \\ 2 & 2 & -1.273 \\ 3 & 5 & 3 \end{pmatrix}$$

one forms the spectrum

$$\sigma(A_{x_1}) = \{-3.125\cdot 10^{-15} + i\cdot 1.142\cdot 10^{-7},\ -3.125\cdot 10^{-15} - i\cdot 1.142\cdot 10^{-7}, -1\},$$

$$\sigma(A_{x_2}) = \{1,\ -2.944\cdot 10^{-8}, 2.944\cdot 10^{-8}\},\ \sigma(A_{x_3}) = \{-1,\ -4.133\cdot 10^{-8}, 4.133\cdot 10^{-8}\},$$

$$\sigma(A_{x_4}) = \{-3.88\cdot 10^{-8}, 3.88\cdot 10^{-8}, -1\},\ \sigma(A_{x_6}) = \{-1, i\cdot 3.269\cdot 10^{-8}, -i\cdot 3.269\cdot 10^{-8}\}$$

that is equal to a given set.

The matrix 3). Let's evaluate the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},\ R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 2 \\ 2 & 0 & -1 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix},\ R_5 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix},\ R_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}.$$

The matrices R_2 and R_5 are singular and hence the solutions X_2 and X_5 do not exist. With the remaining solutions

$$X_1 = (-2, -5, -4)^T,\ X_3 = (-1, -2, -2)^T,\ X_4 = (-2, -5, -2)^T,\ X_6 = (-2, -4, -2)^T$$

the matrices

$$A_{x_1} = \begin{pmatrix} -2 & -5 & -4 \\ 1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix},\ A_{x_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{pmatrix},\ A_{x_4} = \begin{pmatrix} -2 & 1 & 0 \\ -5 & 2 & 2 \\ -2 & 1 & 0 \end{pmatrix},\ A_{x_6} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}$$

take the given spectrum

$$\sigma(A_{x_1}) = \{1, 0, -1\},\ \sigma(A_{x_3}) = \{1, -1, 0\},\ \sigma(A_{x_4}) = \{1, 2.148\cdot 10^{-15}, -1\};\ \sigma(A_{x_6}) = \{1, -1, 0\}.$$

Example 2. Spectrum transformation with linear replacement of elements in different rows and columns. Consider the matrix

$$\begin{pmatrix} . & . & 13 \\ . & 22 & 23 \\ . & . & . \end{pmatrix}$$

and the equations

$$a_{11} + X_{22} + a_{33} = d_1;$$
$$a_{11}X_{22} - a_{12}a_{21} + a_{11}a_{33} - X_{13}a_{31} + X_{22}a_{33} - X_{23}a_{32} = d_2;$$
$$a_{11}X_{22}a_{33} + a_{12}X_{23}a_{31} + X_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}X_{23}a_{32} - X_{13}X_{22}a_{31} = d_3.$$

From the first equation we define the unknown

$$X_{22} = d_1 - a_{11} - a_{33}$$

at once. Two others are reduced to a linear equation for X_{13} and X_{23} with the matrix

$$\begin{pmatrix} a_{31} & a_{32} \\ a_{21}a_{32} - a_{31}X_{22} & a_{12}a_{31} - a_{11}a_{32} \end{pmatrix}$$

and the vector

$$\begin{pmatrix} -d_2 + (a_{11} + a_{33})X_{22} - a_{12}a_{21} + a_{11}a_{33} \\ d_3 - a_{11}a_{33}X_{22} + a_{12}a_{21}a_{33} \end{pmatrix}.$$

With the set and matrix 1) from example 1, the solution

$$(X_{13}, X_{22}, X_{23})^T = (2.434, -10, -14.38)^T$$

for the matrix

$$A_{x_{15}} = \begin{pmatrix} 1 & 2 & 2.434 \\ 4 & -10 & -14.38 \\ 7 & 8 & 9 \end{pmatrix}$$

forms the given spectrum $\sigma(A_{x_{15}}) = \{1, -1, -8.065\cdot 10^{-14}\}$.

All calculations was made in MathCAD.

8. CONCLUSION

A method for obtaining a matrix spectrum equal to a given set of numbers without transformation to a Frobenius form is stated. Calculating tool is a system of equations, which having obtained by replacement of arbitrary matrix elements by unknowns. Their number is equal to the size obtained from relationships between matrix elements in the form of main minors and elements of a given set.

The method has many variants for choosing replacing elements and equations to calculating replacing elements from linear to non-linear with an order equal to the size.

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