International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 5, May 2015, PP 1-7
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Inequalities of Pečarić-Rajić type in quasi 2-normed space

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Abstract: *C. Park [8] introduced the term of quasi 2-normed space, furthermore he proved few properties of quasi 2-norm. M. Kir and M. Acikgoz [2] elaborated the procedure for completing a quasi 2-normed space, and in [4] are considered families of quasi-norms generated by quasi 2-norm and are also proven few statements about them. In this paper are proven inequalities of Pečarić-Rajić type in quasi 2-normed space. Further, in* $p-$*normed space is considered the case for* $n\geq 2$ *vectors, and in quasi-normed space are separately considered the cases for* $n=2$ *vectors and* $n\geq 3$ *vectors.*

Keywords: *quasi 2-norm, p-norm, Pečarić-Rajić inequality*

2010 Mathematics Subject Classification. 26D15, 26D10, 46B20

1. Introduction

S. Gähler [1] introduced the 2-normed spaces. One of the axioms of the 2-norm is the parallelepiped inequality, which is actually a fundamental one in the theory of 2-normed spaces. Precisely this inequality (analogously as in the normed spaces), C. Park has replaced with a new condition, which actually means that he gave the following definition for quasi 2-normed space.

Definition 1 ([8]). Let L be a real vector space with $\dim L\geq 2$. *Quasi* 2-*norm* is a real function $\|\cdot,\cdot\|: L\times L\to[0,\infty)$ such that:

a) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent,
b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$,
c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$, and
d) It exists a constant $C\geq 1$ such that $\|x+y,z\|\leq C(\|x,z\|+\|y,z\|)$, for all $x,y,z\in L$.

The ordered pair $(L,\|\cdot,\cdot\|)$ is called a *quasi* 2-*normed space*. The smallest possible number C such that it satisfies the condition *d*) is called a *modulus of concavity* of the quasi 2-norm $\|\cdot,\cdot\|$.

Further, M. Kir and M. Acikgoz [2] give few examples of trivial quasi 2-normed spaces and consider the question about completing the quasi 2-normed space, and in [4] is proven the following Lemma which will be used while proving the inequalities of Pečarić-Rajić type.

Lemma 1 ([4]). If L be a quasi 2-normed space with modulus of concavity $C\geq 1$, then for each $n>1$ and for all $z,x_1,x_2,...,x_n\in L$

$$\|\sum_{i=1}^{n}x_i,z\|\leq C^{1+[\log_2(n-1)]}\sum_{i=1}^{n}\|x_i,z\|. \tag{1}$$

holds.

Further, C. Park gave a characterization of quasi 2-normed space, i.e. proved the following theorem.

Theorem 1 ([8]). Let $(L,\|\cdot,\cdot\|)$ be a quasi 2-normed space. It exists $p,\ 0<p\leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ over L such that

$$|||\, x+y,z\,|||^p \le |||\, x,z\,|||^p + |||\, y,z\,|||^p \,, \tag{2}$$

holds, for all $x,y,z\in L$.

Definition 2 ([8]). Quasi 2-norm defined as in Theorem 1 is called $(2,p)-$*norm*, and a quasi 2-normed space L is called $(2,p)-$*normed space*.

2. INEQUALITIES IN QUASI 2-NORMED SPACE

Theorem 2. Let $(L,\|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C\ge 1$. If $\alpha_i\in\mathbf{R}$, $z,x_i\in L$, for $i=1,2$, then

$$\|\sum_{i=1}^{2}\alpha_i x_i,z\|\le C\min_{k\in\{1,2\}}\{|\alpha_k|\cdot\|\sum_{i=1}^{2}x_i,z\|+C\sum_{i=1}^{2}|\alpha_i-\alpha_k|\cdot\|x_i,z\|\}, \tag{3}$$

$$\max_{k\in\{1,2\}}\{\tfrac{|\alpha_k|}{C}\cdot\|\sum_{i=1}^{2}x_i,z\|-C\sum_{i=1}^{2}|\alpha_k-\alpha_i|\cdot\|x_i,z\|\}\le\|\sum_{i=1}^{2}\alpha_i x_i,z\|. \tag{4}$$

Proof. For each $k\in\{1,2\}$ it holds true that

$$\begin{aligned}\|\sum_{i=1}^{2}\alpha_i x_i,z\| &= \|\alpha_k\sum_{i=1}^{2}x_i+\sum_{i=1}^{2}(\alpha_i-\alpha_k)x_i,z\|\\ &\le C\|\alpha_k\sum_{i=1}^{2}x_i,z\|+C\|\sum_{i=1}^{2}(\alpha_i-\alpha_k)x_i,z\|\\ &\le C|\alpha_k|\cdot\|\sum_{i=1}^{2}x_i\|+C^2\sum_{i=1}^{2}|\alpha_i-\alpha_k|\cdot\|x_i\|.\end{aligned}$$

If we take a minimum, for the right side in the latter when $k\in\{1,2\}$, we get the inequality (3).

For each $k\in\{1,2\}$ it is satisfied that

$$\begin{aligned}\|\sum_{i=1}^{2}\alpha_k x_i,z\| &= \|\sum_{i=1}^{2}\alpha_i x_i+\sum_{i=1}^{2}(\alpha_k-\alpha_i)x_i,z\|\\ &\le C\|\sum_{i=1}^{2}\alpha_i x_i,z\|+C\|\sum_{i=1}^{2}(\alpha_k-\alpha_i)x_i,z\|\\ &\le C\|\sum_{i=1}^{2}\alpha_i x_i,z\|+C^2\sum_{i=1}^{2}|\alpha_k-\alpha_i|\cdot\|x_i,z\|,\end{aligned}$$

i.e.

$$\tfrac{|\alpha_k|}{C}\|\sum_{i=1}^{2}x_i,z\|-C\sum_{i=1}^{2}|\alpha_k-\alpha_i|\cdot\|x_i,z\|\le\|\sum_{i=1}^{2}\alpha_i x_i,z\|$$

If we take a maximum for the left side in the latter, when $k\in\{1,2\}$, we get the inequality (4).

Corollary 1. Let $(L,\|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C\ge 1$. If $z,x_i\in L$, for $i\in\{1,2\}$ are such that the sets $\{z,x_i\}$, $i=1,2$ are linearly independent, then

$$\|\sum_{i=1}^{2}\tfrac{x_i}{\|x_i,z\|},z\|\le C\min_{k\in\{1,2\}}\{\tfrac{1}{\|x_k,z\|}[\|\sum_{i=1}^{2}x_i,z\|+C\sum_{i=1}^{2}|\,\|x_k,z\|-\|x_i,z\|\,|]\}, \tag{5}$$

$$\max_{k\in\{1,2\}}\{\tfrac{1}{\|x_k,z\|}[\tfrac{1}{C}\|\sum_{i=1}^{n}x_i,z\|-C\sum_{i=1}^{n}|\,\|x_k,z\|-\|x_i,z\|\,|]\}\le\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i,z\|},z\|. \tag{6}$$

Proof. Let $\alpha_i = \frac{1}{\|x_i, z\|}, i = 1,2$. If in the inequalities (3) and (4) we substitute the above expression for $\alpha_i, i = 1,2$ then we obtain the inequalities (5) and (6), respectively.

Corollary 2. Let $(L, \|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C \geq 1$. If $z, x_i \in L$, for $i \in \{1,2\}$, then

$$\| \sum_{i=1}^{2} \| x_i \| x_i, z \| \leq C \min_{k\in\{1,2\}} \{\| x_k, z \| \cdot \| \sum_{i=1}^{2} x_i, z \| + C \sum_{i=1}^{2} | \| x_i, z \| - \| x_k, z \| | \cdot \| x_i, z \|\}, \tag{7}$$

$$\max_{k\in\{1,2\}} \{\frac{\|x_k, z\|}{C} \cdot \| \sum_{i=1}^{2} x_i, z \| - C \sum_{i=1}^{2} | \| x_k, z \| - \| x_i, z \| | \cdot \| x_i, z \|\} \leq \| \sum_{i=1}^{2} \| x_i \| x_i, z \|. \tag{8}$$

Proof. Let $\alpha_i = \| x_i, z \|, i = 1,2$. If we substitute the above expression for $\alpha_i, i = 1,2$, in the inequalities (3) and (4), then we obtain the inequalities (7) and (8), respectively.

Corollary 3. Let $(L, \|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C \geq 1$. If $z, x_i \in L$, for $i \in \{1,2\}$, then

$$C(C \sum_{i=1}^{2} \| x_i, z \| - \| \sum_{i=1}^{2} x_i, z \|) \min_{k\in\{1,2\}} \| x_k, z \| \leq C^2 \sum_{i=1}^{2} \| x_i, z \|^2 - \| \sum_{i=1}^{2} \| x_i, z \| x_i, z \|, \tag{9}$$

$$C(C \sum_{i=1}^{2} \| x_i, z \|^2 - \| \sum_{i=1}^{2} \| x_i, z \| x_i, z \|) \leq (C^2 \sum_{i=1}^{2} \| x_i, z \| - \| \sum_{i=1}^{2} x_i, z \|) \max_{k\in\{1,2\}} \| x_k, z \|. \tag{10}$$

Proof. Let $\min_{k\in\{1,2\}} \| x_k, z \| = \| x_{k_0}, z \|$ and $\alpha_i = \| x_i, z \|, \quad i = 1,2$. Then, the proof of the theorem 2, implies the following inequality

$$\| \sum_{i=1}^{n} \| x_i, z \| x_i, z \| \leq C \| x_{k_0}, z \| \cdot \| \sum_{i=1}^{n} x_i, z \| - C^2 \| x_{k_0}, z \| \sum_{i=1}^{n} \| x_i, z \| + C^2 \sum_{i=1}^{n} \| x_i, z \|^2$$

which is equivalent to the inequality (9).

Let $\max_{k\in\{1,2\}} \| x_k, z \| = \| x_{k_0}, z \|$ and $\alpha_i = \| x_i, z \|, \quad i = 1,2,...,n$. Then, the proof of the theorem 2, implies the following inequality

$$\| x_{k_0}, z \| \cdot \| \sum_{i=1}^{n} x_i, z \| \leq C \| \sum_{i=1}^{n} \| x_i, z \| x_i, z \| + C^2 \| x_{k_0}, z \| \sum_{i=1}^{n} \| x_i, z \| - C^2 \sum_{i=1}^{n} \| x_i, z \|^2$$

which is equivalent to the inequality (10).

Theorem 3. Let $(L, \|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C \geq 1$ and $n > 2$. If $\alpha_i \in \mathbf{R}$, $z, x_i \in L$, for $i \in \{1,2,...,n\}$, then

$$\| \sum_{i=1}^{n} \alpha_i x_i, z \| \leq C \min_{k\in\{1,...,n\}} \{| \alpha_k | \cdot \| \sum_{i=1}^{n} x_i, z \| + C^{1+[\log_2(n-2)]} \sum_{i=1}^{n} | \alpha_i - \alpha_k | \cdot \| x_i, z \|\}, \tag{11}$$

$$\max_{k\in\{1,...,n\}} \{\frac{|\alpha_k|}{C} \| \sum_{i=1}^{n} x_i, z \| - C^{1+[\log_2(n-2)]} \sum_{i=1}^{n} | \alpha_k - \alpha_i | \cdot \| x_i, z \|\} \leq \| \sum_{i=1}^{n} \alpha_i x_i, z \|. \tag{12}$$

Proof. For each $k \in \{1,2,...,n\}$ it is true that

$$\begin{aligned}\|\sum_{i=1}^{n}\alpha_i x_i, z\| &= \|\alpha_k \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}(\alpha_i-\alpha_k)x_i, z\| \\ &\le C\|\alpha_k \sum_{i=1}^{n} x_i, z\| + C\|\sum_{i\neq k}(\alpha_i-\alpha_k)x_i, z\| \\ &\le C|\alpha_k|\cdot\|\sum_{i=1}^{n} x_i, z\| + C\cdot C^{1+[\log_2(n-2)]}\sum_{i\neq k}|\alpha_i-\alpha_k|\cdot\|x_i, z\| \\ &= C(|\alpha_k|\cdot\|\sum_{i=1}^{n} x_i, z\| + C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\alpha_i-\alpha_k|\cdot\|x_i, z\|).\end{aligned}$$

If we take the minimum for the right side in the latter when $k\in\{1,2,...,n\}$, we get the inequality (11). For each $k\in\{1,2,...,n\}$ it is true that

$$\begin{aligned}\|\sum_{i=1}^{n}\alpha_k x_i, z\| &= \|\sum_{i=1}^{n}\alpha_i x_i + \sum_{i=1}^{n}(\alpha_k-\alpha_i)x_i, z\| \\ &\le C\|\sum_{i=1}^{n}\alpha_i x_i, z\| + C\|\sum_{i\neq k}(\alpha_k-\alpha_i)x_i, z\| \\ &\le C\|\sum_{i=1}^{n}\alpha_i x_i, z\| + C\cdot C^{1+[\log_2(n-2)]}\sum_{i\neq k}|\alpha_k-\alpha_i|\cdot\|x_i, z\| \\ &\le C(\|\sum_{i=1}^{n}\alpha_i x_i, z\| + C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\alpha_k-\alpha_i|\cdot\|x_i, z\|),\end{aligned}$$

i.e. the inequality

$$\tfrac{|\alpha_k|}{C}\|\sum_{i=1}^{n} x_i, z\| - C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\alpha_k-\alpha_i|\cdot\|x_i, z\| \le \|\sum_{i=1}^{n}\alpha_i x_i, z\|$$

If we take a maximum for the left side in the latter when $k\in\{1,2,...,n\}$, we get the inequality (12).

Corollary 4. Let $(L,\|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C\ge 1$ and $n>2$. If $z, x_i\in L$, for $i\in\{1,2,...,n\}$ are such that the sets $\{z, x_i\}, i=1,2,...,n$ are linearly independent, then

$$\|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i,z\|}, z\| \le C\min_{k\in\{1,...,n\}}\{\tfrac{1}{\|x_k,z\|}(\|\sum_{i=1}^{n} x_i, z\| + C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\,\|x_k,z\|-\|x_i,z\|\,|)\}, \tag{13}$$

$$\max_{k\in\{1,...,n\}}\{\tfrac{1}{\|x_k,z\|}|(\tfrac{1}{C}\|\sum_{i=1}^{n} x_i, z\| - C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\,\|x_k,z\|-\|x_i,z\|\,|)\} \le \|\sum_{i=1}^{n}\tfrac{x_i}{\|x_i,z\|}, z\|. \tag{14}$$

Proof. Let $\alpha_i=\frac{1}{\|x_i,z\|}, i=1,2,...,n$. If we substitute the above expression for $\alpha_i, i=1,2,...,n$, in the inequalities (11) and (12) then we obtain the inequalities (13) and (14), respectively.

Corollary 5. Let $(L,\|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C\ge 1$ and $n>2$. If $z, x_i\in L$, for $i\in\{1,2,...,n\}$, then

$$\|\sum_{i=1}^{n}\|x_i\|x_i, z\| \le C\min_{k\in\{1...,n\}}\{\|x_k,z\|\cdot\|\sum_{i=1}^{n} x_i, z\| + C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\,\|x_i,z\|-\|x_k,z\|\,|\cdot\|x_i,z\|\}, \tag{15}$$

$$\max_{k\in\{1,...,n\}}\{\tfrac{\|x_k,z\|}{C}\|\sum_{i=1}^{n} x_i, z\| - C^{1+[\log_2(n-2)]}\sum_{i=1}^{n}|\,\|x_k,z\|-\|x_i,z\|\,|\cdot\|x_i,z\|\} \le \|\sum_{i=1}^{n}\|x_i\|x_i, z\|. \tag{16}$$

Proof. Let $\alpha_i = \| x_i, z \|, i = 1, 2, ..., n$. If we substitute the above expression for $\alpha_i, i = 1, 2, ..., n$, in the inequalities (13) and (14) then we obtain the inequalities (15) and (16), respectively.

Corollary 6. Let $(L, \|\cdot,\cdot\|)$ be a real quasi 2-normed space with modulus of concavity $C \geq 1$ and $n > 2$. If $z, x_i \in L$, for $i \in \{1, 2, ..., n\}$, then

$$\|\sum_{i=1}^{n} \| x_i, z \| x_i, z \| - C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|^2 \leq C(\| \sum_{i=1}^{n} x_i, z \| - C^{1+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|) \min_{k \in \{1,...,n\}} \| x_k, z \|, \tag{17}$$

$$(\| \sum_{i=1}^{n} x_i, z \| - C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|) \max_{k=1,..,n} \| x_k, z \| \leq C \| \sum_{i=1}^{n} \| x_i, z \| x_i, z \| - C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|^2 \tag{18}$$

holds true.

Proof. Let $\min_{k \in \{1,2,...,n\}} \| x_k, z \| = \| x_{k_0}, z \|$ and $\alpha_i = \| x_i, z \|, \quad i = 1, 2, ..., n$. Then the proof of theorem 3, implies the following inequality

$$\| \sum_{i=1}^{n} \| x_i, z \| x_i, z \| \leq C \| x_{k_0}, z \| \cdot \| \sum_{i=1}^{n} x_i, z \| + C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|^2 - C^{2+[\log_2(n-2)]} \| x_{k_0}, z \| \sum_{i=1}^{n} \| x_i, z \|$$

which is equivalent to the inequality (17).

Let $\max_{k \in \{1,2,...,n\}} \| x_k, z \| = \| x_{k_0}, z \|$ and $\alpha_i = \| x_i, z \|, \quad i = 1, 2, ..., n$. Then the proof of theorem 3, implies the following inequality

$$\| x_{k_0}, z \| \cdot \| \sum_{i=1}^{n} x_i, z \| \leq C \| \sum_{i=1}^{n} \| x_i, z \| x_i, z \| + C^{2+[\log_2(n-2)]} \| x_{k_0}, z \| \sum_{i=1}^{n} \| x_i, z \| - C^{2+[\log_2(n-2)]} \sum_{i=1}^{n} \| x_i, z \|^2$$

which is equivalent to the inequality (18).

Remark 3. The inequalities (5), (6), (13) and (14) are actually inequalities of Pečarić-Rajić type in quasi 2-normed space.

Theorem 5. Let L be $(2, p)-$normed space and $n \geq 2$. If $\alpha_i \in \mathbf{R}$, $z, x_i \in L$, за $i \in \{1, 2, ..., n\}$, then

$$\| \sum_{i=1}^{n} \alpha_i x_i, z \|^p \leq \min_{k \in \{1,2,...,n\}} \{ | \alpha_k |^p \| \sum_{i=1}^{n} x_i, z \|^p + \sum_{i=1}^{n} | \alpha_i - \alpha_k |^p \| x_i, z \|^p \}, \tag{19}$$

$$\max_{i \in \{1,2,...,k\}} \{ | \alpha_k |^p \| \sum_{i=1}^{n} x_i, z \|^p - \sum_{i=1}^{n} | \alpha_k - \alpha_i |^p \| x_i, z \|^p \} \leq \| \sum_{i=1}^{n} \alpha_i x_i, z \|^p. \tag{20}$$

Proof. For each $k \in \{1, 2, ..., n\}$ it holds true that

$$\begin{aligned} \| \sum_{i=1}^{n} \alpha_i x_i, z \|^p &= \| \alpha_k \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (\alpha_i - \alpha_k) x_i, z \| \\ &\leq \| \alpha_k \sum_{i=1}^{n} x_i, z \|^p + \| \sum_{i=1}^{n} (\alpha_i - \alpha_k) x_i, z \|^p \\ &\leq | \alpha_k |^p \| \sum_{i=1}^{n} x_i, z \|^p + \sum_{i=1}^{n} | \alpha_i - \alpha_k |^p \| x_i, z \|^p. \end{aligned}$$

If we take a minimum for the right side of the latter, when $k \in \{1, 2, ..., n\}$, we get the inequality (19).

For each $k \in \{1, 2, ..., n\}$ the following is satisfied

$$\|\sum_{i=1}^{n}\alpha_k x_i, z\|^p = \|\sum_{i=1}^{n}\alpha_i x_i + \sum_{i=1}^{n}(\alpha_k-\alpha_i)x_i, z\|^p$$
$$\leq \|\sum_{i=1}^{n}\alpha_i x_i, z\|^p + \|\sum_{i=1}^{n}(\alpha_k-\alpha_i)x_i, z\|^p$$
$$\leq \|\sum_{i=1}^{n}\alpha_i x_i, z\|^p + \sum_{i=1}^{n}|\alpha_k-\alpha_i|^p \cdot \|x_i, z\|^p,$$

i.e. it holds true that

$$|\alpha_k|^p \|\sum_{i=1}^{n} x_i, z\|^p - \sum_{i=1}^{n}|\alpha_k-\alpha_i|^p \|x_i, z\|^p \leq \|\sum_{i=1}^{n}\alpha_i x_i, z\|^p.$$

If we take a maximum for the left side of the latter, when $k \in \{1,2,...,n\}$, we get the inequality (20).

Corollary 7. Let L be $(2,p)-$normed space and $n \geq 2$. If $z, x_i \in L$, for $i \in \{1,2,...,n\}$ are such that the sets $\{z, x_i\}, i=1,2,...,n$ are linearly independent, then

$$\|\sum_{i=1}^{n}\frac{x_i}{\|x_i,z\|}, z\|^p \leq \min_{k\in\{1,2,...,n\}}\{\frac{1}{\|x_k,z\|^p}[\|\sum_{i=1}^{n}x_i, z\|^p + \sum_{i=1}^{n}|\|x_k,z\| - \|x_i,z\||^p]\}, \tag{21}$$

$$\max_{i\in\{1,2,...,k\}}\{\frac{1}{\|x_k,z\|^p}[\|\sum_{i=1}^{n}x_i, z\|^p - \sum_{i=1}^{n}|\|x_k,z\| - \|x_i,z\||^p]\} \leq \|\sum_{i=1}^{n}\frac{x_i}{\|x_i,z\|}, z\|^p. \tag{22}$$

Proof. Let $\alpha_i = \frac{1}{\|x_i,z\|}, i=1,2,...,n$. If we substitute the above expression for $\alpha_i, i=1,2,...,n$ in the inequalities (19) and (20), then we obtain the inequalities (21) and (22), respectively.

Corollary 8. Let L be $(2,p)-$normed space and $n \geq 2$. If $z, x_i \in L$, for $i \in \{1,2,...,n\}$, then

$$\|\sum_{i=1}^{n}\|x_i,z\|x_i, z\|^p \leq \min_{k\in\{1,...,n\}}\{\|x_k,z\|^p\|\sum_{i=1}^{n}x_i,z\|^p + \sum_{i=1}^{n}|\|x_i,z\| - \|x_k,z\||^p\|x_i,z\|^p\}, \tag{23}$$

$$\max_{i\in\{1,...,k\}}\{\|x_k,z\|^p\|\sum_{i=1}^{n}x_i,z\|^p - \sum_{i=1}^{n}|\|x_i,z\| - \|x_k,z\||^p\|x_i,z\|^p\} \leq \|\sum_{i=1}^{n}\|x_i,z\|x_i, z\|^p. \tag{24}$$

Proof. Let $\alpha_i = \|x_i, z\|, i=1,2,...,n$. If we substitute the above expression for $\alpha_i, i=1,2,...,n$ in the inequalities (21) and (22), then we obtain the inequalities (23) and (24), respectively.

Remark 4. The inequalities (21) and (22) are actually inequalities of Pečarić-Rajić type in $(2,p)-$normed space.

3. CONCLUSION

The above proved inequalities are only a few possible generalizations of plenty of potential generalizations of well known inequalities into 2-normed spaces. It is logically to be asked the following question:

Whether and which other inequalities can be generalized into 2-normed spaces?

Giving and proving such the generalizations can be subject of lots of further researches.

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