International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 4, April 2015, PP 39-44
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Vertex- Edge Domination Polynomials of Lollipop Graphs $L_{n,1}$

A. Vijayan
Associate Professor,
Department of Mathematics,
Nesamony Memorial Christian College,
Marthandam, Tamil Nadu, India
dravijayan@gmail.com

T. Nagarajan
Assistant Professor,
Department of Mathematics,
Sivanthi Aditanar College,
Pillayarpuram, Nagercoil, Tamil Nadu, India
nagarajant64@gmail.com

Abstract: *Let G = (V, E) be a simple Graph. The vertex-edge domination polynomial of graph G is $D_{ve}(G, x) = \sum_{i=\gamma_{ve}(G)}^{|V(G)|} d_{ve}(G, i)\, x^i$, where $d_{ve}(G, i)$ is the number of vertex-edge dominating sets of G with cardinality i and $\gamma_{ve}(G)$ is the vertex-edge domination number of G. In this paper we derived a formula for finding the vertex-edge domination polynomial of Lollipop Graph $L_{n,1}$ and some interesting results are established.*

Keywords: *Lollipop Graph, Vertex-edge dominating sets, vertex-edge domination number, Vertex-edge domination polynomial, vertex-edge dominating roots.*

1. Introduction

Let G = (V, E) be a simple graph of order n. A set S ⊆ V is a dominating set of G, if every vertex in V\S is adjacent to atleast one vertex in S. The domination number of a graph, denoted by γ(G), is the minimum cardinality of the dominating sets in G. A set of vertices in a Graph G is said to be a vertex-edge dominating set, if for all edges e ∈ E(G), there exists a vertex v ∈ S such that v dominates e. Otherwise, for a graph G = (V, E), a vertex u ∈ V(G) ve - dominates an edge vw ∈ E(G) if (i) u = v or u = w (u is incident to vw) or (ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

The minimum cardinality of a ve-dominating set of G is called the vertex-edge domination number of G, and is denoted by $\gamma_{ve}(G)$.

The Lollipop Graph is the Graph obtained by joining a complete Graph K_n to a path graph P_1 with a bridge and it is denoted by $L_{n,1}$.

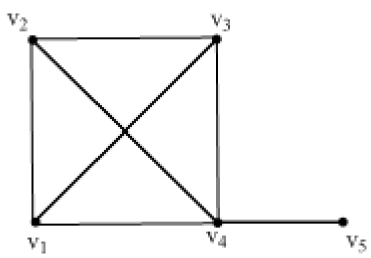
Let $L_{n,1}$ be the Lollipop Graph with n + 1 vertices. In the next section, we construct the families of the vertex-edge dominating sets of Lollipop Graphs. In section 3, we use the results obtained in section 2 to study the vertex-edge domination polynomial of Lollipop Graphs.

2. Vertex Edge Dominating Sets of Lollipop Graphs

Definition: 2.1

The lollipop Graph is the Graph obtained by joining a complete Graph K_n to a path Graph P_1 with a bridge and it is denoted by $L_{n,1}$.

Example 2.2



Vertex-edge dominating sets of cardinality 1 are

$\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$

$d_{ve}(L_{4,1}, 1) = 4$

Vertex-edge dominating sets of cardinality 2 are

$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v4\}, \{v_2, v_5\},$

$\{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}.$

$\therefore d_{ve}(L_{4,1}, 2) = 10$

The number of vertex-edge dominating sets of cardinality 3 is

$$d_{ve}(L_{4,1}, 3) = \binom{5}{3} = 10$$

The number of vertex-edge dominating sets of cardinality 4 is

$$d_{ve}(L_{4,1}, 4) = \binom{5}{4} = 5$$

The number of vertex-edge dominating sets of cardinality 5 is

$$d_{ve}(L_{4,1}, 5) = \binom{5}{5} = 1.$$

Theorem 2.3

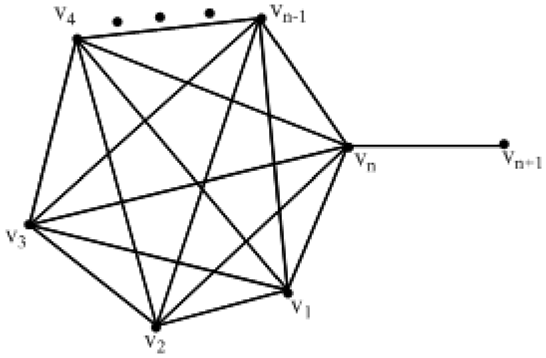
Let $L_{n,1}$ be the Lollipop Graph with n + 1 vertices, Then the vertex-edge dominating sets of the lollipop Graph is

$$d_{ve}(L_{n,1}, n) = \begin{cases} \binom{n+1}{r} - 1, & r = 1 \\ \binom{n+1}{r}, & 1 < r \leq n+1 \end{cases}$$

Proof:

Let $L_{n,1}$ be a Lollipop Graph with n + 1 vertices. Let the vertices of $L_{n,1}$ as $v_1, v_2, \ldots, v_n, v_{n+1}$, where v_i is of degree n , $1 \leq i < n$. v_n is a vertex of degree n + 1 and v_{n+1} is a vertex of degree 1.

$V(L_{n,1}) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ as given in fig.



$\{v_1\}, \{v_2\}, \ldots, \{v_n\}$ are the vertex-edge dominating sets of cardinality 1.

$\therefore$ The minimum cardinality is 1

$\therefore \gamma_{ve}(L_{n,1}) = 1$

$\therefore$ Number of vertex-edge dominating sets of cardinality 1 is $\binom{n+1}{1} - 1 = n$

Any two vertices of $V(L_{n,1}) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ cover all the vertices and edges of $L_{n,1}$.

$\therefore$ Number of vertex-edge dominating sets of cardinality 2 are $\binom{n+1}{2}$ continuing like this, we get,

Number of vertex-edge dominating sets of cardinality n + 1 are $\binom{n+1}{n+1}$.

Hence,

$$d_{ve}(L_{n,1}, n) = \begin{cases} \binom{n+1}{r} - 1, & r = 1 \\ \binom{n+1}{r}, & 1 < r \leq n+1 \end{cases}$$

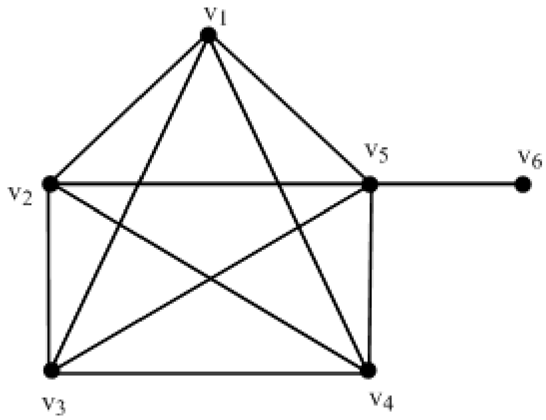
3. VERTEX-EDGE DOMINATION POLYNOMIAL OF LOLLIPOP GRAPHS

Definition: 3.1

Let $d_{ve}(L_{n,1}, i)$ be the number of vertex-edge dominating sets of Lollipop Graph $L_{n,1}$ with cardinality i. Then, the vertex-edge domination polynomial of $L_{n,1}$ is

$$D_{ve}(L_{n,1}, x) = \sum_{i=\gamma_{ve}(L_{n,1})}^{|V(L_{n,1})|} d_{ve}(L_{n,1}, i)\, x^{i}.$$

Example: 3.2



Vertex-edge dominating sets of cardinality 1 are

$\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}$

$d_{ve}(L_{5,1}, 1) = 5$

Vertex-edge dominating sets of cardinality 2 are

$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}$

$\{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}$

$\{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}$

$d_{ve}(L_{5,1}, 2) = 15$

The number of vertex-edge dominating sets of cardinality 3 is

$$d_{ve}(L_{5,1}, 3) = \binom{6}{3} = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$$

The number of vertex-edge dominating sets of cardinality 4 is

$$d_{ve}(L_{5,1}, 4) = \binom{6}{4} = \frac{6 \times 5}{1 \times 2} = 15$$

The number of vertex-edge dominating sets of cardinality 5 is

$$d_{ve}(L_{5,1}, 5) = \binom{6}{5} = 6$$

The number of vertex-edge dominating sets of cardinality 6 is

$$d_{ve}(L_{5,1}, 6) = \binom{6}{6} = 1$$

Vertex edge domination polynomial of $L_{5,1}$ is

$$\begin{aligned} D_{ve}(L_{5,1}, x) &= \sum_{i=\gamma_{ve}(L_{5,1})}^{|v(L_{5,1})|} d_{ve}(L_{5,1}, i)\, x^{i} \\ &= 5x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \\ &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 - x - 1 \\ &= (1 + x)^6 - (1 + x) \end{aligned}$$

Vertex-edge domination polynomial of $L_{4,1}$ is

$$D_{ve}(L_{4,1}, x) = \sum_{i=\gamma_{ve}(L_{4,1})}^{|v(L_{4,1})|} d_{ve}(L_{4,1}, i)x^i$$

$$= \sum_{i=1}^{5} d_{ve}(L_{4,1}, i)x^i$$

$$= 4x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 - 1 - x$$

$$= (1 + x)^5 - (1 + x)$$

Vertex-edge domination polynomial of $L_{5,1}$ is

$$D_{ve}(L_{5,1}, x) = 5x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 - x^6 - 1 - x$$

$$= (1 + x)^6 - (1 + x)$$

In general,

Vertex-edge domination polynomial of $L_{n,1}$, n > 3 is

$$D_{ve}(L_{n,1}, x) = (1 + x)^{n+1} - (1 + x).$$

Theorem : 3.3

Let $L_{n,1}$ be the Lollipop Graph with n + 1 vertices, then vertex-edge domination polynomial of the Lollipop Graph is $D_{ve}(L_{n,1}, x) = (1 + x)^{n+1} - (1 + x)$, n > 3.

Proof:

$$D_{ve}(L_{n,1}, x) = \sum_{i=\gamma_{ve}(L_{n,1})}^{|v(L_{n,1})|} d_{ve}(L_{n,1}, i)x^i$$

$$= \sum_{i=1}^{n+1} d_{ve}(L_{n,1}, i)x^i$$

$$= d_{ve}(L_{n,1}, 1)\, x^1 + \sum_{i=2}^{n+1} d_{ve}(L_{n,1}, i)x^i$$

$$= \left[\binom{n+1}{1} - 1\right]x + \sum_{i=2}^{n+1}\binom{n+1}{i}x^i \quad \text{(Theorem 2.3)}$$

$$= \binom{n+1}{1}x - x + \binom{n+1}{2}x^2 + \binom{n+1}{3}x^3 + \ldots + \binom{n+1}{n+1}x^{n+1}$$

$$= -x + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \binom{n+1}{3}x^3 + \ldots + \binom{n+1}{n+1}x^{n+1}$$

$$= -x - 1 + 1 + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \binom{n+1}{3}x^3 + \ldots + \binom{n+1}{n+1}x^{n+1}$$

$$= -x - 1 + (1 + x)^{n+1}$$

$$= (1 + x)^{n+1} - (1 + x), \; n > 3.$$

Proposition: 3.4

Let $L_{n,1}$ be the Lollipop Graph with n + 1 vertices, Then $D_{ve}(L_{n,1}, -1) = 0$.

Proof:

From theorem 3.3,

the vertex-edge domination polynomial of Lollipop Graph $L_{n,1}$ is

$$D_{ve}(L_{n,1}, x) = (1 + x)^{n+1} - (1 + x)$$

$$\therefore \quad D_{ve}(L_{n,1}, -1) = (1 - 1)^{n+1} - (1 - 1) = 0$$

Result: 3.5

$$\frac{d^{n+1}}{dx^{n+1}} D_{ve}(L_{n,1}, x) = (n + 1)\,!$$

Proof:

We know that

$D_{ve}(L_{n,1}, x) = (1 + x)^{n+1} - (1 + x)$

D.w.r. to x,

$\frac{d}{dx} D_{ve}(L_{n,1}, x) = (n + 1)(1 + x)^{n} - 1$

D.w.r. to x,

$\frac{d^2}{dx^2} D_{ve}(L_{n,1}, x) = (n + 1)n(1 + x)^{n-1.}$

$\frac{d^n}{dx^n} D_{ve}(L_{n,1}, x) = (n + 1)n(n - 1) \ldots (n - (n - 2))(1 + x)^{n-(n-1)}$

$= (n + 1)n(n - 1) \ldots 2(1 + x)$

$\therefore \frac{d^{n+1}}{dx^{n+1}} D_{ve}(L_{n,1}, x) = (n + 1)n(n - 1) \ldots 2 \times 1$

$= (n + 1)!$

Theorem: 3.6

The vertex-edge dominating roots of Lollipop Graph $L_{n,1}$ are -1,

$$\frac{\cos 2(k+1)\pi}{n} - 1 + i\frac{\sin 2(k+1)\pi}{n}, k = 0, 1, 2, \ldots, n - 1.$$

Proof:

The vertex-edge dominating roots of the Lollipop Graph $L_{n,1}$ are obtained by putting

$D_{ve}(L_{n,1}, x) = 0$

$\therefore (1 + x)^{n+1} - (1 + x) = 0$

$(1 + x)[(1 + x)^n - 1] = 0$

$\Rightarrow 1 + x = 0, (1 + x)^n - 1 = 0$

$\Rightarrow x = -1, (1 + x)^n = 1$

$\therefore 1 + x = 1^{1/n}$

$= (\cos 2\pi + i \sin 2\pi)^{1/n}$

$= [\cos(2k\pi + 2\pi) + i\sin(2k\pi + 2\pi)]^{1/n}$

where k is an integer.

$= [\cos 2(k + 1)\pi + i \sin 2(k + 1)\pi]^{1/n}$

$= \frac{\cos 2(k+1)\pi}{n} + i\frac{\sin 2(k+1)\pi}{n}, k = 0, 1, 2, \ldots, n - 1$

$x = \frac{\cos 2(k+1)\pi}{n} - 1 + i\frac{\sin 2(k+1)\pi}{n}, k = 0, 1, 2, \ldots, n - 1.$

$\therefore$ The vertex-edge dominating roots of lollipop Graph $L_{n,1}$ and -1, $\frac{\cos 2(k+1)\pi}{n} - 1 + \frac{i \sin 2(k+1)\pi}{n}$, $k = 0, 1, 2, \ldots, n - 1$.

4. CONCLUSION

The vertex-edge domination polynomial of a Graph is one of the algebraic representation of the Graph. This paper induces the concept of vertex-edge domination polynomial of Lollipop Graphs $L_{n,1}$. Similarly we can find vertex-edge dominating sets and vertex-edge domination polynomials of some specified Graphs.

REFERENCES

[1]. S. Alikhani and Y.H. Peng, Dominating sets and Domination polynomials of Paths, International Journal of Mathematics and Mathematical science, (2009).

[2]. S. Alikhani and Y.H.Peng, Introduction to Domination Polynomial of a Graph, arXiv : 0905. 225[V] [math.co] 14 may (2009).

[3]. J.A.Bondy and V.S.R. Murthy, Graph Theory with applications, Elsevier science publishing co, sixth printing (1984).

[4]. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Newyork (1978).

[5]. A. Vijayan and T. Nagarajan Vertex-edge domination polynomial of graphs. International journal of mathematical Archive5(2), (2014), 281-292.