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# On Riemann Hypothesis and Robin's Inequality

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**Abstract:** *Identified as one of the 7 Millennium Problems, the Riemann zeta hypothesis has successfully evaded mathematicians for over 100 year. Simply stated, Riemann conjectured that all of the nontrivial zeroes of the zeta function have real part equal to ½. There are various propositions equivalent to Riemann Hypothesis. In this piece of work, we investigate one of these propositions, simply called Robin's Inequality. We illustrate the list of known natural numbers that fail to satisfy Robin's Inequality, and we prove that* $N!$ *and* $p_k\#$ *satisfy Robin's inequality for every* $N \geq 8$ *and* $k \geq 4$*, respectively.*

**Keywords:** *Riemann Hypothesis, Robin's Inequality, Sum of divisors function, Primorial.*

## 1. INTRODUCTION

The sum-of-divisors function $\sigma$ is defined by

$$\sigma(N) := \sum_{d \setminus N} d.$$

By the Fundamental-Number-Theory, every natural number *N* can be uniquely written as the product with its prime factors

$$N = p_1^{\alpha_1}.p_2^{\alpha_2}...p_k^{\alpha_k}, \qquad p_i \setminus N, \alpha_i \geq 0, \text{ this implies}$$

$$\sigma(N) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdot \frac{p_3^{\alpha_3+1}-1}{p_3-1} \cdots \frac{p_k^{\alpha_k+1}-1}{p_k-1}.$$

In 1913, Thomas Gronwall [6] proved that

$$\limsup_{N\to\infty} \sigma(N) = e^{\gamma} N \log\log N = (1.78107...) \cdot N \log\log N, \qquad (N > 1)$$

Where $\gamma$ is the Euler-Mascheroni constant, defined as the limit

$$\gamma := \lim_{N\to\infty} (H_N - \log N) = 0.57721...,$$

Where $H_N$ denotes the *N*th harmonic number

$$H_N := \sum_{i=1}^{N} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{N}.$$

The interpretation of Gronwall's result is simply that the order of $\sigma(N)$ is 'very nearly' *N*, and it is a consequence of Merten's Theorem [1], which says that if *p* denotes a prime number, then

$$\lim_{x\to\infty} \frac{1}{\log x} \prod_{p \leq x} \frac{p}{p-1} = e^{\gamma}.$$

The first encounter to Riemann Hypothesis, was in 1915 due to Ramanujan [5] who by assuming Riemann Hypothesis provided the following strong result

*If Riemann Hypothesis is true, then*

$$\sigma(N) < e^{\gamma} N \log\log N, \qquad (N \gg 1).$$

Where $N \gg 1$ means for all sufficiently large *N*.

Therefore, the matter arises is how *sufficiently large should be N*. The first response was given by Guy Robin [2] in 1984, stating that

*Riemann Hypothesis is true, if and only if*

$$\sigma(N) < e^{\gamma} N \log\log N, \qquad (N \ge 5041).$$

Furthermore, Robin proved, unconditionally, that

$$\sigma(N) < e^{\gamma} N \log\log N + \frac{0.6483N}{\log\log N} \qquad (N > 1).$$

Recently, Young-Ju Choie et. al [7] proved that 1, 3, 5 and 9 are the only odd positive numbers that do not satisfy Robin's Inequality; therefore, the rest of the proof of Riemann Hypothesis via Robin's criterion is all about **even** positive numbers.

In this paper we are investigating certain even numbers: Factorials and Primorials.

First, we recall the following definitions:

**Definition1.1**. *(Factorial) For every non negative number n, n factorial is defined by*

$$n! := 1\cdot 2\cdot 3\cdots n$$

**Definition 1.2.** *(Primorial) Let $p_k$ be the kth prime, then the primorial is defined by the product of the first consecutive prime numbers up to $p_k$*

$$p_k\# := \prod_{i=1}^{k} p_i = p_1\cdot p_2\cdots p_k = 2\times 3\times 5\times \ldots p_k\,.$$

Now, let *A* be the set of all *known* numbers that fail to satisfy Robin's Inequality;

$$A = \{1,2,3,4,5,6,8,9,10,12,16,18,20,24,30,36,48,60,72,84,120,180,240,360,720,840,2520,5040\}$$

**Remark**

1. Every element in *A* is a divisor of 5040.
2. For $1 \le n \le 7$, $n!$ belongs to *A*.
3. For $1 \le k \le 3$, $p_k\#$ belongs to *A*.

Now, based of 2 and 3 of the previous Remark, what the one can say about $n!$ if $n \ge 8$ and $p_k\#$ for $k \ge 4$?

## 2. Robin's Inequality on Factorial

**Lemma 2.2.** [4] *For $x > 1$ we have*

$$\prod_{p\le x} \frac{p}{p-1} \le e^{\gamma}\left(\log x + \frac{1}{\log x}\right).$$

**Theorem2.1.** *For every natural number $n \ge 8$, we have n! satisfies Robin's Inequality.*

**Proof.** To prove our result, we consider two cases:

*Case1*: $n \ge 11$

First, we show that: $e^{\gamma}\left(\log n+\dfrac{1}{\log n}\right)<e^{\gamma}\log\log n!$ or simply:

$$\left(\log n+\frac{1}{\log n}\right)<\log\log n! \tag{1}$$

By induction; let $n$ = 11, we have:

$$\left(\log 11+\frac{1}{\log 11}\right)= \ 2.8149...< \ 2.8623...= \ \log\log 11!.$$

Assume that (1) holds true for n, we need to show that:

$$\left(\log(n+1)+\frac{1}{\log(n+1)}\right)<\log\log(n+1)!, \text{ or equivalently: } (n+1).\mathrm{e}^{\frac{1}{\log(n+1)}} < \ \log(n+1)!.$$

That is

$$n\mathrm{e}^{\frac{1}{\log(n+1)}}+\mathrm{e}^{\frac{1}{\log(n+1)}} < \ \log(n+1)+\log n!.$$

Now, we have

$$n\mathrm{e}^{\frac{1}{\log(n+1)}} < n\mathrm{e}^{\frac{1}{\log n}} < \log n! \text{ (By assumption), and } \mathrm{e}^{\frac{1}{\log(n+1)}} < \log n!, \text{ for every } n \ \geq 11.$$

Therefore; the inequality (1) holds true for every $n\geq \ 11$.

Now, let $N=n!=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot p_3^{\alpha_3}\cdots p_k^{\alpha_k}$ , we know that:

$$\frac{\sigma(N)}{N}=\frac{p_1^{\alpha_1+1}-1}{p_1^{\alpha_1}\ \ p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2^{\alpha_1}\ \ p_1-1}\cdot\frac{p_3^{\alpha_3+1}-1}{p_3^{\alpha_3}\ \ p_1-1}\cdots\frac{p_k^{\alpha_k+1}-1}{p_k^{\alpha_k}\ \ p_k-1}$$

$$\leq \ \frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdot\frac{p_3}{p_3-1}\cdots\frac{p_k}{p_k-1}\leq \ e^{\gamma}\left(\log p_k+\frac{1}{\log p_k}\right), \qquad \text{(by Lemma 2.1.)}$$

Then, since $p_k\leq n$ , together with inequality (1), we conclude that:

$$\frac{\sigma(N)}{N}=\frac{\sigma(n!)}{n!}\leq e^{\gamma}\left(\log p_k+\frac{1}{\log p_k}\right)\leq e^{\gamma}\left(\log n+\frac{1}{\log n}\right)\leq e^{\gamma}\log\log n!=e^{\gamma}\log\log N$$

This shows that: $N=n!$ satisfies Robin's Inequality for every $n\geq \ 11$.

*Case2*: $8\leq n<11$

It is enough to verify that 8! 9! and 10! Satisfy Robin's inequality; the following table shows the desired result:

| $n$ | $n!$ | $\dfrac{\sigma(n!)}{n!}$ | $e^{\gamma}\log\log n!$ |
|---|---|---|---|
| 8 | 40320 | 3.9464… | 4.2030… |
| 9 | 362880 | 4.0813… | 4.5382… |
| 10 | 3628800 | 4.2256… | 4.8326… |

This completes the Proof of Theorem 1.1.

**Corollary2.1.** Let N be an even number of the form $N = p_1^{\alpha_1}.p_2^{\alpha_2}...p_k^{\alpha_k}$, with $p_1 = 2 < p_2 < ... < p_k$

and $p_k \geq 11$, if $N \geq p_k!$ then $N$ satisfies Robin's Inequality.

Proof. The proof can be derived readily as follows:

$$\frac{\sigma(N)}{N} \leq \prod_{i=1}^{k} \frac{p_i}{p_i - 1} < e^{\gamma}\left(\log p_k + \frac{1}{\log p_k}\right) < e^{\gamma} \log\log p_k! < e^{\gamma} \log\log N \qquad \square$$

The Corollary above requires that the given even number must be greater than the factorial of its greatest prime factor and this one should be more than or equal to 11. Whereas, if a given even number is less the factorial of its greatest prime factor, further constraint is required that is $p_k e^{\frac{1}{\log p_k}} < \log N$

## 3. Robin's Inequality on Primorial

**Lemma3.1.** [3]. *asymptotically, primorials* $p_n\#$ *grow according to*:

$$p_n\# = e^{(1+o(1))\, n\log n}, \text{ where } o(.) \text{ is the little-}o \text{ notation}$$

*For the nth prime number $p_n$ the primorial $p_n$# is defined as the product of the first n primes*

*Further, the nth prime number satisfies the following inequality:*

$$n\log n + n\log\log n - n < p_n < n\log n + n\log\log n, \qquad n \geq 6 \tag{2}$$

**Theorem3.1.**
*For every natural number $k \geq 4$, we have the $k$-th Primorial satisfies Robin's Inequality.*

**Proof.**

*Case1*: $k \geq 6$

$$\frac{\sigma(p_k\#)}{p_k\#} = \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \cdot \prod_{i=1}^{k} 1 - \frac{1}{p_i^2} < \frac{3}{4} e^{\gamma}\left(\log p_k + \frac{1}{\log p_k}\right),$$

By the means of assertion (2) of Lemma 3.1, we have

$$p_k e^{\frac{1}{\log p_k}} < \left(k\log k + k\log\log k\right) . e^{\frac{1}{k\log k + k\log\log k - k}} < e^{\frac{4}{3}p_k\#} = \left(e^{k\log k}\right)^{\frac{4}{3}} = \left(k\log k\right)^{\frac{4}{3}},$$

*that readily shows that $p_k\#$ satisfy Robin's Inequality for every $k \geq 6$.*

*Case2:* $k = 4$ or 5

In this case it is sufficient to verify by simple calculations as follows:

| $k$ | $p_k\#$ | $\frac{\sigma(p_k\#)}{p_k\#}$ | $e^{\gamma}\log\log(p_k\#)$ |
|---|---|---|---|
| 4 | 210 | 2.7428… | 2.9842… |
| 5 | 2310 | 2.9922… | 3.6437… |

This completes the Proof of Theorem 2.1.

**Corollary3.1.** Let $N$ *be an even number of the form:* $N = q_1.q_2...q_k$ , *with* $k \geq 4$ *and* $q_i$, $i = 1, 2, ...$

*Not necessarily the first consecutive prime numbers* then $N$ *satisfies Robin's Inequality.*

**Proof.** $\dfrac{\sigma(N)}{N} \leq \dfrac{\sigma(p_k\#)}{p_k\#} < e^{\gamma}\log\log p_k\# \leq e^{\gamma}\log\log N$. ( $p_k$ *is the kth prime number*)

The above result simply means that any product of at least 4 prime numbers must satisfy Robin's Inequality.

**Corollary3.2.** *Given that* $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_k^{\alpha_k} = (p_k\#)^x$.

*If* $x > \dfrac{p_k e^{\frac{1}{\log(p_k)}}}{\log(p_k\#)}$ *, then N satisfies Robin's Inequality.*

**Corollary3.3.** *Given that* $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_k^{\alpha_k}$*, then* $\dfrac{\sigma(N)}{N} \leq \dfrac{\pi^2}{6}\cdot\dfrac{\sigma(p_k\#)}{p_k}$*, for every k.*

*Moreover, if* $k \geq 4$*. Then* $\dfrac{\sigma(N)}{N} \leq \dfrac{\pi^2}{6}\cdot e^{\gamma}\log\log p_k\#$

**Proof.** $\dfrac{\sigma(N)}{N}\cdot\dfrac{p_k\#}{\sigma(p_k\#)} \leq \prod_{i=1}^{k}\dfrac{p_i}{p_i-1}\cdot\prod_{i=1}^{k}\dfrac{p_i}{p_i+1} \leq \prod_{i=1}^{k}\dfrac{p_i^2}{p_i^2-1} < \prod_{p\ prime}\dfrac{p^2}{p^2-1} = \xi(2) = \dfrac{\pi^2}{6}$

**Remark.** Both results in Corollary 2.3 and Corollary 2.4 require N to be large enough, at least comparing to the primorial of its prime factors.

**Proposition.** We notice that the least number such that $p_k\#$ satisfying Robin's Inequality is *k = 4*, and the least number such that *n!* Satisfying Robin's Inequality is *n= 8.* We have 4/8 = ½, that is exactly the real part of non-trivial zeroes of the Riemann Zeta function. We may wish to figure out the connection, *if any*, to Zeta function by studying the behavior of

$\xi(z) = \xi(a+bi)$ where $\mathrm{Re}(z) = a = \dfrac{k\ such\ that\ p_k\#\ satisfies\ Robin's\ Inequality}{n\ such\ that\ n!\ satisfies\ Robin's\ Inequality}$, and conclude that:

$$\xi(z) = 0 \Rightarrow \mathrm{Re}(z) = \frac{LEAST\ k\ such\ that\ p_k\#\ satisfies\ Robin's\ Inequality}{LEAST\ n\ such\ that\ n!\ satisfies\ Robin's\ Inequality} = \frac{1}{2}$$

## 4. Conclusion

Robin's Inequality remains one of the crucial elementary criterions equivalent to Riemann Hypothesis. At this stage, conclusions about odd numbers is already solved, while regarding even numbers some important results have been provided in this work by investigating factorial and primorials. The one wishes to develop certain connection of any given even number to some factorial in its neighborhood or restructure the definition of the sum of divisors function to be dependent on a factorial closed to the given even number. We do assume that elementary approach to Riemann Hypothesis remains possible and Robin's criterion is still a plausible approach toward the main objective: Riemann Hypothesis is Riemann Theorem.

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