International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 3, March 2015, PP 1-11
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Optimal Control Problem for a Class of Bilinear Systems via Shifted Legendre Polynomials

Omar BALATIF
Faculty of sciences Ben M'Sik, Department of Mathematics and Computer Science, LAMS, Hassan II University, Casablanca, Morocco.
balatif.maths@gmail.com

Mostafa RACHIK
Faculty of sciences Ben M'Sik, Department of Mathematics and Computer Science, LAMS, Hassan II University, Casablanca, Morocco.
m_rachik@yahoo.fr

Mohamed EL HIA
Faculty of sciences Ben M'Sik, Department of Mathematics and Computer Science, LAMS, Hassan II University, Casablanca, Morocco.
elhia_mohamed@yahoo.fr

Omar RAJRAJI
Faculty of sciences Ben M'Sik, Department of Mathematics and Computer Science, LAMS, Hassan II University, Casablanca, Morocco.

Abstract: *In this paper we study the optimal control problem for a class of bilinear systems via Shifted Legendre Polynomials (SLPs). This method is based on approximating the system state variables and the control variable by SLPs series in finite length with unknown parameters. The optimal control problem is replaced by a parameter optimization problem, which consists of the minimization or maximization of the performance index, subject to algebraic constraints. An example has been considered to clarify the proposed method. We analyzed a model for cancer chemotherapy that aims at minimizing the damage done to bone marrow cells during the chemotherapy*

Keywords: *Bilinear systems, Optimal control, Orthogonal functions, Shifted Legendre Polynomials.*

1. Introduction

Bilinear systems are a special class of nonlinear systems, in which nonlinear terms are constructed by multiplication of control vector and state vector. Through nearly half a century, they have received great attention by researchers. The importance of such systems lies in the fact that many important processes, not only in engineering [17], but also in biology [24], socio-economics [18], and chemistry [3], can be modeled by bilinear systems. An overview of the available control strategies for bilinear systems can be found in [6-19]. Also, for more information about modeling and control of bilinear systems, we can see the thesis [5] and the references therein. Besides, optimal control is one of the most active subjects in the control theory.

An optimal control problem consists of finding a control function u^* which minimizes a given functional cost (performance index) while satisfying the system state equations and constraints. It has successful applications in many disciplines, namely, economics, environment, management, engineering, etc. As we know, a nonlinear optimal control problem does not have usually an analytical solution contrary to the linear case, and this reason motivates many researchers to try to find a numerical solution to this problem. In the literature, several papers address the solution of optimal control problems of nonlinear systems via Orthogonal Functions technique(OFs)[10]. We find that not much work has been reported on this approach. Lee and Chang [9] appear to be the first to study optimal control problems of nonlinear systems using general orthogonal polynomials.

The OFs technique has been developed for solving the problems (identification, analysis and control) of continuous time dynamical systems. The basic idea of this technique is that it converts calculus (differential or integral) to algebra[10]. i.e, the optimal control problem is replaced by a parameter optimization problem which consists of the minimization or maximization of the performance index subject to algebraic equations.

Very recently, applications of OFs technique has been extended to different types of systems, i.e. systems described by integrodifferential equations [11], multi-delay systems [16], distributed parameter systems [16], delay systems with reverse time functions [16a] and to singular systems [15]. There are three classes of sets of orthogonal functions that are widely used. The first includes sets of piecewise constant basis functions (such as the Walsh functions, block pulse functions, etc). The second consists of sets of orthogonal polynomials (such as Legendre polynomials and Chebyshev polynomials, etc). The third is the widely used sets of sine-cosine functions in Fourier series[4-7].

In this work we propose a method to solve this optimal control problem for a class of bilinear system by converting it directly into a parameter optimization problem using the orthogonal functions thecnique (OF)[4]. This method is based on approximating the system state variables by Shifted Legendre Polynomials Shilfted Legendre Polynomials (SPLs) series in finite length with unknown parameters.

The paper is organized as follows: The next section briefly deals with SLPs and their properties. Section 3 discusses optimal control of a class of bilinear systems via SLPs, and presents a recursive algorithm to solve the control problem. An example is considered in Section 4 to demonstrate the method. Finally Section 5 concludes the paper.

2. SHIFTED LEGENDRE POLYNOMIALS AND THEIR PROPERTIES [4-8]

The well-know Legendre polynomials $p_i(z),\ i = 0,1,\ldots$, are defined on the interval $z \in [-1,1]$ and are satisfy the recurrence relation

$$p_{i+1}(z) = \frac{2i+1}{i+1} z p_i(z) - \frac{i}{i+1} p_{i-1}(z) \qquad \text{for } i = 1,2,3,\ldots \tag{1}$$

with

$$p_0(z) = 1 \ \ and \ \ p_1(z) = z. \tag{2}$$

The Rodriguez 's formula of Legendre polynomials is

$$p_n(z) = \frac{1}{n!\,2^n} \frac{d^n}{z^n} [z^2 - 1] \tag{3}$$

The Legendre polynomials are a basis for the set of polynomials, appropriate for use on the interval $[-1,1]$. The first few Legendre polynomials are:

$$\begin{aligned} p_0(z) &= 1. \\ p_1(z) &= z. \\ p_2(z) &= \frac{3z^2 - 1}{2}. \\ p_3(z) &= \frac{5z^3 - 3z}{2}. \\ p_4(z) &= (35z^4 - 30z^2 + 3)/8. \end{aligned} \tag{4}$$

For practical use of Legendre polynomials on the time interval $t \in [t_0, t_f]$, it is necessary to shift the defining domain by the following variable substitution :

$$z = \frac{t - t_0}{t_f - t_0} - 1, \qquad t_0 \le t \le t_f \tag{5}$$

Let $s_i(t)$ denote the shifted Legendre polynomials $p_i(2\frac{t-t_0}{t_f-t_0} - 1)$. Then, from (1) it can be readily show that the shifted Legendre polynomials $s_i(t)$ are defined by the recurrence relationship

$$s_{i+1}(t) = \frac{2i+1}{i+1}\left(2\frac{t-t_0}{t_f-t_0} - 1\right)s_i(t) - \frac{i}{i+1} s_{i-1}(t), \quad \text{for } i = 1,2,3,\ldots \tag{6}$$

with

$$s_0(t) = 1 \ \ and \ s_1(t) = 2\frac{t - t_0}{t_f - t_0} - 1 \tag{7}$$

The elementary properties of SLPs are as follows

Orthogonality: The shifted Legendre polynomials form a complete set and they are orthogonal on the interval $[t_0, t_f]$ with

$$\int_{t_0}^{t_f} s_i(t)s_j(t)dt = \frac{t_f - t_0}{2i+1}\delta_{ij} \tag{8}$$

where δ_{ij} is the Kronecker delta.

The derivative: The derivative of SLPs is [20]

$$\dot{s}_i(t) = (t_f - t_0)\sqrt{i+1/2} \sum_{r=0}^{i-q_i-1/2} \sqrt{2r+q_i+1/2}\, s_{2r+q_i}(t), \qquad t \in [t_0, t_f] \tag{9}$$

where $q_i = 0$ if i is odd and $q_i = 1$ if i is even.

Function approximation: A function $f(t)$ that is square-integrale on the time interval $[t_0, t_f]$ can be represented in terms of SLPs as

$$f(t) \approx \sum_{i=0}^{m-1} f_i s_i(t) = f^T s(t) \tag{10}$$

where

$$f = [f_0, f_1, \dots, f_{m-1}]^T \tag{11}$$

is called Legendre spectrum of $f(t)$, and

$$s = [s_0, s_1, \dots, s_{m-1}]^T \tag{12}$$

is called SLPs vector. f_i given in Eq (10) is given by

$$f_i = \frac{2i+1}{t_f - t_0}\int_{t_0}^{t_f} f(t)s_i(t)dt \tag{13}$$

The multiplication: The product of two SLPs $s_i(t)$ and $s_j(t)$can be expressed as

$$s_i(t)s_j(t) \cong \sum_{v=1}^{m-1} \psi_{ijv} s_v(t) \tag{14}$$

where

$$\psi_{ijv} = \frac{2v+1}{t_f - t_0}\int_{t_0}^{t_f} s_i(t)s_j(t)s_v(t)dt \tag{15}$$

Let $\pi_{ijv} = \int_{t_0}^{t_f} s_i(t)s_j(t)s_v(t)dt$, then

$$\psi_{ijv} = \frac{2v+1}{t_f - t_0}\pi_{ijv} \tag{16}$$

Notice that

$$\pi_{ijv} = \pi_{ivj} = \pi_{jiv} = \pi_{vij} = \pi_{vji} \tag{17}$$

The recursive formula for computing π_{ijv} is given as follows: The product of $s_i(t)$ and $s_j(t)$with $i \geq j$ can be approximated by SLPs series as [22]

$$s_i(t)s_j(t) = \sum_{i=0}^{j} \frac{a_l a_{j-l} a_{i-j+l}}{a_{i+l}} \frac{2(i-j+2l)+1}{2(i+1)+1}\, s_{i-j+2l}(t) \tag{18}$$

where a_l are obtained by the recurrence formulas

$$a_0 = 1, \quad a_{l+1} = \frac{2l+1}{l+1}a_l, \quad for \quad l = 0,1,2,\dots,j \tag{19}$$

Multiplying (18) by $s_v(t)$then integrating from t_0 to t_f, and finally using the orthogonal property (8), we obtain, for $i \geq j$

$$\pi_{ijv} = \begin{cases} \dfrac{a_l a_{j-l} a_{i-j+l}}{a_{i+l}} \dfrac{t_f - t_0}{2(l+1)+1} & if\ k = i-j+2l, l = 0,1,2,\dots,j \\ 0 & if\ k = i-j+2l, l = 0,1,2,\dots,j \end{cases} \tag{20}$$

3. OPTIMAL CONTROL PROBLEM FOR BILINEAR SYSTEM

3.1 Problem Statement

We consider the bilinear continuous system described as follow

$$\dot{x}_k(t) = (Ax(t))_k + u(t)(Bx(t))_k \tag{21}$$

with the initial conditions $x_k(t^0) = x_k^0$, $k \in \{1,2,\ldots,n\}$.

where $x_k, u \in \mathbb{R}$, A and B are $n \times n$ matrices. We assumed that the process starts from t_0 and ends at fixed time $t_f \succ 0$.

The optimal control problem that is considered in this paper can be stated as follows: Find the optimal control u^* which minimizes the cost functional

$$J(u) = \int_{t_0}^{t_f} \left[\sum_{l=1}^{l=n} \lambda_l x_l^2 + ru^2 \right] dt \tag{22}$$

In other words, we seek the optimal control u^* such that

$$J(u^*) = min\{J(u):\ u \in U\} \tag{23}$$

where U is the set of admissible controls defined by

$$U = \{u(t) : u\ is\ Lebesgue\ mesurable, \alpha \leq u(t) \leq \beta, t \in [t^0, t_f]\ and\ (\alpha,\beta) \in \mathbb{R}^2\} \tag{24}$$

Subject to

$$\dot{x}_k(t) = (Ax(t))_k + u(t)(Bx(t))_k \tag{25}$$

with the initial conditions $x_k(t^0) = x_k^0$, where $k \in \{1,2,\ldots,n\}$, $\lambda_l \geq 0, r > 0$, the parameters λ_l and r are the cost coefficients, they are selected to weigh the relative importance of x_l and u. And t_0 and t_f are the initial and final times.

Integrating Eq. (21) with respect to t, we get

$$x_k(t) = x_k^0 + \int_{t_0}^{t} [(Ax(s))_k + u(s)(Bx(s))_k] ds \tag{26}$$

This solution is bounded. Indeed, the general form of the solution (21) is

$$\forall t \in [t_0, t_f], \qquad x(t) = x_0 + \int_{t_0}^{t} [Ax(s) + u(s)Bx(s)] ds \tag{27}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $x_0 = x(t_0)$.

So, $\forall t \in [t_0, t_f]$,

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|A + u(s)B\| \|x(s)\| ds \\ &\leq \|x_0\| + \int_{t_0}^{t} (\|A\| + |u(s)| \|B\|) \|x(s)\| ds \\ &\leq \|x_0\| + \int_{t_0}^{t} (\|A\| + c\|B\|) \|x(s)\| ds \\ &\leq C_1 + \int_{t_0}^{t} C_2 \|x(s)\| ds \end{aligned} \tag{28}$$

where $c = sup(|\alpha|, |\beta|)$, $C_1 = \|x_0\|$ and $C_2 = \|A\| + c\|B\|$.

Using Gronwall inequality, see [23], we obtain $\forall t \in [t_0, t_f]$,

$$\begin{aligned} \|x(t)\| &\le C_1 + \int_{t_0}^{t} C_2 \|x(s)\| ds \\ &\le C_1 \exp\left(\int_{t_0}^{t} C_2 ds\right) \\ &\le C_1 \exp\left(C_2(t_f - t_0)\right). \end{aligned} \tag{29}$$

Then the boundedness of the solution (21).

The existence of the optimal control can be obtained using a result by Fleming and Rishel in [6] (see Corollary 4.1).

Theorem.1: Consider the control problem with system (21). There exists an optimal control $u^* \in U$ such that

$$J(u^*) = min\{J(u): u \in U\}, \tag{30}$$

if the following conditions are met:

(1) The set of controls and corresponding state variables is nonempty.

(2) The control set U is convex and closed.

(3) The right-hand side of the state system is bounded by a linear function in the state and control variables.

(4) The integrand of the objective functional is convex on U.

(5) There exist constants $c_1, c_2 \succ 0$ and $\beta \succ 1$ such that the integrand $L(x_1, x_2, \dots, x_n, u)$ of the objective functional satisfies

$$L(x_1, x_2, \dots, x_n, u) \ge c_1 + c_2(|u|^2)^{\beta/2}. \tag{31}$$

To prove that the set of controls and corresponding state variables is nonempty, we will use a simplified version of an existence result in Boyce and DiPrima ([20], Theorem 7.1.1):

Theorem.2: Let $\dot{x}_i = F_i(t; x_1, \dots, x_n)$ for $i \in \{1, \dots, n\}$ be a system of n differential equations with initial conditions $x_i(t_0) = x_i^0$ for $i \in \{1, \dots, n\}$. If each of the functions $F_1, \dots, F_n$ and the partial derivatives $\partial F_1/\partial x_1, \dots, \partial F_1/\partial x_n$, $\partial F_2/\partial x_1, \dots, \partial F_2/\partial x_n, \dots, \partial F_n/\partial x_1, \dots, \partial F_n/\partial x_n$, are continuous in $\mathbb{R}^{n+1}$ space, then there exists a unique solution $x_1, \dots, x_n$ that satisfies the initial conditions.

Proof: (Theorem.1) We use Theorem.2 to prove that the set of controls and corresponding state variables is nonempty. Let $\dot{x}_1 = F_1(t; x_1, \dots, x_n), \dots, \dot{x}_n = F_n(t; x_1, \dots, x_n)$, where the $F_1, \dots, F_n$ form the right hand side of the system of equations (21). Let $u(t) = c$, for some constant, and since all parameters are constants, $F_1, \dots, F_n$ are linear. Thus, they are continuous everywhere. Additionally, the partial derivatives $\partial F_1/\partial x_1, \dots, \partial F_1/\partial x_n$, $\partial F_2/\partial x_1, \dots, \partial F_2/\partial x_n, \dots, \partial F_n/\partial x_1, \dots, \partial F_n/\partial x_n$ are all constants, and so they are also continuous everywhere.

Therefore, there exists a unique solution $x_1, \dots, x_n$ that satisfies the initial conditions. Therefore, the set of controls and corresponding state variables is nonempty, and condition 1 is satisfied.

The control set is convex and closed by definition. Since the state system is bilinear in u, the right side of (21) satisfies condition 3, using the boundedness of the solution. The integrand in the objective functional (22) is convex on U. It rest to show that there exists constants $c_1, c_2 \succ 0$ and $\beta \succ 1$ such that the integrand $L(x_1, \dots, x_n, u)$ of the objective functional satisfies

$$L(x_1, x_2, \dots, x_n, u) \ge c_1 + c_2(|u|^2)^{\beta/2}.$$

The state variables being bounded, let $c_1 = \frac{1}{n} inf(p_1 x_1, \dots, p_n x_n)$, $c_2 = r$ and $\beta = 2$. Then it follows that : $\sum_{i=1}^{n} p_i x_i + \frac{r}{2} u^2 \ge c_1 + c_2(|u|^2)$.

Now, we present an approximation for the state variables and the control variable by a finite series of SLPs as follows

$$x_k(t) = \sum_{j=1}^{m} \alpha_j^{(k)} s_j(t)\,, \quad k \in \{1,2,\ldots,n\} \tag{32}$$

$$u(t) = \sum_{j=1}^{m} \beta_j s_j(t) \tag{33}$$

Where $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, $\alpha^k = (\alpha_1^k, \alpha_2^k, \ldots, \alpha_m^k)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ are unknowns. For simplicity we have used the same degree of expansion for the state and control, where the choice of m depends on the required accuracy.

3.2 The Performance Index Approximation

Using the approximations for the state variables and the control variable from Equations (32) and (33), we can expressing x_k^2 and u^2 in terms of SLPs. Then the value of the approximated performance index in (22) can be given as follows:

Theorem 3: The value of the approximated performance index given in (22) is given by

$$J(\alpha,\beta) = \sum_{i=1}^{m} \frac{t_f - t_0}{2i+1} \left[\sum_{l=1}^{l=n} \lambda_l \left[\alpha_i^{(l)}\right]^2 + r[\beta_i]^2 \right] \tag{34}$$

where $\alpha^l = (\alpha_1^l, \alpha_2^l, \ldots, \alpha_m^l)$ for $l = 1,2,\ldots,n$, and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$.

Proof : The term x_k^2 can be expanded into

$$\begin{aligned} x_k^2 &= \left[\sum_{i=1}^{m} \alpha_i^{(k)} s_i(t)\right]\left[\sum_{j=1}^{m} \alpha_j^{(k)} s_j(t)\right] \\ &= \left[\alpha_1^{(k)} s_1(t)\right]^2 + 2\alpha_1^{(k)}\alpha_2^{(k)} s_1(t)s_2(t) + 2\alpha_1^{(k)}\alpha_3^{(k)} s_1(t)s_3(t) + \cdots + 2\alpha_1^{(k)}\alpha_m^{(k)} s_1(t)s_m(t) \\ &\quad + \left[\alpha_2^{(k)} s_2(t)\right]^2 + 2\alpha_2^{(k)}\alpha_3^{(k)} s_2(t)s_3(t) + \quad \ldots \quad + 2\alpha_2^{(k)}\alpha_m^{(k)} s_2(t)s_m(t) \\ &\quad + \left[\alpha_3^{(k)} s_3(t)\right]^2 + \quad \ldots \quad + 2\alpha_3^{(k)}\alpha_m^{(k)} s_3(t)s_m(t) \\ &\quad \vdots \\ &\quad + \left[\alpha_m^{(k)} s_m(t)\right]^2 \end{aligned} \tag{35}$$

Now, substituting the expression of $x_k^2\,(t)$ in Equation (22) and using integration property of SLPs, The integration of all terms in (35), $s_i(t)s_j(t)$, when $i \neq j$ is zero. Then

$$\int_{t_0}^{t} x_k^2 dt = \frac{t_f - t_0}{2i+1}\left[\left[\alpha_1^{(k)}\right]^2 + \left[\alpha_2^{(k)}\right]^2 + \cdots + \left[\alpha_m^{(k)}\right]^2\right]. \tag{36}$$

This gives proof of the first part of the theorem. Following the same procedure, integration of the second part u^2 can be computed.
Hence

$$J(\alpha,\beta) = \sum_{i=1}^{m} \frac{t_f - t_0}{2i+1} \left[\sum_{l=1}^{l=n} \lambda_l \left[\alpha_i^{(l)}\right]^2 + r[\beta_i]^2 \right]$$

3.3 Approximation of the System Dynamics

Using the result developed in the previous section, we have

Theorem 4: The approximation of the System Dynamics given in (21) is given by

$$\begin{cases} \sum_{j=1}^{m} \left[\alpha_j^{(k)} w_j(q_j) - \left[\sum_{l=1}^{l=n} a_{kl}\alpha_j^{(l)}\right] s_j(t) - g_j(k) s_j(t) \right] = 0\,, \quad t \in [t_0, t_f]. \\ k = 1,2,\ldots,n. \end{cases} \tag{37}$$

where

$$w_j(q_j) = \sqrt{j+1/2} \sum_{r=0}^{j-q_j-1/2} \sqrt{2r+q_j+1/2}\, s_{2r+q_j}(t) \tag{38}$$

where $q_j = 0$ if j is odd and $q_j = 1$ if j is even.
And

$$g_j(k) = \frac{2j+1}{t_f - t_0} \sum_{l=1}^{m} \sum_{v=1}^{m} \pi_{ivj} \beta_i \left[\sum_{l=1}^{l=n} b_{kl} \alpha_v^{(l)} \right] \tag{39}$$

Proof: To approximate the term $\dot{x}_k(t)$ by SLPs series, we use the derivative of SLPs

$$\dot{s}_i(t) = (t_f - t_0)\sqrt{i+1/2} \sum_{r=0}^{i-q_i-1/2} \sqrt{2r+q_i+1/2}\, s_{2r+q_i}(t), \qquad t \in [t_0, t_f]$$

where $q_i = 0$ if i is odd and $q_i = 1$ if i is even.
Hence, for $k \in \{1,2,\ldots,n\}$ we have

$$\dot{x}_k(t) = \sum_{j=1}^{m} \alpha_j^{(k)} w_j(q_j) \tag{40}$$

where

$$w_j(q_j) = \sqrt{j+1/2} \sum_{r=0}^{j-q_j-1/2} \sqrt{2r+q_j+1/2}\, s_{2r+q_j}(t) \tag{41}$$

where $q_j = 0$ if j is odd and $q_j = 1$ if j is even.

The term $(Ax(t))_k$ can be expanded as follow

$$(Ax(t))_k = \sum_{l=1}^{l=n} a_{kl} x_l(t) = \sum_{j=1}^{m} \left[\sum_{l=1}^{l=n} a_{kl} \alpha_j^{(l)} \right] s_j(t) \tag{42}$$

Using the product propriety of two SLPs, the term $u(t)(Bx(t))_k$ can be expanded as follow

$$\begin{aligned} (Bx(t))_k \quad &= u(t) = \sum_{j=1}^{m} \beta_j s_j(t) \sum_{j=1}^{m} \left[\sum_{l=1}^{l=n} b_{kl} \alpha_j^{(l)} \right] s_j(t) \\ &= \sum_{j=1}^{m} g_j(k) s_j(t) \end{aligned} \tag{43}$$

Therefore

$$\begin{aligned} g_j(k) \quad &= \frac{2j+1}{t_f - t_0} \int_{t_0}^{t_f} \sum_{l=1}^{m} \sum_{v=1}^{m} \beta_i \left[\sum_{l=1}^{l=n} b_{kl} \alpha_v^{(l)} \right] s_i(t) s_v(t) s_j(t) dt \\ &= \frac{2j+1}{t_f - t_0} \sum_{l=1}^{m} \sum_{v=1}^{m} \pi_{ivj} \beta_i \left[\sum_{l=1}^{l=n} b_{kl} \alpha_v^{(l)} \right] \end{aligned} \tag{44}$$

Substituting the expression of $\dot{x}_k(t)$, $(Ax(t))_k$ and $u(t)(Bx(t))_k$ in equation (21), we have the approximation of the System Dynamics

$$\begin{cases} \sum_{j=1}^{m} \left[\alpha_j^{(k)} w_j(q_j) - \left[\sum_{l=1}^{l=n} a_{kl} \alpha_j^{(l)} \right] s_j(t) - g_j(k) s_j(t) \right] = 0, \quad t \in [t_0, t_f]. \\ k = 1,2,\ldots,n. \end{cases}$$

where

$$w_j(q_j) = \sqrt{j+1/2} \sum_{r=0}^{j-q_j-1/2} \sqrt{2r+q_j+1/2}\, s_{2r+q_j}(t)$$

where $q_i = 0$ if i is odd and $q_i = 1$ if i is even.

And

$$g_j(k) = \frac{2j+1}{t_f - t_0} \sum_{l=1}^{m} \sum_{v=1}^{m} \pi_{ivj} \beta_i \left[\sum_{l=1}^{l=n} b_{kl} \alpha_v^{(l)} \right]$$

The optimal control problem is replaced by a parameter optimization problem. The problem now is to find the minimum value of $J(\alpha, \beta)$ given by (22), subject to the equality constraints $F(\alpha, \beta) = 0$ given by (21), i.e.,

$$Minimize\ J = J(\alpha, \beta) \qquad (45)$$
$$Subject\ to\ F(\alpha, \beta) = 0$$

Many mathematical programming techniques can be used to solve this parameter optimization problem, such as the Lagrange Multipliers, the penalty function, etc. In this work, we use the Lagrange Multipliers method.

Let

$$\begin{aligned} \frac{\partial L}{\partial \alpha_i}(\alpha, \beta, \lambda) &= 0, \\ \frac{\partial L}{\partial \beta_i}(\alpha, \beta, \lambda) &= 0, \\ \frac{\partial L}{\partial \lambda_i}(\alpha, \beta, \lambda) &= 0, \end{aligned} \qquad (46)$$

for $i = 1,2,..,m$.

4. NUMERICAL SIMULATION: OPTIMAL CONTROL FOR A MODEL IN CANCER CHEMOTHERAPY

In [21] Panetta lays out a simple model of proliferating and quiescent cell populations, and the effects of paclitaxel(the paclitaxel belongs to the group of drugs that fight cancer. It works by slowing or stopping the growth of cancer cells in the body) on the proliferating population. For a given cell population, there are populations of proliferating (P) and quiescent (Q) cells. The population of proliferating cells increases, the cells transition from one state to the other, some portion of each population is lost due to naturally occurring causes, and some proliferating cells are killed by paclitaxel.

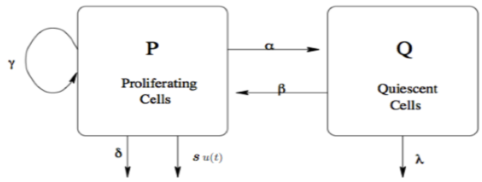

Figure 1. *Cells transitioning between resting and growth states [17]]*

From this model, Panetta derived the following bilinear system [17]:

$$\dot{x}(t) = Ax(t) + u(t)Bx(t) \qquad (47)$$

with the initial conditions $x(0) = (P_0, Q_0)$, where $x = (P, Q)$, A and B are 2×2 matrices given by

$$A = \begin{pmatrix} \gamma - \alpha - \delta & \beta \\ \alpha & -(\beta + \lambda) \end{pmatrix} \quad and \quad B = \begin{pmatrix} -s & 0 \\ 0 & 0 \end{pmatrix}$$

where all parameters are non negative and defined as follows

Table 1. *Parameter definitions*

Parameter	Definition
α	The transition rate from proliferating to quiescent state
β	The transition rate from quiescent to proliferating state
γ	The rate at which the proliferating cell population is increasing
λ	The quiescent cell loss rate
δ	The natural cell decay rate of proliferating cells
su	The rate of death by chemotherapy
P	The proliferating (cell-cycle) cell population
Q	The quiescent (resting) cell population

$u(t)$, represents the chemotherapy dose strength. It can vary between 0 (no chemotherapy) and 1 (maximal chemotherapy). (Note: Maximal chemotherapy is essentially a sub-lethal dose, or the maximum that can be given that will not kill the patient).

Bone marrow, whose blood cell production is an essential function in the body, has a high proliferative fraction [17]. This makes it very susceptible to damage by paclitaxel, which is a major concern when it is used in the treatment of cancers. As this model can be used to represent bone marrow or cancer cell populations, Panetta used the bone marrow version in attempts to find the ideal treatment schedule that destroys the least bone marrow.

The goal is to minimize the cancer cell population $(P+Q)$ while also minimizing chemotherapy strength. These have competing effects, as the cancer cell population will shrink with increased chemotherapy. Recall, however, that chemotherapy is highly cytotoxic, and so using as little as possible will result in better overall health for the patient. By minimizing an objective functional comprised of the cell population and the chemotherapy dose, we gain insight into the ideal treatment schedule and dosing that balances damage to the tumor with chemotherapy strength.

Mathematically, the problem is to minimize the objective functional

$$J(u)=\int_{t_0}^{t_f}\lambda_1 P^2+\lambda_2 Q^2+ru^2dt \tag{48}$$

where the parameters $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ and r denote the weight on cost. And t_f represents the duration of the chemotherapy program.

Using the proposed method, our problem can be reformulated as follows

Minimize:

$$J(\alpha,\beta)=\sum_{i=1}^{m}\frac{t_f-t_0}{2i+1}\left[\lambda_1\left[\alpha_i^{(1)}\right]^2+\lambda_2\left[\alpha_i^{(2)}\right]^2+r[\beta_i]^2\right] \tag{49}$$

Subject to:

$$\begin{cases}\sum_{j=1}^{m}\left[\alpha_j^{(1)}w_j(q_j)-\left[(\gamma-\alpha-\delta)\alpha_j^{(1)}+\beta\alpha_j^{(2)}\right]s_j(t)-g_j(1)s_j(t)\right]=0, & t\in[t_0,t_f].\\ \sum_{j=1}^{m}\left[\alpha_j^{(2)}w_j(q_j)-\left[\alpha\alpha_j^{(1)}-(\beta+\lambda)\alpha_j^{(2)}\right]s_j(t)-g_j(2)s_j(t)\right]=0, & t\in[t_0,t_f].\end{cases} \tag{50}$$

where

$$w_j(q_j)=\sqrt{j+1/2}\sum_{r=0}^{j-q_j-1/2}\sqrt{2r+q_j+1/2}\,s_{2r+q_j}(t) \tag{52}$$

where $q_j=0$ if j is odd and $q_j=1$ if j is even.

And

$$\begin{aligned}g_j(1)&=-\frac{2j+1}{t_f-t_0}\sum_{i=1}^{m}\sum_{v=1}^{m}\pi_{ivj}\beta_i s\alpha_v^{(1)}.\\ g_j(2)&=0.\end{aligned} \tag{53}$$

This problem can be solved using the Lagrange Multipliers method. The numerical simulations are carried out using Matlab and using the initial conditions $P_0=100$ and $Q_0=200$ and the parameter values $\alpha=0.8252, \beta=0.05, \gamma=0.89, \delta=0.3184, \rho=0$ and $s=1$. Note that the initial conditions and the parameter values are taken from [1]. Also, $t_0=0$, t_f =80.

Now, for $m=3$, we present the graphs for proliferating and quiescent cells with and without control, the graph of optimal control and the graph of the performance index.

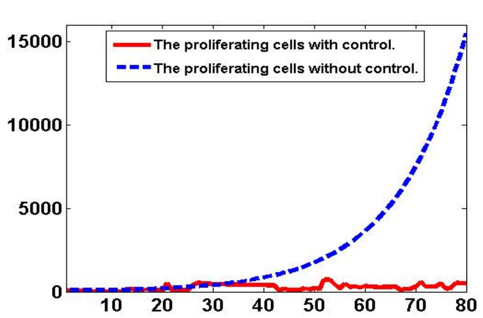

Figure 2. *The proliferating cells with and without control*

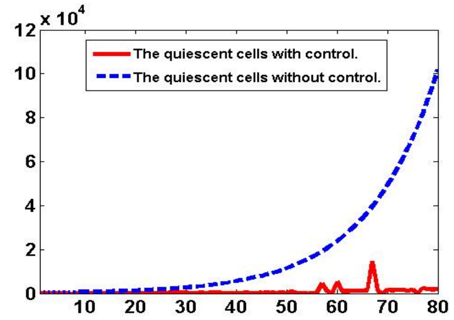

Figure 3. *The quiescent cells with and without control*

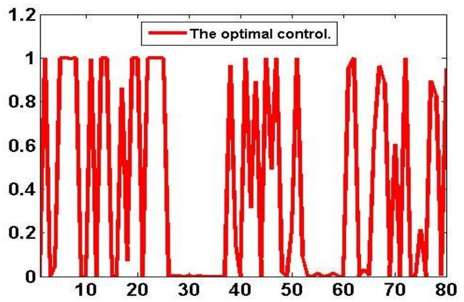

Figure 4. *The optimal control*

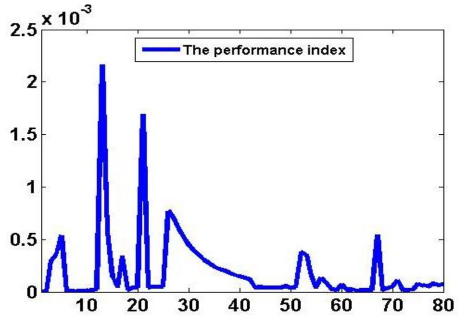

Figure 5. *The performance index*

These graphs, allow us to compare changes in the cancer cell population P and Q before and after the introduction of the control.

Figure 2 shows the effect of the optimal control in decreasing more rapidly the number of proliferating cells during the treatment program. Also, figure 3 shows that before treatment, the quiescent cells increase rapidly. While, we notice that after the treatments the number of quiescent cells decreases greatly. We show, in figure 5, that the values of the performance index decrease greatly during the treatment program. Finally, figure 4 gives a representation of the optimal control u^*.

5. Conclusion and Perspectives

An approximate optimal control method has been developed for a class of bilinear system by using the shifted Legendre polynomials. The control variable and state variables are approximated by SLPs series. Then the system dynamics has transformed into systems of algebraic equations in unknown parameters which can conveniently be solved. An example has been considered to clarify the proposed method. We analyzed a model for cancer chemotherapy that aims at minimizing the damage done to bone marrow cells during the chemotherapy, using the proposed method. The perspectives of this work are to find the error estimates of the approximation of the control variable and the states variables, and extend the technique for a multi-input bilinear system.

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