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Method of Structural Matching and its Application to Lagerstrom's Model Equation

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Abstract: *Here it is constructed by the method of structural matching of an asymptotic solution Lagerstrom's model equation andthis solution will converge uniformly for small parameter though one is other small parameter, than primary.*

Keywords: *Method of matching, Method of structural matching, inner and outer solutions, Asymptotic expansion, Function Green's.*

1. Introduction

1950 y. Lagerstrom A.P. [1] for the investigation of the Nav'e-Stokes equation supposed next equation in the small number of Reinolds

$$\frac{d^2\eta(r)}{dr^2}+\frac{n-1}{r}\eta(r)+\eta(r)\frac{d\eta(r)}{dr}=0,\ \eta(\varepsilon)=0, \eta(\infty)=1,$$

here $0<\varepsilon$ - small parameter,*n*-dimension ofspace, $r\in[\varepsilon,\infty)$ - the independent variable, $\eta(r)$ - unknown function.

Existence and uniqueness of this equation was proved in [2-3]. Expansion asymptotic of the solution of this equation proved by method of matching (MM) in [4-11], by method of the integral equation in [12], by method of fictitious parameter in [13], for the different meanings *n*.It is important to note that the rule of matching was proposed by Van Dike [14]. Justification of the MM was made by Il'in A.M [15]. Here we will apply method of structural matching [16-20] for expansion asymptotic of solution of this equation.

2. Method of Structural Matching

It is conveniently to make the next transform $r=\varepsilon x,\ \ \eta(x)=1-y(x)$ in this equation, then we have got

$$\frac{d^2y(x)}{dx^2}+(\frac{n-1}{x}+\varepsilon)\frac{dy}{dx}=\varepsilon y(x)\frac{dy}{dx}, y(1)=1,\ y(\infty)=0\ .$$

For concreteness we will consider case *n=2* only. Then

$$\frac{d^2y(x)}{dx^2}+(\frac{1}{x}+\varepsilon)\frac{dy}{dx}=\varepsilon y(x)\frac{dy}{dx},\ y(1)=1,\ y(\infty)=0. \tag{1}$$

Definition1. The variable *x* is named outer variable.

Definition 2. We will call the solution of this equation (1) that satisfies the condition $y(1)=0, y'(1)=a$, here $a=const$, outer solution. We will select $a=const$ so that the outer solution will exist on maximal interval that is $J(\varepsilon)=[1,\varepsilon^{-1}]$.

3. The Structure of the Outer Solution

The outer solution we will seek in the form

$$y(x,\varepsilon)=y_0(x)+\varepsilon y_1(x)+\ldots+\varepsilon^n y_n(x)+\ldots \quad . \tag{2}$$

Here $y_j(x)$ – as long as do not definite functions and one exist on $J(\varepsilon)$ and will satisfy next conditions : $y_0(1)=1,\ y_0'(1)=a;\ y_k(1)=0,\ y_k'(1)=0\ \ (k=1,2,\ldots)$.

Substituting (3) on (2) we will have next equations for define of functions $y_k(x)$:

$$Ly_0(x)=y_0'(x)+\frac{1}{x}y_0'(x)=1,\ \ y_0(1)=0,\ y_0'(1)=a\ , \tag{3.0}$$

$$Ly_1(x)=-y_0'(x)+y_0(x)y_0'(x),\ \ y_1(1)=y_1'(1)=0\ , \tag{3.1}$$

$$Ly_2(x)=-y_1'(x)+y_0'(x)y_1'(x)+y_1(x)y_0'(x),\ y_2(1)=y_2'(1)=0 \quad (3.2)\ \ ,$$

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$$Ly_m(x)=-y_{m-1}'(x)+\sum_{i+j=m-1}y_i(x)y_j'(x),\ y_m(1)=\ y_m'(1)=0 \quad , \tag{3.n}$$

....

Solution of (3.0) will have the form

$$y_0(x)=a\,lnx+1\,. \tag{4.0}$$

By using (4.0), for define $y_1(x)$ we have next equation

$$Ly_1(x)=a^2x^{-1}lnx,\ y_1(1)=0,\ \ y_1'(1)=0\,.$$

From here we have

$$y_1'(x)=u_1(x)=a^2lnx-a^2+a^2x^{-1}\sim a^2lnx-a^2,\ x\to\infty.$$

From here

$$y_1(x)\sim a^2xlnx-2a^2x,\ x\to\infty. \tag{4.1}$$

For define $y_2(x)$ we have the equation

$$Ly_2(x)\sim a^3ln^2x+a^3\ln x,\ y_2(1)=0,\ \ y_2'(1)=0\,.$$

Solving this equation we have

$$y_2'(x)\sim\frac{a^3}{2}xln^2x-\frac{a^3}{2}xlnx,\ x\to\infty.$$

By integrating this equation we have

$$y_2(x)\sim\frac{a^3}{4}x^2ln^2x+\frac{a^3}{8}x^2\ln x-\frac{3}{8}x^2,\ x\to\infty. \tag{4.2}$$

For define $y_3(x)$ we have next equation

$$Ly_3(x) \sim a^3 x\ln^3 x + a^4 x\ln^2 x,\ x\to\infty;\ y_3(1)=y_3'(1)=0.$$

From here

$$y_3(x)\sim\frac{a^4}{3\cdot 3!}x^3\ln^3 x-\frac{7a^3}{36}x^3\ln^2 x,\ \ x\to\infty. \tag{4.3}$$

So on from $(3.m)$ we have

$$y_m(x)=\frac{a^{m+1}}{m\cdot m!}x^m\ln^m x+a^{m+1}\lambda_m x^m\ln^{m-1}x,\ \ x\to\infty,$$
$$y'(x)=\frac{a^{m+1}}{m!}x^{m-1}\ln^m x+a^{m+1}\gamma_m x^{m-1}\ln^{m-1}x,\ \ x\to\infty. \tag{4.m}$$

Here and further λ_k и γ_k are noted some real numbers.We must prove formula (4.m) by the method of induction.Let (4.m) is correctly, then we will prove that correct next formula:

$$y_{m+1}(x)\sim\frac{a^{m+2}}{(m+1)\cdot(m+1)!}x^{m+1}\ln^{m+1}x+a^{m+2}\lambda_{m+1}x^{m+1}\ln^m x,\ \ x\to\infty,$$
$$y_{m+1}(x)\sim\frac{a^{m+2}}{(m+1)!}x^m\ln^{m+1}x+a^{m+2}\gamma_{m+1}x^m\ln^m x,\ \ x\to\infty.$$

The equation for define of $y_{m+1}(x)$ have the form

$$Ly_{m+1}(x)=(y_0(x)-1)y_m'(x)+y_1(x)y_{m-1}'(x)+\ldots+y_{m-1}(x)y_1'(x)+y_m(x)y_0'(x).$$

By using (4.0), (4.1),...,(4.m) this equation will have next form

$$Ly_{m+1}(x)\sim\frac{a^{m+2}}{m!}x^{m+1}\ln^{m+1}x+a^{m+2}\tilde{\lambda}_m x^{m-1}\ln^m x,\ \ x\to\infty.$$

By integrating this equation we have got (*4.m*).

Consequently the outer solution we can reprisent next form

$$y(x,\varepsilon)\sim 1+a\ln x+a[a\varepsilon x\ln x+\frac{1}{2\cdot 2!}(a\varepsilon x\ln x)^2+\ldots+\frac{1}{m\cdot m!}(a\varepsilon x\ln x)^m+\ldots],x\to\infty, \tag{5}$$

We will select indefinite number a such: $a=\mu=(\ln\frac{1}{\varepsilon})^{-1}$, then the equation: $\varepsilon\mu x\ln x=1$ will have the solution $x=\varepsilon^{-1}$ and the series (5) will have next form

$$y(x,\varepsilon)\sim 1+\mu\ln x+\mu[\mu\varepsilon x\ln x+\frac{\mu^2}{2\cdot 2!}(\varepsilon x\ln x)^2+\ldots+\frac{\mu^m}{m\cdot m!}(\varepsilon x\ln x\ \ )^m+\ldots],x\to\infty. \tag{5'}$$

This series is asymptotic series on the interval $[1,\varepsilon^{-1}]$.

From here we can have got next

Theoremı1. Outer solution (2) is asymptotical series in the interval $I(\varepsilon)=[1,\varepsilon^{-1}]$ that is

$$y(x,\varepsilon)=y_0(x)+\mu y_1(x)+\ldots+\mu^n y_n(x)+\mu^{n+1}R_{n+1}(x,\varepsilon)\ . \tag{6}$$

Here $R_{n+1}(x,\varepsilon)$ is reminder term andfor it we have got the estimate:

$$|R_{n+1}(x,\varepsilon)|\le l\ . \tag{7}$$

Here l constant that do not depend from ε .

4. Inner and Full Solution

Now we will construct the solution of the equation (1) that will satisfy the condition $y(\infty)=0$.

It is make in (1) next transform $t = x\varepsilon$ then we have got

$$u''(t)+\left(\frac{1}{t}+1\right)u'(t)=u(t)u'(t) \tag{8}$$

Here $u(t,\varepsilon)=y(x,\varepsilon)\big|_{x=t\varepsilon^{-1}}$.

Definition 3. The variable t is named inner variable.

Definition 4. We will call the solution of this equation (8) that satisfies the condition $y(\infty)=0$ the inner solution.

It is evidently if $x=1$ then $t=\varepsilon$.

We will rewrite the outer solution (6) in the inner variable t, then:

$$y(x,\varepsilon)|_{(x=t\varepsilon^{-1})} \sim 1+\mu lnt\varepsilon^{-1}+\mu[\mu\ln(t\varepsilon^{-1})+\frac{1}{2\cdot 2!}(\mu\ln(t\varepsilon^{-1}))^2+...+\frac{1}{m\cdot m!}(\mu\ln(t\varepsilon^{-1}))^m+...]. \tag{9}$$

Series (9)is asymptotical series on the interval $\varepsilon \le t \le \varepsilon^{-1}$.

It is appears that the inner solution will existence not only in the neighborhood of the point $t=\infty$, but also everywhere in $I(\varepsilon)=[\varepsilon,\infty)$. So we will solve the equation (8) with next boundary value problem:

$$u(\varepsilon)=1,\ u(\infty)=0 \tag{10}$$

Теорема 2. The solution of the problem (8) and(10) we can representative in the form

$$u(t,\varepsilon)=u_0(t,\mu)+u_1(t,\mu)+...+u_n(t,\mu)+... . \tag{11}$$

Here $u_k(t,\mu)=O(\mu^k),\quad u'_k(t)=O(\mu^k),\quad (k=0,1,2,...)$, that is, $\{u_k(t,\mu)\}$ is the asymptotical sequence .

By inserting (11) in (8) for defining of functions $u_k(t,\mu)$ we have got next equations

$$Mu_0(t):=u''_0(t)+(2+t^{-1})u'_0(t)=0,\ \ u_0(\varepsilon)=1,\ u_0(\infty)=0 \tag{12.0}$$

$$Mu_1(t)=u_0(t)u'_0(t),\ \ u_1(\varepsilon)=\ u_1(\infty)=0 \tag{12.1}$$

$$Mu_2(t)=u_0u'_1(t)+u_1u'_0(t),\ \ u_2(\varepsilon)=\ u_2(\infty)=0 \tag{12.2}$$

$$Mu_3(t)=(u'_1(t))^2+u_0u'_2(t)+u_1u'_1+u_2u'_0(t),\ \ u_3(\varepsilon)=\ u_3(\infty)=0 \tag{12.3}$$

....

The homogenous equation (12.0) will have two linear independent solutions

$$U_1(t)=1,\ U_2(t,\varepsilon)=\alpha_1\int_{\varepsilon}^{t}s^{-1}e^{-s}ds,\ \alpha_1=\alpha_1(\varepsilon)=[\int_{\varepsilon}^{\infty}s^{-1}e^{-s}ds]^{-1}=O(\mu). \tag{13}$$

Trivial boundary value problem $u_0(\varepsilon)=u_0(\infty)=0$ for the equation (13.0) will have only the trivial solution.

Лемма 1. The function of Greene for the problem

$$Mz(t)=0,\quad z(\varepsilon)=z(\infty)=0$$

will have next form

$$G_1(t,s,\varepsilon)=\alpha_1^{-1}U_2(t,\varepsilon)K_1(s,\varepsilon),\ \ \varepsilon\le t\le s,$$
$$G_1(t,s,\varepsilon)=\alpha_1^{-1}U_2(s,\varepsilon)K_1(t,\varepsilon),\ \ s\le t<\infty. \tag{14}$$
$$K_1(t,\varepsilon)=1-U_2(t,\varepsilon)$$

Лемма 2. Задача

$$Mz(t)=f(t),\ \ z(\varepsilon)=0,\quad z(\infty)=0$$

here $f(t)\in C[\varepsilon,\infty)$ will have got the unique solution and one have the form

$$z(t)=\int_\varepsilon^\infty s^2e^s\,G_1(t,s,\varepsilon)\,f(s)ds\,. \tag{15}$$

The solution of the problem (12.0) has the form

$$u_0(t)=K_1(t,\varepsilon)\,.$$

Now we will make transform in (8)

$$u(t)=u_0(t)+z(t) \tag{16}$$

then one rewrite in the form

$$z''(t)+\left(\frac{1}{t}+1\right)z'(t)=[u_0(t)+z(t)][u_0'(t)+z'(t)]$$
$$z(\varepsilon)=z(\infty)=0\,. \tag{17}$$

By using of formula (15) we can rewrite the problem (17) in the form of system integral equation

$$z(t)=\int_\varepsilon^\infty s^2e^s\,G_1(t,s,\varepsilon)[u_0(s)+z(s)][u_0'(s)+z'(s)]ds,$$
$$z'(t)=\int_\varepsilon^\infty s^2e^s\,G_{1t}(t,s,\varepsilon)[u_0(s)+z(s)][u_0'(s)+z'(s)]ds. \tag{18}$$

We will make in (18) the substitution

$$z(t)=K_1(t,\varepsilon)z_1(t),\ \ z'(t)=K_{1t}(t,\varepsilon)z_2(t)$$

then

$$z_1(t)=\int_\varepsilon^\infty\{\,Q_{100}(t,s,\varepsilon)+Q_{110}(t,s,\varepsilon)z_1(s)+Q_{101}(t,s,\varepsilon)z_2(s)+Q_{111}(t,s,\varepsilon)z_1(s)z_2(s)\}ds,$$
$$z_2(t)=\int_\varepsilon^\infty\{Q_{200}(t,s,\varepsilon)+Q_{210}(t,s,\varepsilon)z_1(s)+Q_{201}(t,s,\varepsilon)z_2(s)+Q_{211}(t,s,\varepsilon)z_1(s)z_2(s)\}ds. \tag{19}$$

Here

$$Q_{100}(t,s,\varepsilon)=se^sG_1(t,s,\varepsilon)u_0(s)u_0'(s)K_1^{-1}(t,\varepsilon),$$
$$Q_{200}(t,s,\varepsilon)=se^sG_{1t}(t,s,\varepsilon)u_0(s)u_0'(s)K_1^{-1}(t,\varepsilon),$$
$$Q_{110}(t,s,\varepsilon)=se^sG_1(t,s,\varepsilon)K_1(s,\varepsilon)u_0'(s)K_1^{-1}(t,\varepsilon),$$
$$Q_{210}(t,s,\varepsilon)=se^sG_{1t}(t,s,\varepsilon)K_1(s,\varepsilon)u_0'(s)K_{1t}^{-1}(t,\varepsilon),$$
$$Q_{111}(t,s,\varepsilon)=se^sG_1(t,s,\varepsilon)K_1(s,\varepsilon)K_1'(s,\varepsilon)K_1^{-1}(t,\varepsilon),$$
$$Q_{211}(t,s,\varepsilon)=se^sG_{1t}(t,s,\varepsilon)K_1(s,\varepsilon)K_1'(s,\varepsilon)K_1^{-1}(t,\varepsilon)$$

It is true next

Лemma 3

$$J_{ij}^{(k)}(t,\varepsilon)=\int_{\varepsilon}^{\infty}\left|Q_{kij}(t,s,\varepsilon)\right|ds\le O(\mu),\quad (l=const;\ i,j=0,1;k=1,2).$$

Proof. We will consider only the case $J_{00}(t,\varepsilon)$ .Other cases will be proved analogously. We will have

$$J_{00}^{(1)}(t,\varepsilon)=\int_{\varepsilon}^{\infty}\left|Q_{100}(t,s,\varepsilon)\right|ds\le\int_{\varepsilon}^{t}\left|Q_{100}(t,s,\varepsilon)\right|ds+\int_{t}^{\infty}\left|Q_{100}(t,s,\varepsilon)\right|ds=$$

$$=\int_{\varepsilon}^{t}\alpha_1^{-1}se^{s}U_2(s,\varepsilon)K_1(s,\varepsilon)K_1(t,\varepsilon)s^{-1}e^{-s}\alpha_1K_1^{-1}(t,\varepsilon)ds+\int_{t}^{\infty}\alpha_1^{-1}se^{s}U_2(s,\varepsilon)K_1^{2}(s,\varepsilon)s^{-1}e^{-s}\alpha_2K_2^{-1}(t,\varepsilon)ds\le$$

$$\le\int_{\varepsilon}^{t}U_2(s,\varepsilon)K_1(s,\varepsilon)ds+\int_{t}^{\infty}K_1(s,\varepsilon)\,ds\le\int_{\varepsilon}^{t}K_1(s,\varepsilon)ds+\int_{t}^{\infty}K_1(s,\varepsilon)\,ds\le$$

$$\le\int_{0}^{\infty}K_1(s,\varepsilon)\,ds\left|u=K_1(s,\varepsilon),dv=ds;du=-\alpha_2s^{-1}e^{-s}ds,v=s\right|=$$

$$\le\alpha_1\int_{\varepsilon}^{\infty}e^{-s}ds=e^{-\varepsilon}O[(\ln\varepsilon^{-1})^{-1}]=O(\mu).$$

By using this lemma we easily prove next

Theorem 3 The solution of the equation (19) we can represent in the form

$$\begin{aligned}z_1(t,\varepsilon)&=u_1^{(1)}(t)\mu+u_2^{(1)}(t)\mu^2+\ldots+u_n^{(1)}(t)\mu^n+\ldots,\\ z_2(t,\varepsilon)&=u_1^{(2)}(t)\mu+u_2^{(2)}(t)\mu^2+\ldots+u_n^{(2)}(t)\mu^n+\ldots\end{aligned}\qquad(20)$$

and this series will converge in the small parameter μ

The proof of this theorem we will prove by the method of majorant. Let $\varphi=\sup_{\varepsilon\le t<\infty}\{z_1(t),z_2(t)\}$. By using of the lemma 3 we will estimate (19) then we have got next majorant equation

$$\varphi=l\mu(1+\varphi+\varphi^2)$$

The solution of this equation will expand to the analytical series on power small parameter μ

$$\varphi=\mu\varphi_1+\mu^2\varphi_2+\ldots+\mu^n\varphi_n+\ldots$$

and $\left|u^{(k)}{}_j(t)\right|\le\varphi_j\ (k,j=1,2.,..)$ .Theorem3 and Theorem 2 proved.

The case of $k\ge3$ will consider analogously and true next

Theorem 4. The solution of the problem(8) and(10) we can representative in the form

$$u(t,\varepsilon)=u_0(t,\tilde{\mu}_k)+u_1(t,\tilde{\mu}_k)+\ldots+u_n(t,\tilde{\mu}_k)+\ldots\ .\qquad(21)$$

Here $\tilde{\mu}_3\sim\varepsilon\ln\varepsilon^{-1},\ \tilde{\mu}_j\sim\frac{j-1}{j-2}\varepsilon\ (j\ge4),\varepsilon\to0;\ u_m(t,\tilde{\mu}_k)=O(\mu_k{}^m),u'_m(t)=O(\mu_k{}^m),$ $\tilde{\mu}_k\to0,\ (m=0,1,2,\ldots)$

that is , $u_m(t,\tilde{\mu}_k)$ is the asymptotical sequence . Series (12) will convergent uniformly in the interval $[\varepsilon,\infty)$.

5. Conclusion

Method of structurally matching will help what small parameter power series will expand of the solution Lagerstrom's model equation and one solution is not only asymptotical series but and uniformly convergent for some small parameter.

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