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Super filters in BL-Almost Distributive Lattices

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Abstract: *These are the manuscript preparation guidelines used as a standard template for all journal submissions. Author must follow these instructions while preparing/modifying these guidelines. This guideline is used for all journals. These are the manuscript preparation guidelines used as a standard template for all journal submissions. Author must follow these instructions while preparing/modifying these guidelines. This guideline is used for all journals. These are the manuscript preparation guidelines used as a standard template for all journal submissions. Author must follow these instructions while preparing/modifying these guidelines. This guideline is used for all journals.*

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1. Introduction

In 1981, U.M. Swamy and G.C. Rao introduced the concept of an Almost Distributive Lattice (ADL) (see in [6]) as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. G. Epstein and A. Horn introduced the concept of a B−algebra(see in [2]) as a bounded distributive lattice with center B in which, for any x, y ∈ A, the largest element $x \Rightarrow y$ in B exists with the property $x \wedge (x \Rightarrow y) \leq y$. The connective ⇒ play an important role in building block in the computers that is the comparator or analog-to-digital converter. For this reason, in our paper [4], we introduced the concept a BL-Almost Distributive Lattice(BL−ADL)(see [3]) as a generalization of a BL-Algebra and derived its properties. In this paper, we introduce the notation of *Super filter and studied its properties and derive different equivalent conditions*.

2. Preliminaries

In this section, we give the necessary definitions and important properties of an ADL[6] and B−ADL[3]. For more information in theory of lattice, the reader is referred to G. Birkhoff [1].

Definition 2.1. [6] An algebra $(A, \vee, \wedge, 0)$ of type (2, 2, 0) is called an Almost Distributive Lattice(ADL)(see [5]) if it satisfies the following axioms: for all x, y, z ∈ A.,

(i) $x \vee 0 = x$

(ii) (ii) $0 \wedge x = 0$

(iii) (iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(v) $(x \vee y) \wedge y = y$.

Definition 2.2. [6] Let *A* be an ADL and *F* be a nonempty subset of *A*. Then F is said to be a filter if it satisfies the following:

i. x, y ∈ F implies $x \wedge y \in F$.

ii. ii. x ∈ F and a ∈ A implies $a \vee x \in F$.

For other properties of an ADL, we refer the readers to [5].

Definition 2.3. Let A be an ADL with a maximal element m and $B(A) = \{a \in A \mid a \wedge b = 0 \text{ and } a \vee b \text{ is maximal for some } b \in A\}$. Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of A. We use the symbol B instead of $B(A)$ when there is no ambiguity.

Definition 2.4. Let A be an ADL with Birkhoff center B. Then A is said to be a B−ADL(see [3]) if for any $x, y \in A$, there exists $b \in B$ satisfying the following conditions:

R_1 : $y \wedge x \wedge b = x \wedge b$

R_2 : if $c \in B$ such that $y \wedge x \wedge c = x \wedge c$, then $b \wedge c = c$.

We denote $b \wedge m$ by $x \Rightarrow y$ where there is no ambiguity.

Here afterwards, A stands for a B−ADL $(A, \vee, \wedge, 0)$ with a maximal element m and Birkhoff center B.

Theorem 2.5. [3] If $x \in A$ and $y \in B$, then we have

(i) $x \wedge (x \Rightarrow y) = x \wedge y \wedge m$. (ii) $(x \Rightarrow y) \wedge y = y$.

Theorem 2.6. [3] For any $x, y, z \in A$, we have the following:

(i) $x \wedge m \leq y \wedge m$ if and only if $(x \Rightarrow y) = m$.

(ii) $(0 \Rightarrow x) = m$, $(x \Rightarrow x) = m$ and $(x \Rightarrow m) = m$.

(iii) If $x \wedge m \leq y \wedge m$, then $(z \Rightarrow x) \leq (z \Rightarrow y)$ and $(x \Rightarrow z) \geq (y \Rightarrow z)$.

(iv) $(z \Rightarrow (x \wedge y)) = (z \Rightarrow x) \wedge (z \Rightarrow y) = (z \Rightarrow (y \wedge x))$.

(v) $((x \vee y) \Rightarrow z) = (x \Rightarrow z) \wedge (y \Rightarrow z) = ((y \vee x) \Rightarrow z)$.

For other properties of B−ADL, we refer the readers to [3].

3. SUPER FILTER

Definition 3.1

Let A be an ADL with a maximal element m and Birkhoff center B. A filter F of A is called a super filter of A, if $x!$ in F whenever x in F.Now we prove the following.

Theorem 3.2 Let A be a BL-ADL with a maximal element m and Birkhoff center B. Then we have the following:

1. If θ is a congruence relation on A, then $F_\theta := \{x \in A \mid (x, m) \in \theta\}$ is a super filter in A.
2. If F is a super filter in A, then $\theta_F := \{(x \wedge m, y) \in A \times A \mid (x \to y) \wedge (y \to x) \wedge m \in F\}$ is a congruence relation on A.
3. For any super filter F in A, $F_{\theta_F} = F$.
4. For any congruence θ on A, $\theta_{F_\theta} = \theta$.

Proof. Let $\mathcal{A}$ be the set of all super filters of A and $\mathcal{B}$, the set of all congruence relations on A. Suppose $x, y, z \in A$.

1. Suppose \theta is a congruence relation on A.

Since $(m \wedge m, m) \in \theta$, we get $m \in F_\theta$ and hence $F_\theta \neq \phi$. Let $(x \wedge m, y) \in F_\theta$. Then $(x \wedge m, m)$, $(y \wedge m, m) \in \theta$ and hence $(x \wedge y \wedge m, m) \in \theta$. Thus $x \wedge y \in F_\theta$. Let $x \in F_\theta$ and $a \in A$. Then $(x \wedge m, m) \in \theta$. So that $((x \wedge m) \vee a, m) = ((x \vee a) \wedge m, m \vee a) \in \theta$ and hence $x \vee a \in F_\theta$. we have F_θ is a filter in A. Let $x \in F_\theta$. Then $(x \wedge m, m) \in \theta$. Since θ is a congruence relation on A, we get $(x! \wedge m, m) = ((m \Rightarrow x) \wedge m, m \Rightarrow m) = (m \Rightarrow x, m) \in \theta$. Thus $x! \in F_\theta$. Hence F_θ is a super filter.

2. Suppose F is a super filter in A.

Since $m=(x\to x)\wedge(x\to x)\wedge m\in F$, we get $(x\wedge m,x)\in\theta$ and hence θ_F is reflexive. Let $(x\wedge m,y)\in\theta_F$. Then $(y\to x)\wedge(x\to y)\wedge m=(x\to y)\wedge(y\to x)\wedge m\in F$ and hence

$(y\wedge m,x)\in\theta_F$. Therefore θ is symmetric. Let $(x\wedge m,y) and (y\wedge m,z)\in\theta_F$.

Then $(x\to y)\wedge(y\to x)\wedge m\in F and (y\to z)\wedge(z\to y)\wedge m\in F.$

We have $(x\to z)\wedge m\geq(x\to z)\wedge m\geq(x\to y)\wedge(y\to z)\wedge m.$ So that we get

$(x\to z)\wedge(z\to x)\wedge m\geq[(x\to y)\wedge(y\to x)\wedge(y\to z)\wedge(z\to y)\wedge m]\wedge m$ and hence $(x\to z)\wedge(z\to x)\wedge m\in F.$ Therefore $(x\wedge m,z)\in\theta.$ Hence θ_F is a equivalence relation on A. Now, let $(x\wedge m,y)\in\theta$ and $z\in A.$ Then

$$\begin{aligned}[(x\wedge z)\to(y\wedge z)]\wedge[(y\wedge z)\to(x\wedge z)]\wedge m\\ &=((x\to y)\vee(z\to y))\wedge m\wedge((y\to x)\vee(z\to x))\wedge m\\ &=((x\to y)\vee(z\to y))\wedge m\wedge((y\to x)\vee(z\to x))\wedge m\\ &=(x\to y)\wedge(y\to x)\wedge m\in F\end{aligned}$$

and hence $(x\wedge z\wedge m,y\wedge z)\in\theta_F$. Similarly, we get that $((x\vee z)\wedge m,y\vee z)\in\theta_F$. Now we prove that $((x\to z)\wedge m,y\to z)\in\theta$ and $((z\to x)\wedge m,z\to y)\in\theta.$

Since $[(x\to z)\to(y\to z)]\wedge[(y\to z)\to(x\to z)]\wedge m\geq[(y\to x)\wedge(x\to y)]\wedge m\in F,$ we get

$((x\to z)\wedge m,y\to z)\in\theta.$ Similarly, we get $((z\to x)\wedge m,z\to y)\in\theta_F.$ Let $(x\wedge m,y)\in\theta_F.$ Then $(x\to y)\wedge(y\to x)\wedge m\in F.$ Since F is a super filter, we get $((x\to y)\wedge(y\to x))!\wedge m\in F$ and hence $(x\to y)!\wedge(y\to x)!\in F.$ Thus $(x\Rightarrow y)\wedge(y\Rightarrow x)\in F.$

Since $(x\Rightarrow z)\to(y\Rightarrow z)\wedge(y\Rightarrow z)\to(x\Rightarrow z)\geq(y\Rightarrow x)\wedge(x\Rightarrow y)\in F,$ we get $((x\Rightarrow z)\wedge m,y\Rightarrow z)\in\theta_F.$ Hence θ_F is a congruence relation on A.

3.Now $x\in F_{\theta_F}$ iff $(x\wedge m,m)\in\theta_F$ iff $(x\to m)\wedge(m\to x)\wedge m\in F$ iff $m\wedge x\wedge m\in F$ iff $x\in F$ and hence $F_{\theta_F}=F.$

4. Let $F_{\theta_F}=F.$ Then $(x\to y)\wedge(y\to x)\wedge m\in F_\theta$ and hence $((x\to y)\wedge(y\to x)\wedge m,m)\in\theta.$ Thus $((x\to y)\wedge(y\to x)\wedge y,y)\in\theta.$ Now $(x\to y)\wedge(y\to x)\wedge y=(y\to x)\wedge(x\to y)\wedge y=(y\to x)\wedge y=y\wedge x\wedge m=x\wedge y\wedge m.$ Therefore $(x\wedge y\wedge m,y)\in\theta.$ Similarly, we get $(x\wedge y\wedge m,x)\in\theta$ and hence $(x,y)\in\theta_{F_\theta}.$ Thus $\theta_{F_\theta}\subseteq\theta.$ Now, suppose

$(x\wedge m,y)\in\theta.$ Then $(x\to y,y\to y)=(x\to y,m)\in\theta.$ Similarly, we prove $(y\to x,m)\in\theta$ and hence $((x\to y)\wedge(y\to x)\wedge m,m)\in\theta.$ Thus $(x\to y)\wedge(y\to x)\wedge m\in F_\theta.$ So that $(x\wedge m,y\wedge m)\in\theta_{F_\theta}$ and hence $\theta\subseteq\theta_{F_\theta}.$ Therefore $\theta_{F_\theta}=\theta.$

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