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On the Relationship between Generalized Fractional Hilbert Transform with Some Classical Transforms

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Abstract: *The generalized Fractional Hilbert Transform is the generalization of the Hilbert transform which has many applications in image reconstruction, optics, signal analysis and so on. In this paper, we established the relationship between generalized fractional Hilbert transform with Fourier transform, Laplace transform, Mellin transform, Hartley transform, Hilbert transform, and Stieltjes transform.*

Keywords: *Fourier transform, Laplace transform, Hilbert transform, Mellin transform, Hartley transform, Stieltjes transform, generalized fractional Hilbert transform.*

1. INTRODUCTION

Integral transforms plays an important role for solving problems in several areas of both physics and applied mathematics. The linear canonical transformation (LCT) is a family of integral transforms that generalizes many classical transforms. Many operations, such as the Fresnel transform, Fourier transform (FT), fractional Fourier transform (FRFT), Lorentz transform and scaling operations are the special cases of the LCT. In 1980, Namias [6] first introduced the concept of fractional Fourier transform which is a generalization of Fourier transform. The fractional Fourier transform has many applications in many fields, including signal processing [4], optics [7]. Gori [1] has shown the relation between fractional Fourier transform and the Fresnel transform. The Hilbert transform based on the Fourier transform has applications in many fields, including optical system, modulation and edge detection [3], etc. Hilbert transform plays an important role in the study of singular integral equations [5].

The fractional Hilbert transform is the generalization of the Hilbert transform can produce the image enhancement or the image compression in different ways when both parameters (the angle of fractional Fourier transform and the phase of the fractional Hilbert transform) are varying.

This paper is organized as follows. Section two explains the definition of fractional Hilbert transform on the space of generalized functions. Section three is devoted for proving relations between generalized fractional Hilbert transform with classical Fourier, Laplace, Hilbert and Stieltjes transforms. Lastly the paper is concluded in section four.

2. GENERALIZED FRACTIONAL HILBERT TRANSFORM

For dealing fractional Hilbert transform in the generalized sense, first we define,

2.1 The Testing Function Space $E(\mathbb{R}^n)$

An infinitely differentiable complex valued function φ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$ if for each compact set $K \subset S_\alpha$ where $S_\alpha = \{x \in \mathbb{R}^n, |x| \le \alpha, \alpha > 0\}$ and for $K \in \mathbb{R}^n$,

$$\gamma_{E,K}(\varphi) = \sup_{x \in K} \left| D^k \varphi(x) \right| < \infty$$

Clearly E is complete and so a Frechet space. Moreover, we say that f is a fractional Hilbert transformable if it is a member of $E'(\mathbb{R}^n)$ (the dual space of E).

2.2 Generalized Fractional Hilbert Transform On $E'(\mathbb{R}^n)$

The generalized fractional Hilbert transform of $f(x) \in E'(\mathbb{R}^n)$, where $E'(\mathbb{R}^n)$ is the dual of the testing function space $E(\mathbb{R}^n)$, can be defined as

$$H^{\alpha}[f(x)](t) = \langle f(x), K_{\alpha}(x,t)\rangle, \qquad \text{for each} \quad t \in \mathbb{R}. \tag{1}$$

where $K_{\alpha}(x,t) = \dfrac{e^{-i\frac{\cot\alpha}{2}(t^2-x^2)}}{\pi(t-x)}$ for $\alpha \neq 0, \dfrac{\pi}{2}, \pi$ and $t \neq x$

The right hand side of (1) has meaning as the application of $f \in E'$ to $K_{\alpha}(x,t) \in E$. $H^{\alpha}[f(x)](t)$ is α^{th} order generalized fractional Hilbert transform of the function $f(t)$.

3. Relations between Generalized Fractional Hilbert Transform with Other Classical Transforms

This section is devoted to present relations between generalized fractional Hilbert transform with classical Fourier, Laplace, Mellin, Hartley, Hilbert and Stieltjes transforms.

3.1 Relation between Generalized Fractional Hilbert Transform with Fourier Transform

The Fourier transform defined in [2] is

$$F[f(t)](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iut} dt$$

Result 3.1.1 $H^{\alpha}[f(x) e^{-i\frac{\cot\alpha}{2}x^2}](t) = \sqrt{\dfrac{2}{\pi}} F\left[\dfrac{f(t-p)}{p}\right](0)$

Proof: The generalized fractional Hilbert transform is

$$H^{\alpha}[f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x}\, dx$$

putting $t - x = p$ so that $-dx = dp$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(t-p)}{p}\, dp$$

$$H^{\alpha}[f(x) e^{-i\frac{\cot\alpha}{2}x^2}](t) = \sqrt{\frac{2}{\pi}} F\left[\frac{f(t-p)}{p}\right](0)$$

3.2 Relation between Generalized Fractional Hilbert Transform with Laplace Transform

The Laplace transform defined in [2] is

$$L[f(x)](u) = \int_{0}^{\infty} f(x)\, e^{-ux} dx$$

Result: 3.2.1 $\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} L\left\{L[f(x) e^{i\frac{\cot\alpha}{2}x^2}](u)\right\}(t) = -H^{\alpha}[f(x)](-t)$, for $t > 0$

Proof : $L[f(x) e^{i\frac{\cot\alpha}{2}x^2}](u) = \displaystyle\int_{0}^{\infty} f(x)\, e^{i\frac{\cot\alpha}{2}x^2} e^{-ux} dx$

$$L\left\{L[f(x)e^{i\frac{\cot\alpha}{2}x^2}](u)\right\}(t)=\int_0^\infty\left\{\int_0^\infty f(x)\,e^{i\frac{\cot\alpha}{2}x^2}\,e^{-ux}dx\right\}e^{-ut}du$$

Changing the order of integration

$$=\int_0^\infty f(x)e^{i\frac{\cot\alpha}{2}x^2}\left\{\int_0^\infty e^{-(x+t)u}du\right\}dx$$

$$=\int_0^\infty f(x)e^{i\frac{\cot\alpha}{2}x^2}\,\frac{1}{x+t}dx$$

$$\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}L\left\{L[f(x)e^{i\frac{\cot\alpha}{2}x^2}](u)\right\}(t)=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^\infty\frac{f(x)e^{i\frac{\cot\alpha}{2}x^2}}{-t-x}dx$$

$$\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}L\left\{L[f(x)e^{i\frac{\cot\alpha}{2}x^2}](u)\right\}(t)=-H^\alpha[f(x)](-t) \qquad \text{, for } t>0$$

3.3 Relation between Generalized Fractional Hilbert Transform with Mellin Transform

The Mellin transform defined in [2] is

$$M[f(x)](z)=\int_0^\infty x^{z-1}f(x)\,dx$$

where z is, in general a complex variable

Result: 3.3.1 $M\left\{e^{i\frac{\cot\alpha}{2}p^2}H^\alpha[f(t)\,e^{-i\frac{\cot\alpha}{2}t^2}](p)\right\}(z)=-\cot(\pi z)M[f(t)](z)$

Proof : $M\left\{e^{i\frac{\cot\alpha}{2}p^2}H^\alpha[f(t)\,e^{-i\frac{\cot\alpha}{2}t^2}](p)\right\}(z)=\int_0^\infty p^{z-1}e^{i\frac{\cot\alpha}{2}p^2}H^\alpha[f(t)\,e^{-i\frac{\cot\alpha}{2}t^2}](p)\,dp$

$$=\int_0^\infty p^{z-1}\left\{\frac{1}{\pi}\int_{-\infty}^\infty\frac{f(t)}{p-t}dt\right\}dp$$

For the class of functions such that $f(t)=0$, for $t<0$

$$=\int_0^\infty p^{z-1}\left\{\frac{1}{\pi}\int_0^\infty\frac{f(t)}{p-t}dt\right\}dp$$

Changing the order of integration

$$=\int_0^\infty f(t)\left\{\frac{1}{\pi}\int_0^\infty\frac{p^{z-1}}{p-t}dp\right\}dt$$

$$=\int_0^\infty f(t)\left\{\frac{1}{\pi t}\int_0^\infty\frac{p^{z-1}}{(p/t)-1}dp\right\}dt$$

$$=\int_0^\infty t^{z-1}f(t)\left\{\frac{1}{\pi}\int_0^\infty\frac{u^{z-1}}{u-1}du\right\}dt$$

$$= -\int_0^{\infty} t^{z-1} f(t) \cot(\pi z)\, dt$$

$$M\left\{ e^{i\frac{\cot\alpha}{2}p^2} H^{\alpha}[f(t)\, e^{-i\frac{\cot\alpha}{2}t^2}](p) \right\}(z) = -\cot(\pi z) M[f(t)](z)$$

3.4 Relation between Generalized Fractional Hilbert Transform with Hartley Transform

The Hartley transform of f is defined as

$$H_A[f(t)](x) = \int_{-\infty}^{\infty} cas(xt)\, dt$$

where $cas(xt) = \cos(xt) + \sin(xt)$

Result: 3.4.1 $H_A\left\{ e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}](t) \right\}(-u) = -\operatorname{sgn}(u) H_A[f(x)](u)$

Proof : $H_A\left\{ e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}](t) \right\}(-u) = \int_{-\infty}^{\infty} cas(-ut) e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}](t)\, dt$

$$= \int_{-\infty}^{\infty} cas(-ut) \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} dx \right\} dt$$

Changing the order of integration

$$= \int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{cas(-ut)}{t-x} dt \right\} dx$$

$$= -\int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{[\cos ut - \sin ut]}{x-t} dt \right\} dx$$

$$= -\int_{-\infty}^{\infty} f(x) \operatorname{sgn}(u) (\sin ux + \cos ux)\, dx$$

$$= -\operatorname{sgn}(u) \int_{-\infty}^{\infty} cas(ux) f(x)\, dx$$

$$H_A\left\{ e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}](t) \right\}(-u) = -\operatorname{sgn}(u) H_A[f(x)](u)$$

3.5 Relation between Generalized Fractional Hilbert Transform with Hilbert Transform

The Hilbert transform defined in [2] is

$$H[f(x)](t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} dx,\ t \in \mathbb{R}, t \neq x$$

where integral is a Cauchy principal value.

In this part we defined $\overline{f}(x) = f(x)\, e^{i\frac{\cot\alpha}{2}x^2}$ and $\overline{\overline{f}}(x) = f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}$.

Result: 3.5.1 $H^{\alpha}[f(x)\ g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H\left[\overline{f}(x)\ g(x)\right](t) = e^{-i\frac{\cot\alpha}{2}t^2} H\left[f(x)\ \overline{g}(x)\right](t)$

Proof: $H^{\alpha}[f(x)\ g(x)](t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{f(x)\ g(x)}{t-x} e^{i\frac{\cot\alpha}{2}x^2} dx$

$$H^{\alpha}[f(x)\ g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[\overline{f}(x)\ g(x)]\,(t) = e^{-i\frac{\cot\alpha}{2}t^2} H[f(x)\ \overline{g}(x)]\,(t)\cdot$$

Result: 3.5.2 $H^{\alpha}[f(x)\ ](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[\overline{f}(x)](t)$

Proof: $H^{\alpha}[f(x)\ ](t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{i\frac{\cot\alpha}{2}x^2} dx$

$$H^{\alpha}[f(x)\ ](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[\overline{f}(x)]\,(t)$$

Result: 3.5.3 $H^{\alpha}[\overline{\overline{f}}(x)g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H\left[f(x)g(x)\right](t)$

Proof: $H^{\alpha}[\overline{\overline{f}}(x)g(x)](t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\overline{\overline{f}}(x)g(x)}{t-x} e^{i\frac{\cot\alpha}{2}x^2} dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)e^{-i\frac{\cot\alpha}{2}x^2}}{t-x} g(x) e^{i\frac{\cot\alpha}{2}x^2} dx$$

$$H^{\alpha}[\overline{\overline{f}}(x)g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H\left[f(x)g(x)\right](t)$$

Result: 3.5.4 $H_{\alpha}[\overline{\overline{f}}(x)\overline{\overline{g}}(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[\overline{\overline{f}}(x)g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[f(x)\overline{\overline{g}}(x)](t)$

Proof: $H^{\alpha}[\overline{\overline{f}}(x)\overline{\overline{g}}(x)](t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\overline{\overline{f}}(x)\overline{\overline{g}}(x)}{t-x} e^{i\frac{\cot\alpha}{2}x^2} dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{\overline{\overline{f}}(x)g(x)}{t-x} e^{-i\frac{\cot\alpha}{2}x^2} e^{i\frac{\cot\alpha}{2}x^2} dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{\overline{\overline{f}}(x)g(x)}{t-x} dx$$

$$H^{\alpha}[\overline{\overline{f}}(x)\overline{\overline{g}}(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[\overline{\overline{f}}(x)g(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[f(x)\overline{\overline{g}}(x)](t)$$

Result: 3.5.5 $H^{\alpha}[\overline{\overline{f}}(x)](t) = e^{-i\frac{\cot\alpha}{2}t^2} H[f(x)](t)$

Proof: $H^{\alpha}[\overline{\overline{f}}(x)](t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\overline{\overline{f}}(x)}{t-x} e^{i\frac{\cot\alpha}{2}x^2} dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{-i\frac{\cot\alpha}{2}x^2} e^{i\frac{\cot\alpha}{2}x^2} dx$$

$$H^{\alpha}[\overline{\overline{f}}(x)](t)=e^{-i\frac{\cot\alpha}{2}t^2}H[f(x)](t)$$

3.6 Relation between Generalized Fractional Hilbert Transform with Stieltjes Transform

The Stieltjes transform defined in [2] is

$$S[f(x)](z)=\int_0^{\infty}\frac{f(x)}{x+z}dx$$

where z is, in general a complex variable

Consider the class of functions that are causal. The support for causal function is $[0,\infty)$ and hence for $f(x)=0$.

Result 3.6.1 $H^{\alpha}[f(x)](t)=\dfrac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)$

Proof: $H^{\alpha}[f(x)](t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_0^{\infty}\frac{f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx \quad , for\, t\geq 0$

$$=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\frac{f(x)}{x+(-t)}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H^{\alpha}[f(x)](t)=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)$$

Result 3.6.2 $H^{\alpha}[f(x)](-t)=\dfrac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](t)$

Proof: $H^{\alpha}[f(x)](-t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_0^{\infty}\frac{f(x)}{-t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\frac{f(x)}{x+t}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H^{\alpha}[f(x)](-t)=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](t)$$

Result 3.6.3 When the function $f(x)$ is not causal,

$$H^{\alpha}[f(x)](t)=\frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\left\{S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)-S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](t)\right\}$$

Proof: $H^{\alpha}[f(x)](t)=\dfrac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$=\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\left\{\frac{f(x)}{t-x}+\frac{f(-x)}{t+x}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$= \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\left\{\frac{f(x)}{x-t}-\frac{f(-x)}{t+x}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$= \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\left\{\int_0^{\infty}\frac{f(x)e^{i\frac{\cot\alpha}{2}x^2}dx}{x+(-t)}-\int_0^{\infty}\frac{f(-x)e^{i\frac{\cot\alpha}{2}x^2}dx}{x+t}\right\}$$

$$H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\left\{S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)-S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](t)\right\}$$

Note that for the results 3.6.4 and 3.6.5 given in [2, p. 266], the average denominator across the branch cut on the negative real axis has been taken. That is, $xe^{i\pi}$ is employed on the upper side of the branch cut and $xe^{-i\pi}$ is used on the lower side of the branch cut.

Result 3.6.4 $H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](te^{i\pi}) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](te^{-i\pi})$

$$+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](t)\text{ , } for\ t>0$$

Proof: $H^{\alpha}[f(x)](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\left\{\frac{f(x)}{t-x}+\frac{f(-x)}{t+x}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$= \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}\int_0^{\infty}f(x)\left\{\frac{1}{x+te^{i\pi}}+\frac{1}{x+te^{-i\pi}}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\frac{f(-x)}{t+x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](te^{i\pi}) - \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](te^{-i\pi})$$

$$+\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](t)$$

Result 3.6.5

$$H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](|t|e^{i\pi}) + \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](|t|e^{-i\pi})$$

$$-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)\text{ , } for\ t<0$$

Proof: $H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\left\{\frac{f(x)}{-t+x}-\frac{f(-x)}{t+x}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi}\int_0^{\infty}f(-x)\left\{\frac{1}{x+|t|e^{i\pi}}+\frac{1}{x+|t|e^{-i\pi}}\right\}e^{i\frac{\cot\alpha}{2}x^2}dx-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_0^{\infty}\frac{f(x)}{-t+x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$H^{\alpha}[f(x)](t) = \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi} S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](|t|e^{i\pi}) + \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{2\pi} S[f(-x)e^{i\frac{\cot\alpha}{2}x^2}](|t|e^{-i\pi})$$

$$-\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} S[f(x)e^{i\frac{\cot\alpha}{2}x^2}](-t)$$

Note that $\alpha = \frac{\pi}{2}$, we get relations between classical Hilbert transform with classical Fourier, Laplace, Mellin, Hartley and Stieltjes transform.

4. Conclusion

In this paper relation between the generalized fractional Hilbert transform with classical Fourier, Laplace, Mellin, Hartley, Hilbert and Stieltjes transforms are established which will be useful in solving differential equations occurring in signal processing and in many other scientific disciplines.

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