International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 10, October 2015, PP 14-27
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Oscillations in Damped Driven Pendulum: A Chaotic System

Kulkarni P. R.

P. G. Department of Mathematics,
N. E. S. Science College, Nanded, India.
pramodrkul@gamil.com

Dr. Borkar V. C.

Department of Mathematics and statistics,
Yeshwant Mahavidyalaya, Nanded, India.
borkarvc@gmail.com

Abstract: *In this paper, we have discussed the solutions of a system of n differential equations as a continuous dynamical system. Then we have discussed the nature of oscillations of a damped driven pendulum. We have analyzed the nature of fixed and periodic points of a damped driven pendulum for certain ranges of parameters. We have proved that oscillations of the pendulum are chaotic for certain ranges of parameters through the period doubling phenomenon. For the analysis of the solutions, mathematical softwares like MATLAB and Phaser Scientific Software are used.*

Keywords: *chaos, dynamical system, fixed points, orbits, periodic points, stability period doubling*

1. INTRODUCTION

A wide range of physical phenomena where there is a change in one quantity that occurs due to a change in one or more quantities can be mathematically modeled in terms of differential equations. Differential equations can be used to describe the motions of objects like satellites, water molecules in a stream, waves on strings and surfaces, etc. In this section we will take a review of some basic terminology associated with a system of differential equations.

1.1 System of Differential Equations [6]

Let $x_1,\ x_2, \dots, x_n$ be differentiable functions of a variable t, usually called as time, on an interval I of the real numbers. Let $f_1,\ f_2, \dots, f_n$ be functions of $x_1,\ x_2, \dots, x_n$ and t. Then the n differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1,\ x_2, \dots, x_n, \text{t}),\\ \frac{dx_2}{dt} &= f_2(x_1,\ x_2, \dots, x_n, \text{t}),\\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1,\ x_2, \dots, x_n, \text{t}) \end{aligned} \tag{1}$$

are called as a system of differential equations. This system can also be expressed as

$X' = F(X, t)$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ and $F = (f_1,\ f_2, \dots, f_n)$.

The system $X' = F(X, t)$, where F can depend on the independent variable t is called as a non-autonomous system. Any non-autonomous system (1) with $X \in R^n$ can be written as an autonomous system $X' = F(X)$ (2)

with $X \in R^{n+1}$ simply by letting $x_{n+1} = t$ and $x'_{n+1} = 1$. The fundamental theory for the systems (1) and (2) does not differ significantly.

1.2 Phase-Plane Analysis

If $X: I \to R^n$ is defined by $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and if $X(t)$ satisfies the system (1), then $X(t)$ is said to be a solution of the system (1). If $t_0 \in R$ and X is s solution for all $t \geq t_0$, then $X(t_0)$ is an initial condition of a solution X. As $x_1, x_2, \ldots, x_n$ are functions of the variable t, it follows that as t increases, $X(t)$ traces a curve in R^n called as the *trajectory* or the *orbit* and in this case, the space R^n is called as the *phase space* of the system. The phase space is completely filled with trajectories since each point $X(t_0)$ can serve as an initial point. The system $X' = F(X)$ is said to be a linear system if the function F is linear. In this case, the system can be expressed as

$X' = A.X$ where A is an $n \times n$ matrix. The function F is also called as a *vector field*. The vector field always dictates the velocity vector X' for each X. A picture which shows all qualitatively different trajectories of the system is called as a *phase portrait*.[13] A second order differential equation which can be expressed as a system of two differential equations can be treated as a vector field on a plane or also called as a *phase plane*. The general form of a vector field over the plane is $x'_1 = f_1(x_1, x_2)$, $x'_2 = f_2(x_1, x_2)$ which can be compactly written in vector notations as

$X' = F(X)$, where $X = (x_1, x_2)$ and $F(X) = (f_1(X), f_2(X))$. For non-linear systems, it is quite difficult to obtain the trajectories by analytical methods and though the trajectories are obtained by explicit formulas, they are too complicated to provide the information about the solution. Hence qualitative behaviors of the trajectories obtained by numerical solution methods are often studied. To obtain a phase portrait, we plot the variable x_1 against the variable x_2 and study the qualitative behavior of the solution. A theorem concerning the uniqueness of the solution of a linear system is stated as follows.

1.3 Theorem (The Fundamental Theorem For Linear Systems) [12]

Let A be an $n \times n$ matrix. Then for a given $X_0 \in R^n$, the initial value problem

$$X' = A.X, \qquad X(0) = X_0$$

has a unique solution given by $X(t) = e^{At} X_0$.

Now we state the fundamental theorem for the existence and uniqueness of the solution of a non-linear system.

1.4 Theorem (The Fundamental Existence-Uniqueness Theorem) [12]

Let E be an open subset of R^n containing X_0 and assume that $F \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem $X' = F(X)$, $X(0) = X_0$ has a unique solution $X(t)$ on the interval $[-a, a]$.

1.5 Fixed Point or Stationary Point or Equilibrium Point or Critical Point [6]

A fixed point or an equilibrium point of a system of differential equations is constant solution, that is, a solution X such that $X(t) = X(t_0)$ for all t. If X is a critical point, then we identify the critical point with the vector $X(t_0)$. From the definition, it is clear that X is a fixed point of the system (1) if $X'(t) = 0$.

1.6 Classification of Fixed Points Depending Upon Their Stability [13]

Let X^* be a fixed point of a system $X' = F(X)$.

(i) We say that X^* is an attracting stable fixed point if there is a $\delta > 0$ such that $\lim_{t\to\infty} X(t) = X^*$ whenever $\| X(0) - X^* \| < \delta$.

This definition implies that any trajectory that starts near X^* within a distance δ is guaranteed to converge to X^*eventually.

(ii) X^* is said to be Liapunov stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that

$\| X(t) - X^* \| < \epsilon$ whenever $t \geq 0$ and $\| X(0) - X^* \| < \delta$.

Thus trajectories that start near X^* within δ remain within ϵ for all positive time. Liapunov stability requires that the nearby trajectories stay close for all the time.

(iii) The fixed point X^* is said to be asymptotically stable if it is both attracting and Liapunov stable.

1.7 Hyperbolic and Nonhyperbolic Fixed Point [13, 19]

A fixed point X^* of a system $X' = F(X)$, where $F = (f_1, f_2, \ldots, f_n)$ and $X = (x_1, x_2, \ldots, x_n)$ is called a hyperbolic fixed point if the real part of the eigenvalues of the Jacobian matrix $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ at the fixed point X^* are nonzero. If the real part of either of the eigenvalues are equal to zero, then the fixed point is called as nonhyperbolic.

1.8 Linearization of a Two Dimensional Nonlinear System [19]

Suppose that the nonlinear two dimensional system

$$x_1' = f_1(x_1, x_2), \; x_2' = f_2(x_1, x_2) \tag{3}$$

has a critical point $X^* = (x_1^*, x_2^*)$, where f_1 and f_2 are at least quadratic in x_1 and x_2. We take a linear transformation which moves the fixed point to the origin. Let $\overline{x_1} = x_1 - x_1^*$ and $\overline{x_2} = x_2 - x_2^*$. Then the system (3) takes the linearized form

$$\overline{x_1}' = \overline{x_1}\frac{\partial f_1}{\partial x_1}|_{X=X^*} + \overline{x_2}\frac{\partial f_1}{\partial x_2}|_{X=X^*}, \; \overline{x_2}' = \overline{x_1}\frac{\partial f_2}{\partial x_1}|_{X=X^*} + \overline{x_2}\frac{\partial f_2}{\partial x_2}|_{X=X^*}. \tag{4}$$

1.9 Hartman's Theorem [19]

Suppose that $X^* = (x_1^*, x_2^*)$ is a critical point of the system (3). Then there is a neighborhood of this critical point on which the phase portrait for the nonlinear system resembles that of the linearized system (4). In other words, there is a curvilinear continuous change of coordinates taking one phase portrait to the other, and in a small region around the critical point, the portraits are qualitatively equivalent.

1.10 Limit Set [18]

The set of all points that are limit points of a given solution is called the set of *ω-limit points*, or the *ω-limit set*, of the solution $X(t)$. Similarly, we define the set of *α-limit points*, or the *α-limit set*, of a solution $X(t)$ to be the set of all points Z such that $\lim_{n\to\infty} X(t_n) = Z$ for some sequence $t_n \to -\infty$.

A number of examples of limit set of solution of a differential equation are given in [18]. Now we state the Poincaré-Bendixson theorem which determines all of the possible limiting behaviors of a planar flow.

1.11 Theorem (Poincaré-Bendixson) [18]

Suppose that Ω is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.

2. Oscillations of a Pendulum System

In this section, we will discuss the oscillations of a pendulum subject to a periodic force and a damping force within certain ranges of parameters. We will discuss different types of oscillations of the pendulum for different values of the damping force and driven force and prove that the oscillations are chaotic using the period doubling phenomenon.

2.1 Oscillations in the Absence of Damping Force And Periodic Force

Consider a pendulum of mass m and length L swinging back and forth. For the present case, suppose that there is no damping force and no periodic driven force acting on the pendulum. The only force action on the pendulum is the weight mg acting downward, where g is the acceleration due to gravity. Let $\theta(t)$ denote the angle made by the pendulum with the normal at time t. In this case, the motion of the pendulum is governed by the second order differential equation $mL^2\frac{d^2\theta}{dt^2} = -mgL\,sin\theta$ (5)

Because of the trigonometric term $sin\theta$, the equation (5) is nonlinear. To find an exact solution of (5) is not possible. However, numerical solutions can be obtained by different methods. We linearize the system by considering very small oscillations of the system so that $sin\theta \approx \theta$. Thus the system takes the form $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$. The exact solution of this system is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right), \tag{6}$$

where A and B are constants which can be determined by using the initial conditions.

The solution (6) is just the equation of a simple harmonic motion. Taking $g = L,\ \theta = x_1$ and $\frac{d\theta}{dt} = x_1' = x_2$, equation (5) can be written as a system of differential equations given by

$$x_1' = x_2 = f_1(x_1,\ x_2), \tag{7}$$

$$x_2' = -\sin x_1 = f_2(x_1,\ x_2) \tag{8}$$

The fixed points of (7), (8) are obtained by solving $x_1' = 0$ and $x_2' = 0$. Thus the fixed points are $X^* = (n\pi,\ 0)$, where n is an integer. The Jacobian matrix of the system (7), (8) is

$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -cosx_1 & 0 \end{bmatrix}$. At the fixed point X^*, the Jacobian matrix $\begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = \pm 1$ and $\lambda = \pm i$. Hence the system has hyperbolic fixed points $((2n+1)\pi,\ 0)$ and nonhyperbolic fixed points $(2n\pi,\ 0)$. By Hartman's theorem, there are neighborhoods of the hyperbolic fixed points in which the phase portraits of the linearized and non-linearized systems are topologically conjugate. By Poincare-Bendixson theorem, the chaos [1,2,4,8,9] does not exist in this two dimensional autonomous system. Some o the trajectories are shown graphically for this system. The graphs are obtained by using the software MATLAB as shown in the Figure 1.

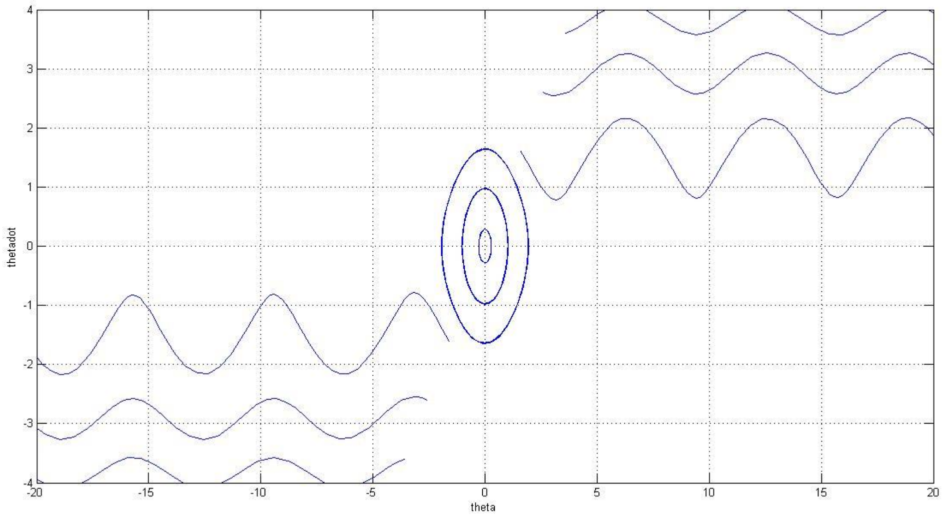

Figure 1

2.2 Damped Oscillations of the Pendulum

Now we suppose that the pendulum is damped. Assume that the damping is proportional to the velocity and it opposes to the motion of the pendulum. The damping force can be caused by air resistance or friction due to any other medium in which the pendulum is immersed. Let d denote the damping parameter. Then the damping force acting on the pendulum is $dL^2\frac{d\theta}{dt}$. Hence the differential equation of motion of the pendulum becomes

$$mL^2\frac{d^2\theta}{dt^2}+dL^2\frac{d\theta}{dt}+mgL\,sin\theta=0$$

$$.e.\quad \frac{d^2\theta}{dt^2}+\frac{d}{m}\frac{d\theta}{dt}+\frac{g}{L}sin\theta=0.$$

For a proper analysis of this equation, we simplify it by the introduction of two variables *viz.* the natural frequency $\omega_N=\sqrt{\frac{g}{L}}$ and the damping constant $b=\frac{d}{2m}$. The differential equation of motion of the pendulum then takes the form

$$\frac{d^2\theta}{dt^2}+2b\,\frac{d\theta}{dt}+\omega_N^2 sin\theta=0. \tag{9}$$

With $\theta=x_1$ and $\frac{d\theta}{dt}=x_1'=x_2,\frac{d^2\theta}{dt^2}=x_2{}'$, equation (9) can be written as a system of differential equations

$$x_1'=x_2, \tag{10}$$

$$x_2'=-2bx_2-\omega_N^2\sin x_1 \tag{11}$$

This is not a linear system, but as discussed in the earlier case where there is no damping, this system is almost linear [6] at the origin. The linearized system

$$x_1'=x_2, \tag{12}$$

$$x_2'=-2bx_2-\omega_N^2 x_1 \tag{13}$$

has the associated matrix $A=\begin{bmatrix}0 & 1\\ -\omega_N^2 & -2b\end{bmatrix}$. The eigenvalues of this matrix are given by

$\lambda_1=-b-\sqrt{b^2-\omega_N^2}$ and $\lambda_2=-b+\sqrt{b^2-\omega_N^2}$. The nature of the fixed point O=(0, 0) depends upon the values of λ_1 and λ_2. Note that the real parts of λ_1 and λ_2 are negative so that the solutions of the linear system (12), (13) are asymptotically stable. Hence by Hartman's theorem, in some neighborhoods of the origin, the solutions of the nonlinear damped system (10), (11) are stable at the origin. As time passes, the solutions spirals and approaches the zero solution and ultimately, the pendulum stops oscillating. This tendency of the solution to spiral is observed as the damping constant d increases from 0 to $(b^2-\omega_N^2)$. As b increases beyond $(b^2-\omega_N^2)$, the trajectory of a solution does not spiral as it converges to the zero solution. The nonlinear system (10), (11) has fixed points $(n\pi,\ 0)$, where n is an integer and the system is almost linear at each fixed point.

The graph of the solution θ plotted against the time t has the nature as shown in the figure 2.

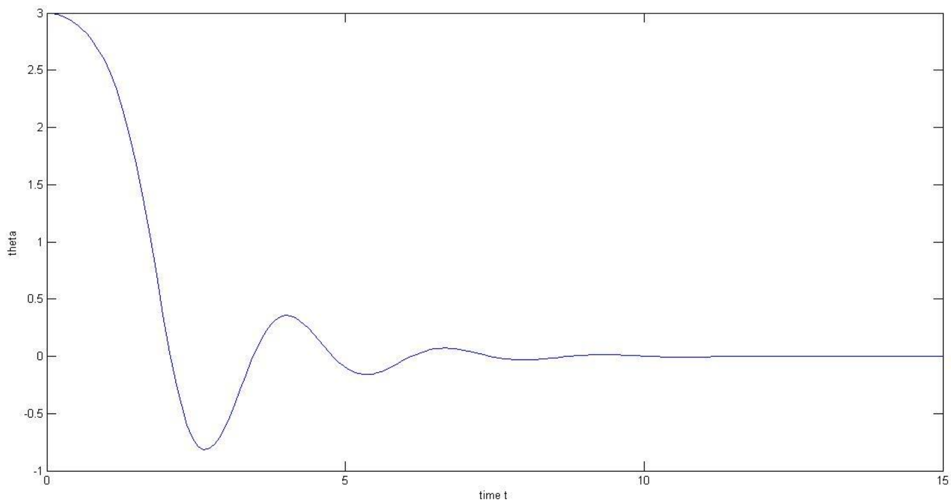

Figure 2

The phase portrait of the damped pendulum is as shown in the figure 3.

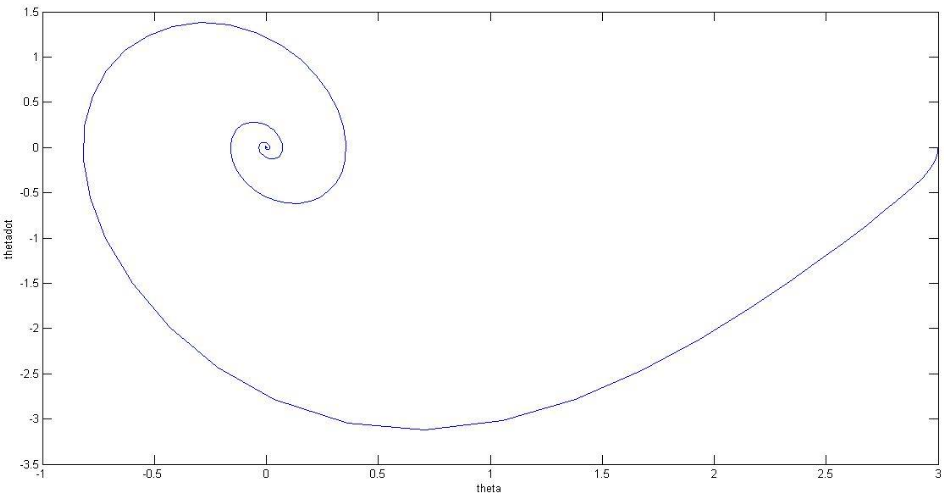

Figure 3

2.3 Oscillations of the Damped Driven Pendulum

Assume that the pendulum is subject to a damping force $dL^2\ \frac{d^2\theta}{dt^2}$ as in the previous case along with a sinusoidal driving force $F(t) = A_D \cos \omega_D\, t$ which varies with time t, where A_D is the amplitude of the driving force and ω_D is the driving angular frequency. The physical pendulum and different forces acting on it are as shown in the figure 4.

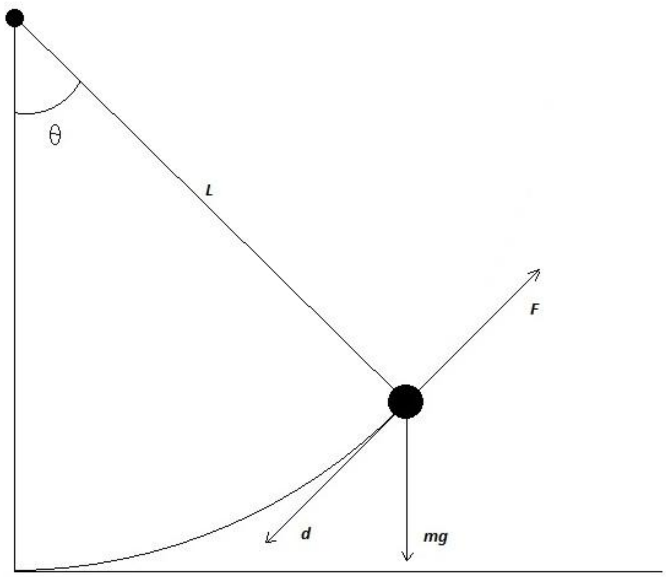

Figure 4. *The physical pendulum and different forces acting on it.*

In this case, the differential equation of motion of the pendulum takes the form

$$mL^2 \frac{d^2\theta}{dt^2} + dL^2 \frac{d\theta}{dt} + mgL\, sin\theta = L\, A_D \cos \omega_D\, t.$$

Dividing by ML^2, we get

$$\frac{d^2\theta}{dt^2} + \frac{d}{M}\frac{d\theta}{dt} + \frac{g}{L}\, sin\theta = \frac{A_D}{ML} \cos \omega_D\, t \tag{14}$$

Taking $\theta = x_1$ and $\frac{d\theta}{dt} = x_1' = x_2, \frac{d^2\theta}{dt^2} = x_2{}'$, equation (14) can be written as a system of differential equations

$$x_1' = x_2, \tag{15}$$

$$x_2' = -\frac{d}{M}x_2 - \frac{g}{L}\sin x_1 + \frac{A_D}{ML}\cos\omega_D t \quad (16)$$

The system of equations (15), (16) appears to be a two dimensional phase-plane system and one may conclude by Poincare-Bendixson theorem that the chaos is not possible in the oscillations of the pendulum, but note that the system is non-autonomous and it can be made a three dimensional autonomous system simply by adding one more variable $x_3 = \omega_D t$ so that the system can be expressed as

$$x_1' = x_2,$$

$$x_2' = -\frac{d}{M}x_2 - \frac{g}{L}\sin x_1 + \frac{A_D}{ML}\cos x_3 \ , x_3' = \omega_D$$

Thus the Poincare-Bendixson theorem is not applicable and chaos may be observed in the system. Denoting $\frac{d}{M} = 2B, \frac{g}{L} = \omega_N^2$ and $r = \frac{A_D}{Mg}$, the system (15), (16) can be expressed as

$$x_1' = x_2, \quad (17)$$

$$x_2' = -2Bx_2 - \omega_N^2\sin x_1 + r\,\omega_N^2\cos\omega_D t \quad (18)$$

This type of system of equations appears in John Taylor's *Classical Mechanics*. [20] We choose $\omega_D = 2\pi$ so that the period of the driving force becomes $\frac{2\pi}{\omega_D} = 1$. The chaos can be observed in the system if ω_N is close to ω_D. It can be verified that the much erratic oscillations of the pendulum are observed when $\omega_N > \omega_D$ as compared to the case $\omega_N < \omega_D$. We will keep the values of the parameters ω_N, ω_D and B as constants and vary the parameter r in search of the chaos. As suggested by John R. Taylor, we will use the parameter values $\omega_D = 2\pi$, $\omega_N = 3\pi$, $B = \frac{3\pi}{4}$ and let r vary. The period doubling phenomenon has been one of the important characterization of chaotic dynamical system. The bifurcation values can be observed from the bifurcation diagram [8] obtained using the MATLAB program. The bifurcation diagram of the pendulum system is as shown in the following figure 5.

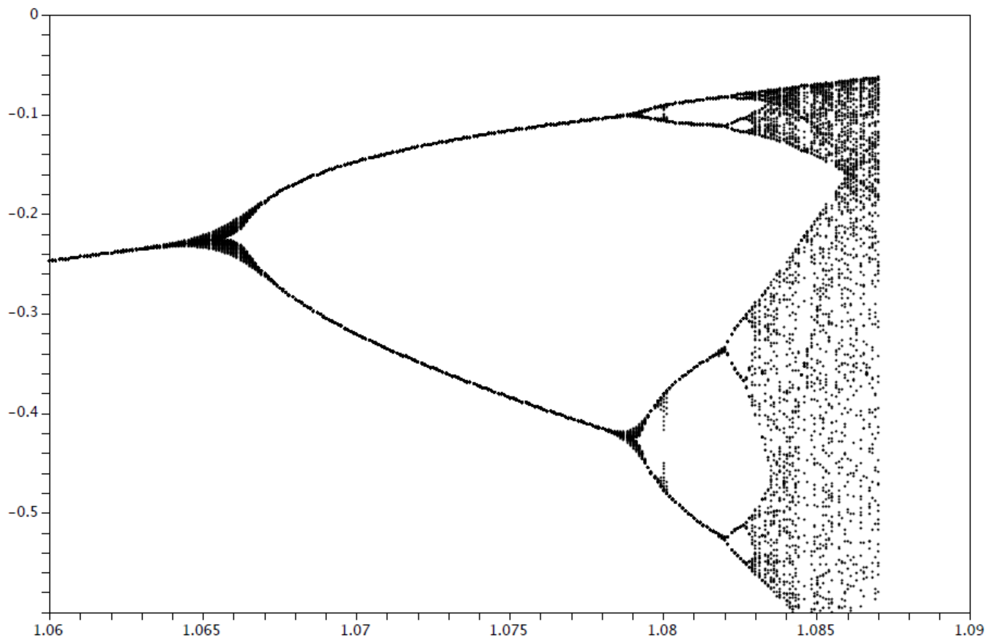

Figure 5

By a careful observation of the bifurcation diagram, we note that for an approximate value $r = 1.065$, the period of oscillations changes from period one to period two. The numerical solutions for θ against the time t for different values of r near 1.065 are plotted in the following figures. The solution for θ against the time t with $r = 1.05$, $\theta = 0.2$, $\dot{\theta} = 0$ is plotted in the figure 6 (a) and 6 (b) below. In the graph we can observe that the period of oscillations of the pendulum is one.

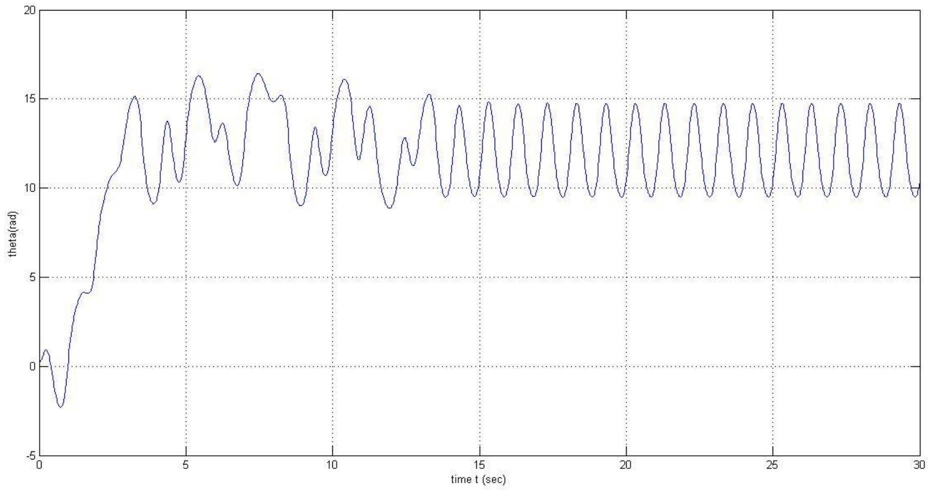

Figure 6 (a): *Solution with* $r = 1.05$, $\theta = 0.2$, $\dot{\theta} = 0$. *After a transient behavior, the solution is observed to be periodic.*

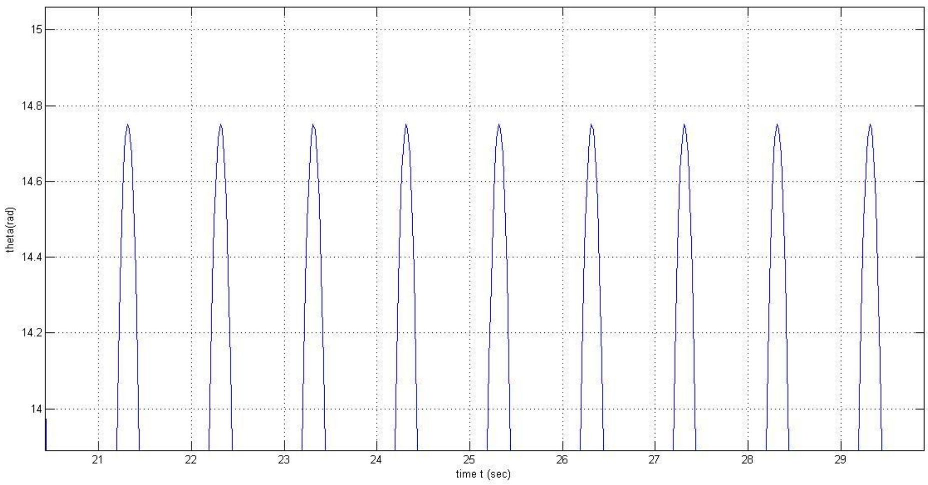

Figure 6 (b): *Zoom in of the solution with* $r = 1.05$, $\theta = 0.2$, $\dot{\theta} = 0$.
The period of oscillations is seen to be one.

The phase plane portrait with the same values $r = 1.05$, $\theta = 0.2$, $\dot{\theta} = 0$ is as shown in the following Figure 7 (a) obtained using MATLAB programming. The phase portrait with same values but obtained using Phaser Scientific Software with Dormand-Prince Algorithm is as shown in the Figure 7 (b). It can be observed that after a transient behavior, there is an orbit of period 1.

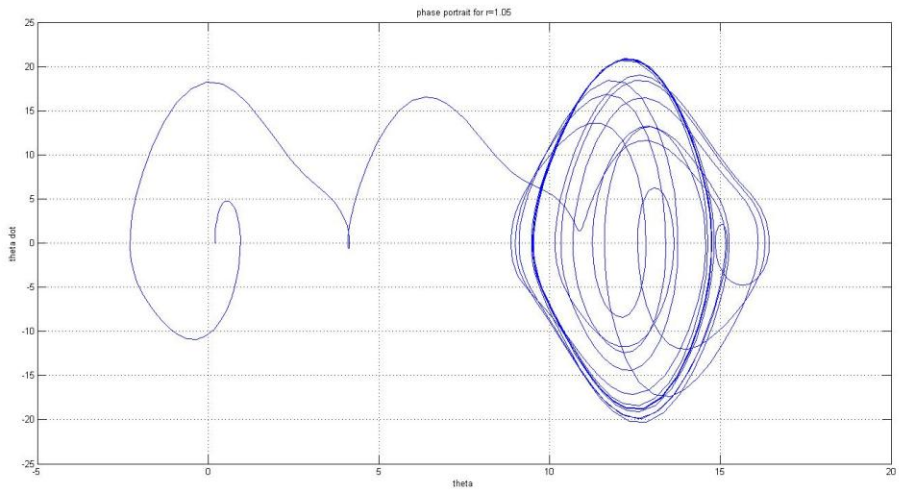

Figure 7 (a) *The phase plane portrait with* $r = 1.05$, $\theta = 0.2$, $\dot{\theta} = 0$

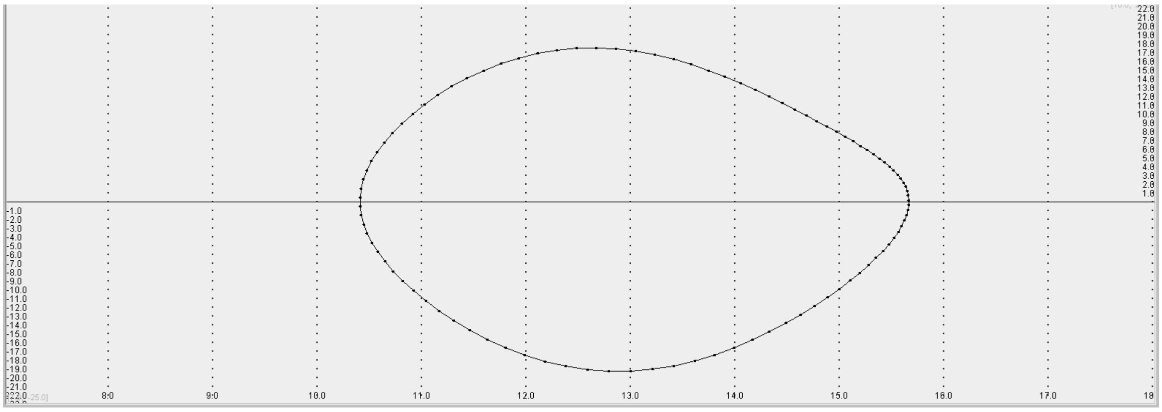

Figure 7 (b)

The solution for θ against the time t with $r = 1.075$, $\theta = 0.2$, $\dot{\theta} = 0$ is plotted in the figure 8 (a) and (b) below. In the graph we can observe that the period of oscillations of the pendulum is two.

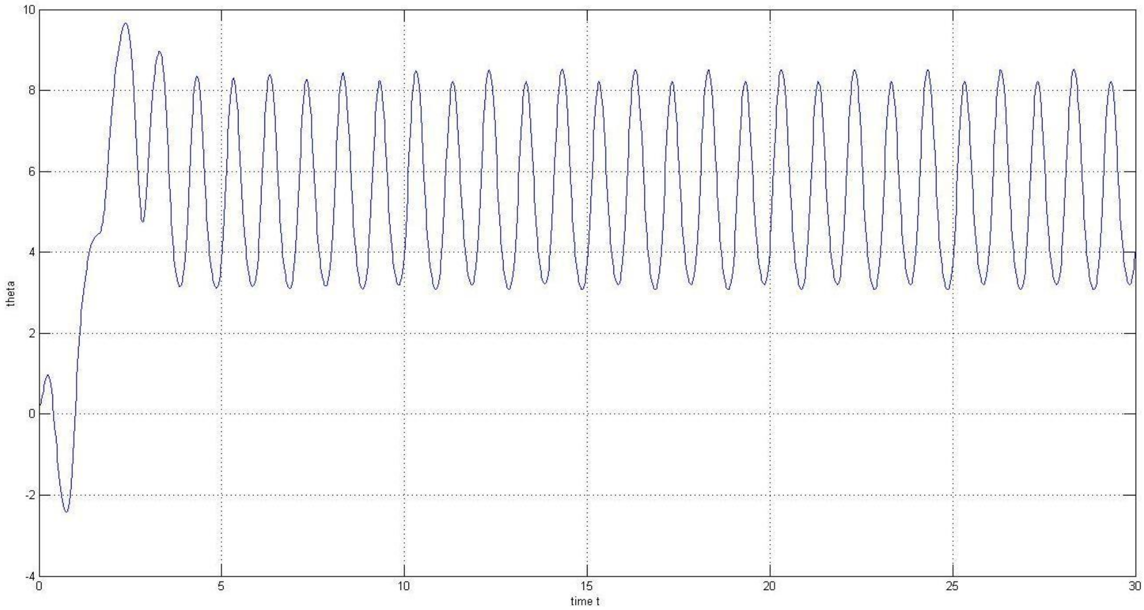

Figure 8 (a) *Solution with* $r = 1.075$, $\theta = 0.2$, $\dot{\theta} = 0$.
Period of oscillation is seen to be two

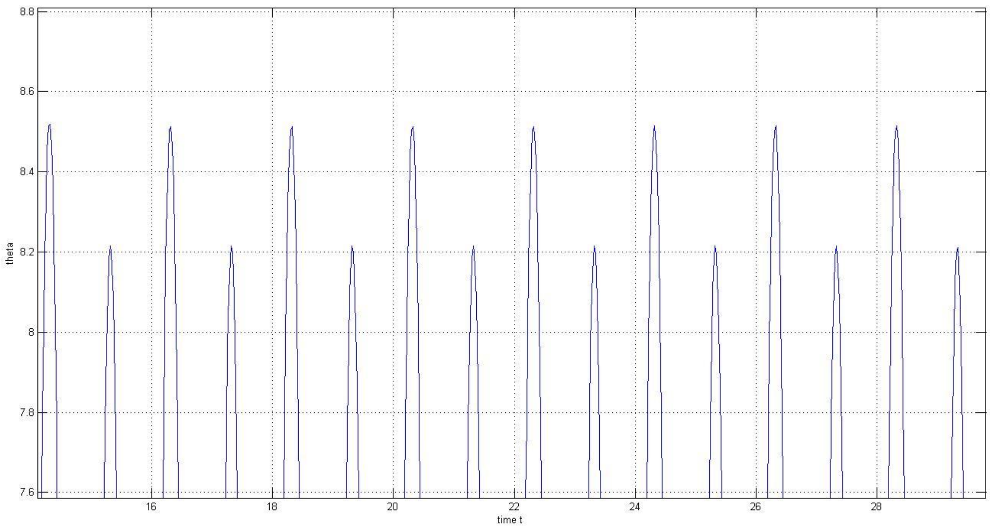

Figure 8 (b). *Zoom in of the solution with* $r = 1.075$, $\theta = 0.2$, $\dot{\theta} = 0$.

The phase plane portrait with the values $r = 1.075,\ \theta = 0.2,\ \dot{\theta} = 0$ is as shown in the following figures 9 (a) and 9 (b) obtained using MATLAB programming and Phaser Scientific Software. It can be observed that after a transient behavior, there is an orbit of period 2.

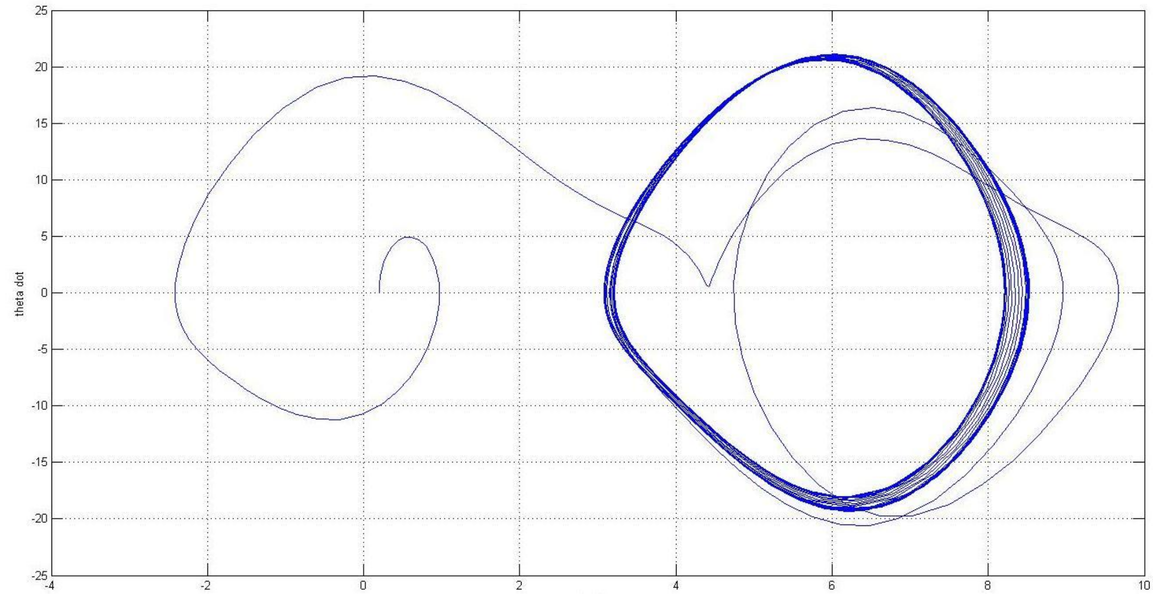

Figure 9 (a)

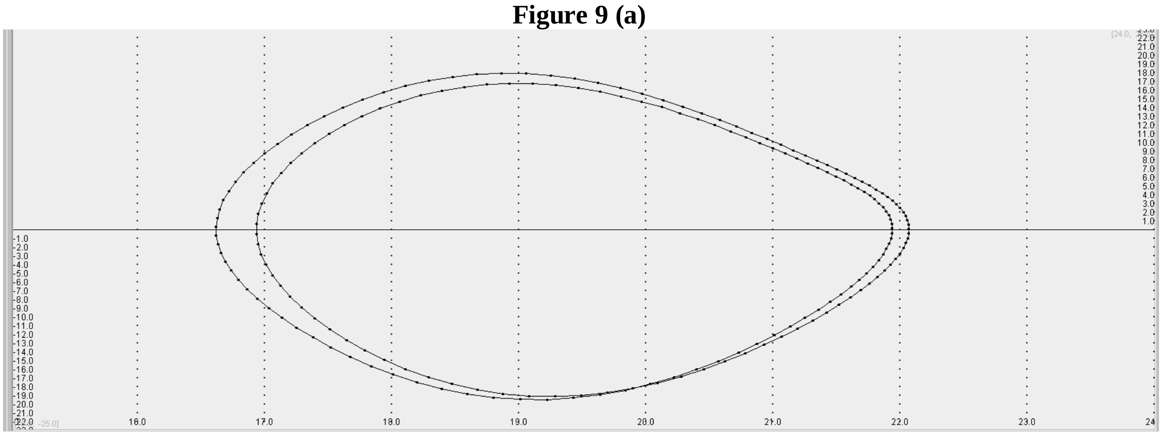

Figure 9 (b)

The solution with $r = 1.081,\ \theta = 0.2,\ \dot{\theta} = 0$ is as shown in the following figure 10 (a) and figure 10 (b). It can be observed that after the transient decay, there is an orbit of period four.

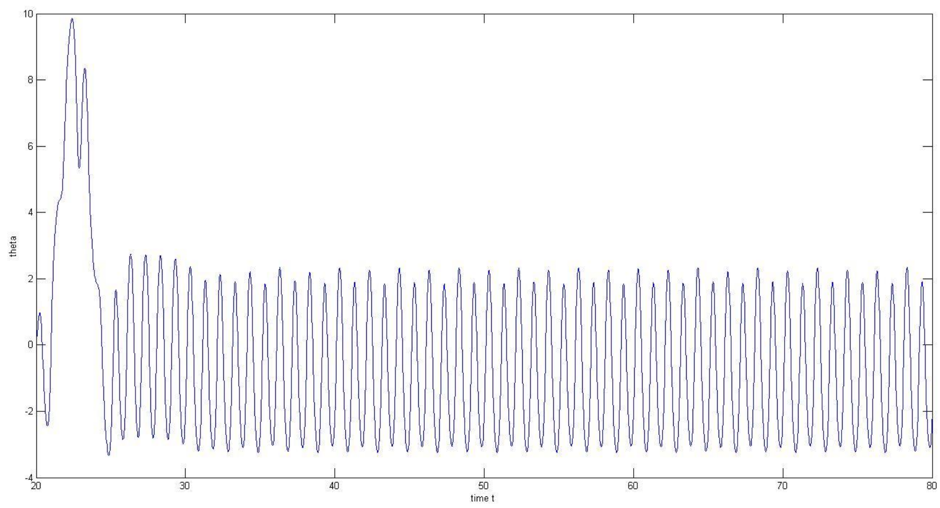

Figure 10 (a)

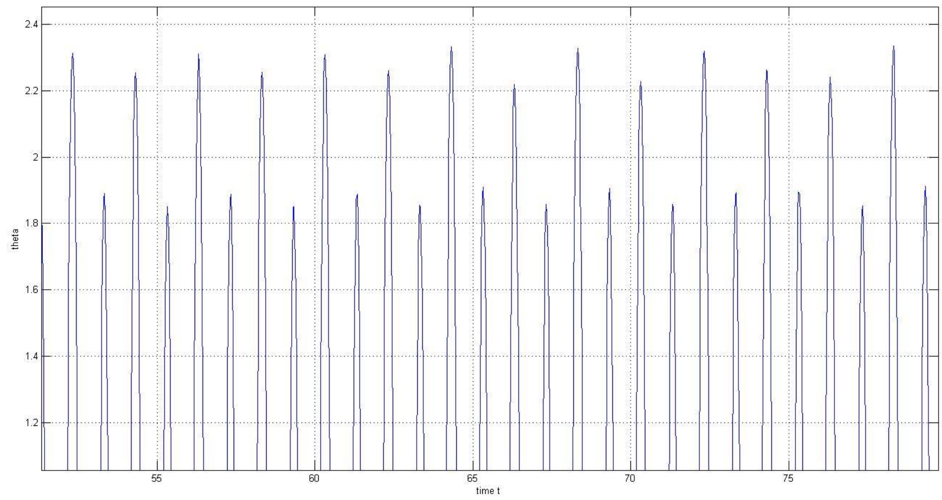

Figure 10 (b)

Zoom in of the solution with $r = 1.081$, $\theta = 0.2$, $\dot{\theta} = 0$.

The phase-plane portrait for $r = 1.081$, $\theta = 0.2$, $\dot{\theta} = 0$ is as shown in the following figures 11 (a), 11 (b) and 11 (c). In the figure 11 (c), we can observe the period four cycle.

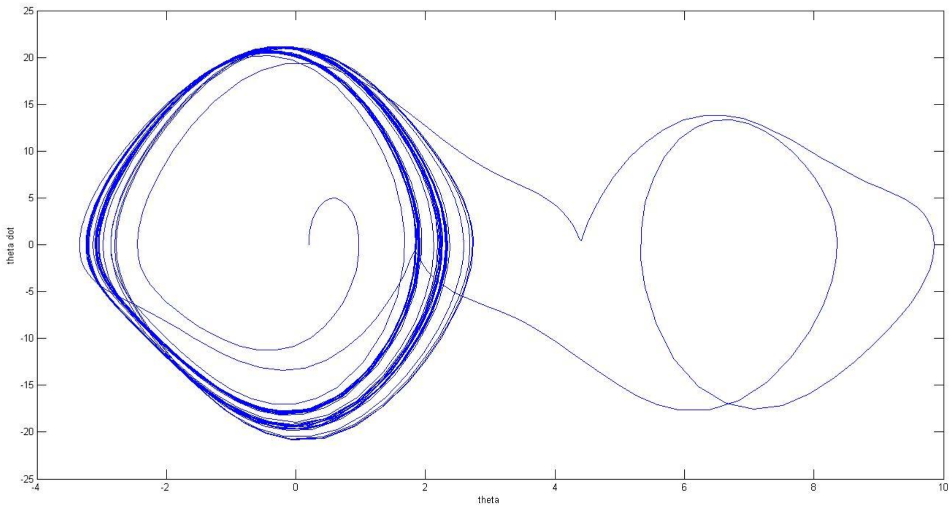

Figure 11 (a). *Phase-plane using MATLAB*

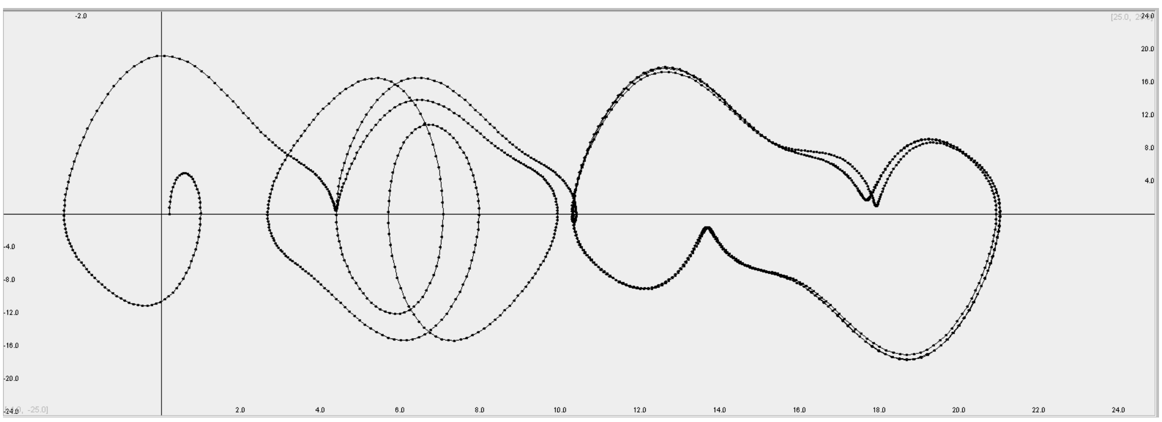

Figure 11 (b) *Phase-plane using Phaser*

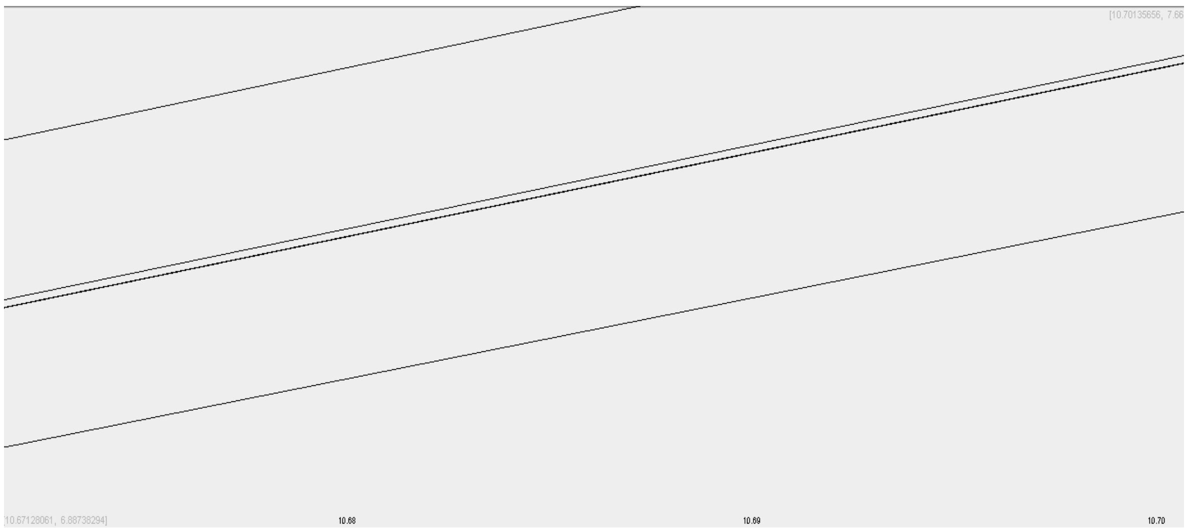

Figure 11 (c) *The zoom-in of the phase portrait [figure 11 (b)], where period four cycle can be observed.*

We can continue in this fashion and obtain the values of r for the oscillations of period eight, period sixteen and so on. Thus we come across a period doubling phenomenon. This period doubling is observed in many systems in nature and it is supposed to be one of the important features of chaos. However, at a certain value of r, as can be observed from the bifurcation diagram and figure 12, the periodic behavior of the pendulum turns out to be non-periodic.

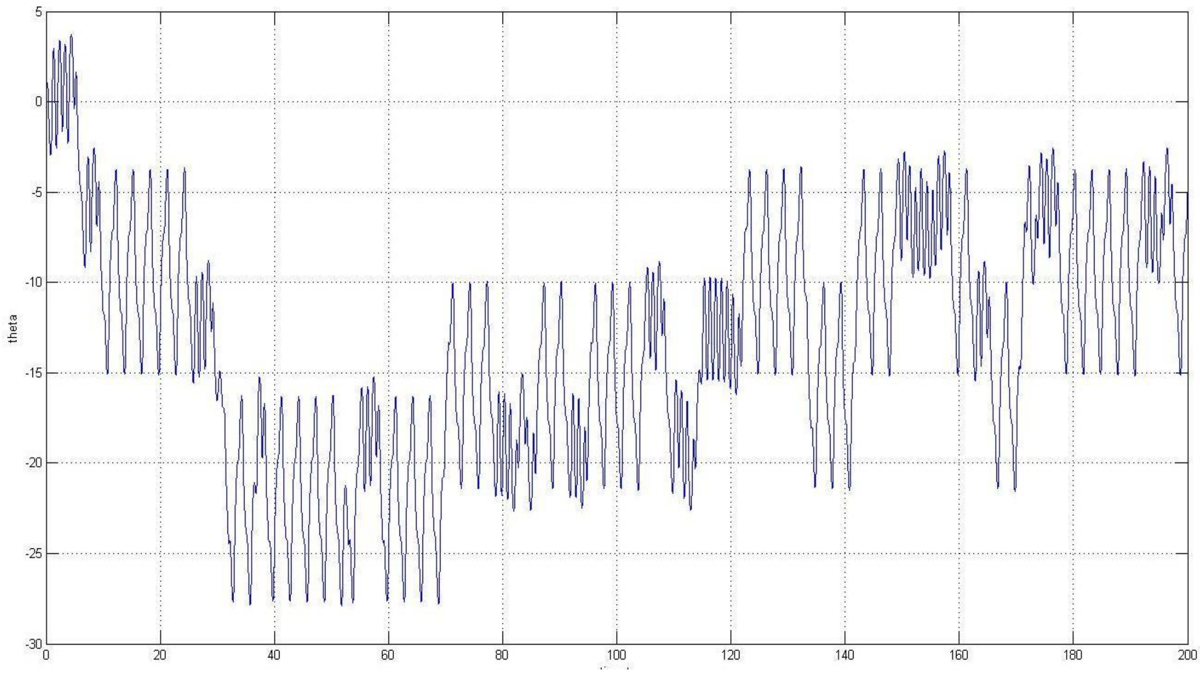

Figure 12. *Graph of the solution θ against time t for* $r = 1.15, \theta = 0.2, \dot{\theta} = 0$

The phase-plane portrait of a non-linear system can sometimes become overcrowded and very difficult to understand. To overcome this difficulty, Poincare maps [19] are extensively used. These maps transform complicated behavior in the phase space to discrete maps in a lower dimensional space. If a dynamical system has a simple attractor, the Poincare map appears as one or more points, with the number of points indicating the period of the solution. The Poincare map for the pendulum system described by the equations (15), (16) for $r = 0.15,\ \theta = 0,\ \dot{\theta} = 0.2$ is as shown in the figure 13. In this figure, we can observe that the system is chaotic for $r = 1.15$.

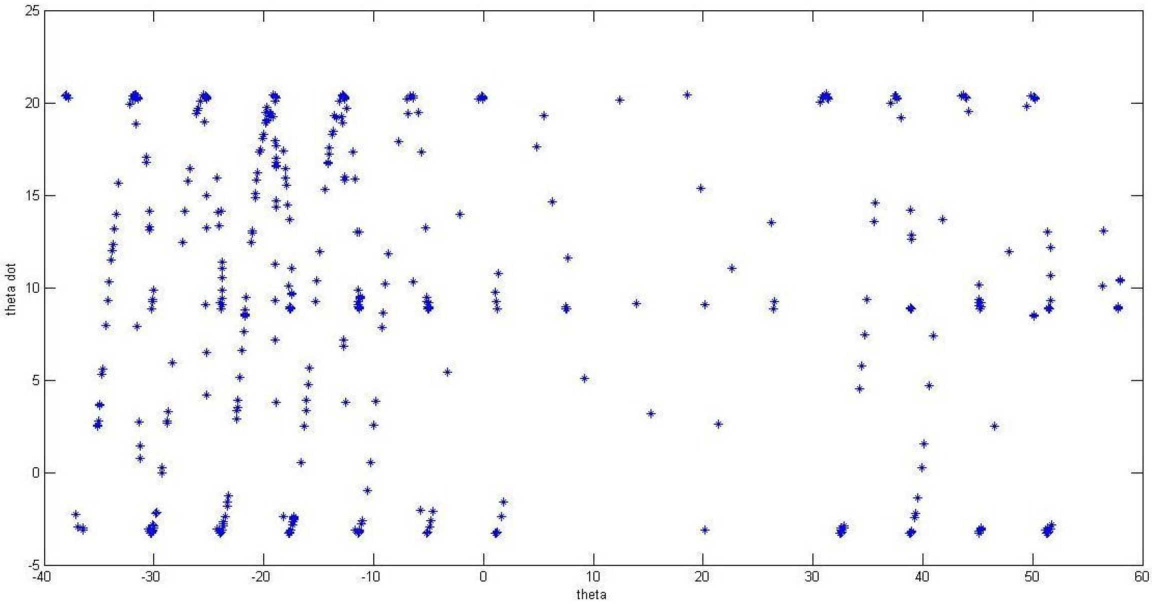

Figure 13. *Poincare map for* $r = 0.15,\ \theta = 0,\ \dot{\theta} = 0.2$

This non-periodic behavior is also an important feature of chaos. Mathematicians all over the world do not agree on a universal definition of chaos, but they agree on the important features of chaos as follows:[1, 3, 13, 19]

'Chaos is a non-periodic long term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.'

In this definition, the term 'long term non-periodic behavior' means that there are trajectories that do not converge to a fixed point or to periodic orbits over a long period of time *i.e.* as the time $t \to \infty$. The random behavior of the system is caused because of the non-linearity of the system and not because of the parameters of the system. Sensitive dependence on the initial conditions is an important feature of the chaotic systems. This condition means that the trajectories that differ with a negligible amount of initial conditions differ very fast as the time passes. In this case, the system has positive Liapunov exponent.

3. Conclusion

The oscillations of the pendulum system exhibit sensitive dependence on initial conditions. When we set to oscillate two pendulums with exactly the same parameters but slightly different initial conditions, the difference between their respective values of θ and $\dot{\theta}$ are observed to decrease exponentially over time, and after a transient decay, their oscillations are indistinguishable. Thus the predictions of such systems is nearly impossible.

References

[1]. Cook P. A., *Nonlinear Dynamical Systems*, Prentice-Hall International (UK) Ltd.1986.

[2]. J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, *On Devaney's Definition of Chaos*, American Mathematical Monthly, Vol. 99,No.4 (Apr.,1992), 332-334.

[3]. Kathleen T Alligood, Tim D. Sauer, James A. Yorke, *Chaos an Introduction to Dynamical Systems* (Springer-Verlag New York, Inc.)

[4]. Devaney, Robert L., *An introduction to Chaotic Dynamical System* (Cambridge, M A : Persuse Books Publishing, 1988).

[5]. Edward R. Scheinerman, *Invitation to Dynamical Systems*.

[6]. Denny Gulick, *Encounters with Chaos,* Mc-Graw Hill, Inc., 1992.

[7]. S. Wiggins, *Introduction to Applied Nonlinear Dynamical System and Chaos*, Springer - Verlang New York, 2003.

[8]. Mr. Kulkarni P. R., Dr. Borkar V. C., *Chaos in the Dynamics of the Family of Mappings* $f_c(x) = x^2 - x + c$, IOSR Journal of Mathematics(IOSR-JM) Volume 10, Issue 4, Version IV (Jul.-Aug. 2014), pp 108-116.

[9]. Kulkarni P. R. and Borkar V. C., *Topological conjugacy and the Chaotic Nature of the Family of Mappings* $f_c(x) = x^2 - x + c$, International Journal of Scientific and Innovative Mathematical Research (IJSIMR), Volume 2, Issue 11, November 2014 PP 868-875.

[10]. George D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, Volulme IX.

[11]. Karl-Hienz Beeker, Michael Dorfer, *Dynamical Systems and Fractals*, Cambridge University Press, Cambridge.

[12]. Lawrence Perko, *Differential Equations and Dynamical Systems*, Third Edition, Springer-Verlag, New York Inc.

[13]. Stevan H. Strogatz, *Non-linear Dynamics and Chaos*, Perseus Books Publishing, LLC.

[14]. Garnett P. Williams, *Chaos Theory Tamed*, Joseph Henry Press, Washington D.C.,1997.

[15]. Marian Gidea, Constantin P. Niculescu, *Chaotic Dynamical Systems an Introduction*, Craiova Universitaria Press, 2002.

[16]. Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* American Mathematical Society.

[17]. Frank C. Hoppensteadt, *Analysis and Simulation of Chaotic Systems,* Second Edition, Springer-Verlag, New York Inc.

[18]. Morris W. Hirsch, Stephan Smale, Robert L. Devaney, *Differential Equations, Dynamical Systems and An Introduction to Chaos,* Second Edition, Elsevier Academic Press, 2004.

[19]. Stephan Lynch, *Dynamical Systems with Applications using MATLAB,* Second Edition, Springer International Publishing, Switzerland, 2004,2014.

[20]. John R. Taylor, *Classical Mechanics*, IInd Edition, University Science Books, 2005.

AUTHORS' BIOGRAPHY

Kulkarni P. R. is working as an assistant professor at the department of Mathematics, N. E. S. Science college, Nanded (Maharashtra). He is a research student working in the Swami Ramanand Teerth Marathwada University, Nanded. His area of research is dynamical systems and it's applications in various fields.

Dr. V. C. Borkar, working as Associate Professor and Head in Department of Mathematics and Statistics Yeshwant Mahavidyalaya Nanded , Under Swami Ramanand Teerth Marathwada University, Nanded (M.S) India. His area of specialization is functional Analysis. He has near about 16 year research experiences. He completed one research project on Dynamical system and their applications. The project was sponsored by UGC, New Delhi.