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Inequalities of Dunkl-Williams and Mercer Type in Quasi-Normed Space

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Abstract: *Inequalities of Dunkl-Williams, Mercer, Pečarić-Rajić and likewise the strictly triangle inequalities are of particular interest in theory of normed spaces. In [2] and [3], are proven the analogous inequalities of the strictly inequalities and the inequalities of Pečarić-Rajić type in the quasi-normed spaces. In this paper will be considered inequalities, which are analogous to the inequalities of Dunkl-Williams and Mercer type in quasi-normed space.*

Keywords: *quasi-norm, p-norm, Dunkl-Williams inequality*

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1. Introduction

The quasi-norm is a generalization of a norm and is defined as following.

Definition 1 ([1], [5]). Let L be a real vector space. A quasi-norm is a real function $\|\cdot\|: L \to \mathbf{R}$ such that it satisfies the following conditions:

i) $\|x\| \geq 0$, for each $x \in L$ and $\|x\| = 0$ if and only if $x = 0$,

ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for each $\lambda \in \mathbf{R}$ and for each $x \in L$,

iii) It exists a constant $C \geq 1$ such that $\|x+y\| \leq C(\|x\| + \|y\|)$, for all $x, y \in L$.

The ordered pair $(L, \|\cdot\|)$ is said to be a quasi-normed space. The smallest possible C as in condition *iii*) is said to be a modulus of concavity of $\|\cdot\|$. The complete quasi-normed space is said to be a quasi-Banach space.

Definition 2 ([1], [5]). A quasi-norm $\|\cdot\|$ is said to be a $p-norm$, $0 < p \leq 1$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p, \tag{1}$$

for all $x, y \in L$. In this case a quasi-normed space is used to be said as $p-$*normed space* and quasi-Banach space is used to be said as $p-$*Banach space*.

In quasi-normed space for the quasi-norms and the $p-$ norms holds true the following theorem. This theorem actually enable rather than quasi-norms to deal with $p-$norms, which is easier in many cases.

Theorem 1 (Aoki-Rolewitz, [1], [5]). Let $(L, \|\cdot\|)$ be a quasi-normed space. Then, there exist $p, 0 < p \leq 1$ and an equivalent quasi-norm $|||\cdot|||$ of L, which is $p-$norm.

2. Main Results

Theorem 2. Let L be a quasi-normed space with modulus of concavity $C \geq 1$. Then for all non-null vectors $x, y \in L$ the following holds true

$$\| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \| \leq 4C \tfrac{\|x-y\|}{\|x\|+\|y\|} + 2(C-1) \tfrac{\max\{\|x\|,\|y\|\}}{\|x\|+\|y\|} . \tag{2}$$

Proof. The definition 1 implies that for all non-null vectors $x, y \in L$ it holds true that

$$\begin{aligned} \| x \| \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \| &= \| x \| \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|x\|} + \tfrac{y}{\|x\|} - \tfrac{y}{\|y\|} \| \\ &\leq C \| x \| \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|x\|} \| + C \| x \| \cdot \| \tfrac{y}{\|x\|} - \tfrac{y}{\|y\|} \| \\ &\leq C \| x - y \| + C \, | \| y \| - \| x \| | . \end{aligned} \tag{3}$$

Further, since Definition 1 we have that

$$\| y \| \leq C \| y - x \| + C \| x \| \text{ and } \| x \| \leq C \| x - y \| + C \| y \| ,$$

which imply the following inequalities

$$\| y \| - \| x \| \leq C \| y - x \| + (C-1) \| x \| \leq C \| x - y \| + (C-1) \max\{\| x \|, \| y \|\} \text{ and}$$

$$\| x \| - \| y \| \leq C \| x - y \| + (C-1) \| y \| \leq C \| x - y \| + (C-1) \max\{\| x \|, \| y \|\} ,$$

i.e. the inequality

$$| \| y \| - \| x \| | \leq C \| x - y \| + (C-1) \max\{\| x \|, \| y \|\} . \tag{4}$$

Now, the inequalities (3) and (4) imply the inequality

$$\| x \| \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \| \leq 2C \| x - y \| + (C-1) \max\{\| x \|, \| y \|\} . \tag{5}$$

Analogously, can be proven the following

$$\| y \| \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \| \leq 2C \| x - y \| + (C-1) \max\{\| x \|, \| y \|\} . \tag{6}$$

Finally, if we summarize the inequalities (5) and (6) and the obtained inequality divide by $\| x \| + \| y \| > 0$ we get the inequality (2).

Theorem 3. Let L be a $p-$normed space, $0 < p \leq 1$. Then for all non-null vectors $x, y \in L$ holds true that

$$\| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \|^p \leq 2 \tfrac{\|x-y\|^p + | \|y\| - \|x\| |^p}{\|x\|^p + \|y\|^p} . \tag{7}$$

Proof. The definition 2, i.e. the properties of $p-$norm imply that for all non-null vectors $x, y \in L$ it hold true that

$$\begin{aligned} \| x \|^p \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \|^p &= \| x \|^p \| \tfrac{x}{\|x\|} - \tfrac{y}{\|x\|} + \tfrac{y}{\|x\|} - \tfrac{y}{\|y\|} \|^p \\ &\leq \| x \|^p \| \tfrac{x}{\|x\|} - \tfrac{y}{\|x\|} \|^p + \| x \|^p \| \tfrac{y}{\|x\|} - \tfrac{y}{\|y\|} \|^p \\ &\leq \| x - y \|^p + | \| y \| - \| x \| |^p \end{aligned} \tag{8}$$

and

$$\begin{aligned} \| y \|^p \cdot \| \tfrac{x}{\|x\|} - \tfrac{y}{\|y\|} \|^p &= \| y \|^p \| \tfrac{x}{\|x\|} - \tfrac{x}{\|y\|} + \tfrac{x}{\|y\|} - \tfrac{y}{\|y\|} \|^p \\ &\leq \| y \|^p \| \tfrac{x}{\|x\|} - \tfrac{x}{\|y\|} \|^p + \| y \|^p \| \tfrac{x}{\|y\|} - \tfrac{y}{\|y\|} \|^p \\ &\leq \| x - y \|^p + | \| y \| - \| x \| |^p . \end{aligned} \tag{9}$$

Finally, if we summarize the inequalities (8) and (9) and the obtained inequality divide by $\| x \|^p + \| y \|^p > 0$, we get the inequality (7).

Remark 1. The inequalities (2) and (7) are actually inequalities of Dunkl-Williams type in quasi-normed and $p-$normed space, $0<p\leq 1$, respectively.

Theorem 4. Let L be a quasi-normed space with modulus of concavity $C\geq 1$. The following statements are equivalent:

1) For all non-null vectors $x,y\in L$ it is true that

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leq 2C\frac{\|x-y\|}{\|x\|+\|y\|}+(C-1)\frac{\max\{\|x\|,\|y\|\}}{\|x\|+\|y\|}. \quad (10)$$

2) If $x,y\in L$ are such that $\|x\|=\|y\|=1$, then

$$\left\|\frac{x+y}{2}\right\|\leq C\|(1-t)x+ty\|+\frac{C-1}{2}\max\{1-t,t\}, \quad (11)$$

for each $t\in[0,1]$.

Proof. $1)\Rightarrow 2)$. Let assume that the statement 1) holds true. Let $x,y\in L$ be such that $\|x\|=\|y\|=1$. Clearly, for $t=0$ and $t=1$, the inequality (11) holds true. If $t\in(0,1)$, then 1) implies the following

$$\begin{aligned}
\left\|\frac{x+y}{2}\right\| &= \frac{1-t}{2}\left(1+\frac{t}{1-t}\right)\|x+y\| \\
&= \frac{1-t}{2}\left(\|x\|+\left\|\frac{t}{1-t}y\right\|\right)\left\|\frac{x}{\|x\|}-\frac{\frac{t}{t-1}y}{\left\|\frac{t}{t-1}y\right\|}\right\| \\
&= \frac{1-t}{2}\left(\|x\|+\left\|\frac{t}{1-t}y\right\|\right)\left(2C\frac{\left\|x-\frac{t}{t-1}y\right\|}{\|x\|+\left\|\frac{t}{t-1}y\right\|}+(C-1)\frac{\max\{\|x\|,\left\|\frac{t}{t-1}y\right\|\}}{\|x\|+\left\|\frac{t}{t-1}y\right\|}\right) \\
&= C(1-t)\left\|x-\frac{t}{t-1}y\right\|+\frac{(C-1)(1-t)}{2}\max\{1,\frac{t}{1-t}\} \\
&= C\|(1-t)x+ty\|+\frac{C-1}{2}\max\{1-t,t\},
\end{aligned}$$

i.e. the inequality (11) holds true.

$2)\Rightarrow 1)$. Let assume that the statement 2) holds true. Let x and y be arbitrary non-null vectors in L. Then, for $\frac{x}{\|x\|},\frac{-y}{\|y\|}\in L$ holds true that $\left\|\frac{x}{\|x\|}\right\|=\left\|\frac{-y}{\|-y\|}\right\|=1$ and if we take that $t=\frac{\|y\|}{\|x\|+\|y\|}$, then by 2) we get that

$$\begin{aligned}
\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &= 2\left\|\frac{\frac{x}{\|x\|}+\frac{-y}{\|y\|}}{2}\right\| \\
&\leq 2\left(C\left\|\left(1-\frac{\|y\|}{\|x\|+\|y\|}\right)\frac{x}{\|x\|}+\frac{\|y\|}{\|x\|+\|y\|}\cdot\frac{-y}{\|y\|}\right\|+\frac{C-1}{2}\max\{1-\frac{\|y\|}{\|x\|+\|y\|},\frac{\|y\|}{\|x\|+\|y\|}\}\right. \\
&= 2C\frac{\|x-y\|}{\|x\|+\|y\|}+(C-1)\frac{\max\{\|x\|,\|y\|\}}{\|x\|+\|y\|}
\end{aligned}$$

i.e. the inequality (8) holds true.

3. Conclusion

The inequality (10) is actually generalization of Mercer inequality $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leq\frac{2\|x-y\|}{\|x\|+\|y\|}$. Which in normed space is satisfied if and only if the norm is generated by a scalar product. Thus, it is logically to wonder:

Is the inequality (10) *into a quasi-normed space with modulus of concavity* $C\geq 1$ *satisfied if and only if there exists a function* $f:L\times L\to\mathbf{R}$ *so that* $f(x,x)=\|x\|^2$.

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