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On Functions A_f^S, G_f^S and H_f^S

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Abstract: *In this paper we introduce the functions* A_f^S, G_f^S *and* H_f^S *for any multiplicative function* f *and for any regular convolution S and obtain a relation between them.*

1. Introduction

For any arithmetic function f let functions A_f, G_f and H_f be

(1.1) $$A_f(n) = \frac{1}{\tau(n)}\sum_{d|n} f(d)$$

(1.2) $$G_f(n) = \left[\prod_{d|n} f(d)\right]^{\frac{1}{\tau(n)}}$$

and

(1.3) $$\frac{1}{H_f(n)} = \frac{1}{\tau(n)}\sum_{d|n}\frac{1}{f(d)},$$

where $\tau(n)$ is the number of positive divisors of n.

Note that the function $A_f(n)$, $G_f(n)$ and $H_f(n)$ are respectively the arithmetic mean, geometric mean and harmonic mean of the function values of f at various positive divisors of n.

In 1974 A. C. Vasu [2] has considered these functions and proved the following

(1.4) If f is multiplicative so are $A_f(n)$, $G_f(n)$ and $H_f(n)$

(1.5) If f is completely multiplicative then
$G_f^2(n) = A_f(n)\cdot H_f(n) = f(n)$

Let F be the set of all arithmetic functions

In this paper we introduce functions A_f^S, G_f^S and H_f^S for any $f \in F$ corresponding to regular divisors of S_n of n (defined below) and establish results which generalize (1.4) and (1.5)

2. Preliminaries

Let F be the class of all arithmetic functions. For any positive integer n let S_n denote a set of positive divisors of n. For $f, g \in F$ their *S-product or S-convolution* $f\,\overline{S}\,g$, is defined by

$$\left(f\,\overline{S}\,g\right)(n) = \sum_{d \in S_n} f(d) g\left(\frac{n}{d}\right)$$

where the sum is over the divisors $d \in S_n$.

2.1 The S-product is said to be *regular* if it satisfies the following conditions

(i) $(F, +, \overline{S})$ is a commutative ring with unity

(ii) $f\,\overline{S}\,g$ is multiplicative whenever f and g are.

(iii) The arithmetic function $u(n) = 1$ for all *n* has inverse $\mu_s \in F$ relative to $\overline{S}$ (that is, $u\,\overline{S}\,\mu_s = \varepsilon$) and $\mu_s(n) = 0$ or -1 when n is a prime power. μ_s is called the *S-analogue of the Mobius function* μ.

NARKIEWICZ [1] has characterized regular convolutions as follows:

2.2 Theorem. A S-convolution is regular if and only if the sets S_n have the following properties:

(i) $d \in S_m,\, m \in S_n \Leftrightarrow d \in S_n,\, \dfrac{m}{d} \in S_{\frac{n}{d}}$

(ii) $d \in S_n \Rightarrow \dfrac{n}{d} \in S_n$

(iii) $\{1, n\} \subseteq S_n$ for every n

(iv) $S_{mn} = S_m S_n = \{ab : a \in S_m, b \in S_n\}$ whenever $\gcd(m,n) = 1$

(v) For every prime power p^α we have

$S_{p^\alpha} = \{1, p^t, p^{2t}, ..., p^{rt}\}$, $rt = \alpha$ for some positive integer *t* and $p^t \in S_{p^{2t}}$, $p^{2t} \in S_{p^{3t}}$,....

2.3 Definition. If $\overline{S}$ is a regular convolution the elements of S_n will be called *regular divisors of n.*The number of S- divisors of n is denoted by $\tau_s(n)$.

Since the Dirichlet convolution and the unitary convolution are both regular, the elements of D_n (the set of all positive divisors of *n*) and U_n (the set of all unitary divisors of *n*) are regular divisors of *n.*

(2.4): For any prime power p^α, the least positive integer *t* such that $p^t \in S_{p^\alpha}$ is called the *type of* p^α *relative to S* and is denoted by $t_S(p^\alpha)$

If $t = t_S(p^\alpha)$ it follows from the Theorem 2.2 (v) that $t \mid a$ whenever $p^a \in S_{p^\alpha}$. Clearly we have

(2.5) $$\tau_S(n) = \sum_{d \in Sn} 1$$

(2.6) $\tau_S = u\overline{S}u$ where *u* is as in Definition 2.1 (iii) and since *u* is multiplicative, it follows from 2.1 (ii) that τ_S is multiplicative.

Also

(2.7) $$\tau_S(p^\alpha) = \frac{\alpha}{t_S(p^\alpha)} + 1$$ for any prime power p^α.

3. MAIN RESULTS

Suppose S_n is a set of regular divisors of n. For any arithmetic function f we define

A_f^S, G_f^S and H_f^S by

(3.1) $$A_f^S(n)=\frac{1}{\tau_s(n)}\sum_{d\in S_n} f(d)$$

(3.2) $$G_f^S(n)=\left[\prod_{d\in S_n} f(d)\right]^{\frac{1}{\tau_s(n)}}$$

And whenever f is nowhere zero

(3.3) $$\frac{1}{H_f^S(n)}=\frac{1}{\tau_s(n)}\sum_{d\in S_n}\frac{1}{f(d)}$$

Note that $A_f^D(n)=A_f(n)$, $G_f^D(n)=G_f(n)$ and $H_f^D(n)=H_f(n)$ where D_n denotes the set of all positive divisors of *n*.

3.4 Theorem: If f is multiplicative then A_f^S , G_f^S and H_f^S are all multiplicative.

Proof: By (2.6), τ_S is multiplicative. Again if f is multiplicative it follows by (2.1)(ii), that $f\bar{S}u$ is also multiplicative. Therefore $A_f^S(n)=\frac{1}{\tau_S(n)}(f\bar{S}u)(n)$ is also multiplicative. To prove the multiplicativity of G_f^S , note that for any n with $n = p_1^{a_1}.\, p_2^{a_2} \ldots p_r^{a_r}$ we have

(3.5) $$G_f^S(n)=\left[\prod_{d\in S_{p_1^{a_1}.p_2^{a_2}\ldots p_r^{a_r}}} f(d)\right]^{\frac{1}{\tau_s\left(p_1^{a_1}.\,p_2^{a_2}\ldots p_r^{a_r}\right)}}$$

Now using 2.2 (iii) each $d\in S_{p_1^{a_1}.\,p_2^{a_2}\ldots p_r^{a_r}}$ can be written uniquely as $d=d_1d_2...d_r$ where $d_i\in S_{p_i^{ai}}$ for $i=1,2,3,...,r$ and $(d_i,d_j)=1$ for $i\neq j$. Therefore (3.5) can be written as

$$G_f^S(n)=\left[\prod_{i=1}^r\prod_{d_i\in S_{p_i^{\alpha_i}}} f(d_i)\right]^{\frac{1}{\tau_s\left(p_1^{a_1}\right)\cdots\tau_s\left(p_r^{a_r}\right)}}$$

$$=\prod_{i=1}^{r} G_f^S\left(p_i^{a_i}\right)$$

Thus $G_f^S(n)$ is multiplicative

Observe that $H_f^S(n)$ can be defined only for the function f which are nowhere zero. If f is nowhere zero and multiplicative then so is $\frac{1}{f}$. Hence $A_{\frac{1}{f}}^S=H_f^S$.

Therefore H_f^S is multiplicative

3.6 Theorem: If f is completely multiplicative then

(3.7) $$\left[G_f^S(n)\right]^2=A_f^S(n)\cdot H_f^S(n)=f(n)\text{ for all } n.$$

Proof: By Theorem 3.4 either side of (3.7) is multiplicative and therefore it is enough to verify the identity (3.7) in the case $n=p^{\alpha}$.

By (3.3) and the complete mulplicativity of $f(n)$ and since a S-convolution is regular if and only if the set S_n have the following the property, $d \in S_n \Rightarrow \frac{n}{d} \in S_n$

We have

$$\frac{1}{H_f^S(p^\alpha)} = \frac{1}{\tau_s(p^\alpha)} \sum_{d \in S_{p^\alpha}} \frac{1}{f(d)}$$

$$= \frac{1}{\tau_s(p^\alpha)} \sum_{d \in S_{p^\alpha}} \frac{f\left(\frac{p^\alpha}{d}\right)}{f(p^\alpha)}$$

$$= \frac{1}{f(p^\alpha)} \frac{1}{\tau_s(p^\alpha)} \sum_{d \in S_{p^\alpha}} f\left(\frac{p^\alpha}{d}\right)$$

$$= \frac{1}{f(p^\alpha)} \frac{1}{\tau_s(p^\alpha)} \sum_{d \in S_{p^\alpha}} f(d)$$

$$= \frac{1}{f(p^\alpha)} A_f^S(p^\alpha)$$

Thus

(3.8) $$A_f^S(p^\alpha) \cdot H_f^S(p^\alpha) = f(p^\alpha)$$

which gives the second part of the identity (3.7)

Again

$$G_f^S(p^\alpha) = \left[\prod_{d \in S_{p^\alpha}} f(d)\right]^{\frac{1}{\tau_S(p^\alpha)}}$$

$$= \left[\prod_{\substack{0 \le \beta \le \alpha \\ p^\beta \in S_{p^\alpha}}} f(p)^\beta\right]^{\frac{1}{\tau_S(p^\alpha)}},$$

which gives

(3.9) $$G_f^S(p^\alpha) = \left[\{f(p)\}^{\sum_{\substack{0 \le \beta \le \alpha \\ p^\beta \in S_{p^\alpha}}} \beta}\right]^{\frac{1}{\tau_S(p^\alpha)}}$$

But we have $S_{p^\alpha} = \{1, p^t, p^{2t}, ..., p^{rt}\}$ where $rt = \alpha$ and $t = t_S(p^\alpha)$

Therefore by (3.4) we get

(3.10) $$\sum_{\substack{0 \le \beta \le \alpha \\ p^\beta \in S_{p^\alpha}}} \beta = 0 + t + 2t + ... + rt$$

$$= r\left(\frac{r+1}{2}\right)t$$

$$= \frac{\alpha}{2}\left(\frac{\alpha}{t} + 1\right)$$

$$= \frac{\alpha}{2} \tau_S\left(p^\alpha\right)$$

From (3.9) and (3.10) we get

$$G_f^S\left(p^\alpha\right) = \left[f\left(p\right)^{\frac{\alpha}{2}\tau_S\left(p^\alpha\right)} \right]^{\frac{1}{\tau_S\left(p^\alpha\right)}}$$

$$= \left[f\left(p\right)\right]^{\frac{\alpha}{2}}$$

which gives (3.7)

3.11 Remark: In the case $S_n = D_n$, Theorem 3.4 and Theorem 3.6 respectively give (1.4) and (1.5)

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