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# Modeling and Analysis of Optimum Step-Stress Life Testing for Accelerated Life Testing Plans

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**Abstract:** *This paper studies Step Stress Accelerated Life Tests (SSALT). It is assumed that the lifetimes of test units follow a Frechet distribution. The experiment is subjected to two types of relationship between lifetimes and stress (linear and Quadratic). The scale parameter of the baseline distribution at a constant stress level is assumed to have log-linear and Quadratic relationship with stress and a Khamis-Higgins model holds. Numerical examples are presented to illustrate all the methods of inference developed here and a comparison of the maximum likelihood estimator's for different sample sizes is shown .The optimum test plan specifies the optimal stress switching point which is determined by minimizing the generalized asymptotic variance of the MLEs for the model parameters. Tables of optimum times of changing stress level for both plans are also obtained.*

**Keywords:** *Accelerated life testing, Step-stress, Inverse Weibull distribution, Life stress relationship (Linear and Quadratic), Maximum likelihood, Asymptotic variance (AV), Optimal time, Confidence interval*

## 1. INTRODUCTION

Introducing a new product in market require decisions regarding reliability of the product or service. Accelerated testing might be the recommended or required approach. Accelerated testing is an approach for obtaining more information from a given test time than would normally be possible. It does this by subjecting a test unit to more severe conditions than normal in order to obtain failure modes quickly and shorten the testing period. Since higher stresses are used, accelerated testing must be approached with caution to avoid introducing failure modes that will not be encountered in normal use. The design of accelerated studies may include elevated temperature, high or low humidity, intense light, or as appropriate.

More specifically Quantitative Accelerated life test (QALT) includes accelerated time and accelerated stress. Various type of stresses include, the constant-stress ALT with an initial low stress, some of the units may survive too long because the stress is kept at a constant level throughout the life of test units. The second type is called step-stress ALT. Instead of holding the stress at a constant level, this method changes the stress setting at specified times on the surviving units. The third type is the progressive stress ALT. The stress is increased continuously (usually increased linearly) during the test.

In quantitative accelerated life testing, the engineer is interested in predicting the life of the product at normal use conditions. Nelson [1], Meeker and Escobar [2] and Nelson [3] provide a significant review of the literature on how to develop optimum QALT plans. Many authors also have provided the studies for statistical inference model for SSALT based on Cumulative Exposure Model (CEM); e.g., see Xiong [4], Watkins [5], Zhao and Elsayed [6], Balakrishnan el al. [7], Yeo and Tang [8], Xiong and Ji [9] and Xiong and Milliken [10]. Khamis and Higgins [11] proposed a new model for SSALT as an alternative to the CEM, which is based on a time transformation of the exponential CEM.

## 2. PROPOSED MODEL AND BASIC ASSUMPTIONS

The following assumptions are made:

a) There are three stress levels $S_1$, $S_2$ and $S_3$($S_3$> $S_2$> $S_1$).

b) Under any stress, the lifetime of test units follow a Frechet distribution.

c) The scale parameter $\theta_i$ at stress level i, i=1, 2, 3 is a function of stress given by (1) or (2)

$$\log(\theta_i)=\beta_0+\beta_1 S_i \qquad (1)$$

$$\log(\theta_i)=\beta_0+\beta_1 S_i+\beta_2 S_i^2 \qquad (2)$$

where, $\beta_0$ and $\beta_1$<0 are unknown parameters which is estimated by the data.

d) Failures occurs according to a cumulative exposure model by Nelson[1].

Principle of cumulative exposure model, is that, the remaining life of test items depends only on the current cumulative fraction failed and current stress regardless of how the fraction accumulated.

According to Cumulative Exposure-Model [1] the CDF in SSALT for 3-step is given by:

$$F(t)=\begin{cases} F_1(t) & 0 \le t < \tau_1 \\ F_2(t-\tau_1+s_1) & \tau_1 \le t < \tau_2 \\ F_3(t-\tau+s_2) & \tau_2 \le t < \infty \end{cases}$$

$s_0=\tau_0=0$;

$s_i(i>0)$ is the solution of:

$F_{i+1}(s_i)= F_i(\tau_i- \tau_{i-1})$, for i= 1,….,k-1.

The procedure of SSALT for 3-step is as follows:

A random sample of n identical products is placed on test under initial stress level $S_1$ and run

until time $\tau_1$ , and then the stress is increased to $S_2$ for items that have not fail till time reaches $\tau_2$ and then the final stress $S_3$ is given and the test continued until all products fail.

## 3. OPTIMUM QUADRATIC STEP-STRESS TEST

Cumulative Exposure-Model for 3-step is given by:

$$F(t)=\begin{cases} F_1(t) & 0 \le t < \tau_1 \\ F_2(t-\tau_1+s_1) & \tau_1 \le t < \tau_2 \\ F_3(t-\tau_2+s_2) & \tau_2 \le t < \infty \end{cases}$$

$$F(t)=\begin{cases} \exp\left(-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right) & 0 \le t < \tau_1 \\ \exp\left(-\left(\frac{t}{\theta_2}\right)^{-\alpha}\right)\exp\left\{-\left[\left(\frac{\tau_1}{\theta_1}\right)^{-\alpha}-\left(\frac{\tau_1}{\theta_2}\right)^{-\alpha}\right]\right\} & \tau_1 \le t < \tau_2 \\ \exp\left(-\left(\frac{t}{\theta_3}\right)^{-\alpha}\right)\exp\left\{-\left\{\left[\left(\frac{\tau_2}{\theta_2}\right)^{-\alpha}-\left(\frac{\tau_2}{\theta_3}\right)^{-\alpha}\right]-\left[\left(\frac{\tau_1}{\theta_2}\right)^{-\alpha}-\left(\frac{\tau_1}{\theta_3}\right)^{-\alpha}\right]\right\}\right\} & \tau_2 \le t < \infty \end{cases}$$

Hence,

$s_1 \equiv$solution of $F_2(s_1) = F_1(\tau_1)$

On solving for $s_1$, we get, $s_1=\tau_1\left(\frac{\theta_2}{\theta_1}\right)$

& $s_2 \equiv$solution of $F_3(s_2) = F_2(\tau_2- \tau_1+s_1)$

On solving for $s_2$, we get, $s_2 = \tau_1\left(\frac{\theta_3}{\theta_1}\right) + \frac{\theta_3}{\theta_2}\left(\tau_2 - \tau_1\right)$

Hence Probability density function for Cumulative Exposure model is given by:

$$
f(t) = \begin{cases}
\alpha\theta_1^{\alpha} t^{-\alpha-1} \exp\left(-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right) & 0 \le t < \tau_1 \\
\alpha\theta_2^{\alpha} t^{-\alpha-1} \exp\left(-\left(\frac{t}{\theta_2}\right)^{-\alpha}\right) \exp\left\{-\left[\left(\frac{\tau_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\tau_1}{\theta_2}\right)^{-\alpha}\right]\right\} & \tau_1 \le t < \tau_2 \\
\alpha\theta_3^{\alpha} t^{-\alpha-1} \exp\left(-\left(\frac{t}{\theta_3}\right)^{-\alpha}\right) \exp\left\{-\left\{\left[\left(\frac{\tau_2}{\theta_2}\right)^{-\alpha} - \left(\frac{\tau_2}{\theta_3}\right)^{-\alpha}\right] - \left[\left(\frac{\tau_1}{\theta_2}\right)^{-\alpha} - \left(\frac{\tau_1}{\theta_3}\right)^{-\alpha}\right]\right\}\right\} & \tau_2 \le t < \infty
\end{cases}
$$

The log-likelihood equation is given by:

$$
\begin{aligned}
\log l = {} & \sum_{j=1}^{n_1}\left[\log\alpha + \alpha(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2) - (\alpha+1)\log(t_{1j}) - t_{1j}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right)\right] + \sum_{j=1}^{n_2}\left[\log\alpha + \alpha(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2) - (\alpha+1)\log(t_{2j})\right] + \\
& \sum_{j=1}^{n_2}\left[-t_{2j}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) - \tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) + \tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right)\right] + \sum_{j=1}^{n_2}\left[\log\alpha + \alpha(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2) - (\alpha+1)\log(t_{3j})\right] \\
& \sum_{j=1}^{n_3}\left[-t_{3j}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right) - \tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) + \tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)\right] + n_3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) \\
& - n_3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log l}{\partial \beta_0} = {} & n\alpha - \alpha\sum_i\sum_{j=1}^{n_i} t_{ij}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_i + \beta_2 S_i^2\right)\right) - n_2\alpha\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) \\
& + n_2\alpha\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) - n_3\alpha\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) + n_3\alpha\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right) \\
& + n_3\alpha\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) - n_3\alpha\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\frac{\partial \log l}{\partial \beta_1} = {} & \sum_{j=1}^{n_1}\left[\alpha S_1\left(1 - t_{1j}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right)\right)\right] + \sum_{j=1}^{n_2}\left[\alpha S_2 - \alpha S_2(t_{2j}^{-\alpha} - \tau_1^{-\alpha})\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right)\right] \\
& - n_2\alpha S_1\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) + \sum_{j=1}^{n_3}\left[\alpha S_3 - \alpha S_3(t_{3j}^{-\alpha} + \tau_1^{-\alpha} - \tau_2^{-\alpha})\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)\right] \\
& - n_3\alpha S_2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) + n_3\alpha S_2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) = 0
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\frac{\partial \log l}{\partial \beta_2} = {} & \sum_{j=1}^{n_1}\left[\alpha S_1^2\left(1 - t_{1j}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right)\right)\right] + \sum_{j=1}^{n_2}\left[\alpha S_2^2 - \alpha S_2^2(t_{2j}^{-\alpha} - \tau_1^{-\alpha})\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right)\right] \\
& - n_2\alpha S_1^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) + \sum_{j=1}^{n_3}\left[\alpha S_3^2 - \alpha S_3^2(t_{3j}^{-\alpha} + \tau_1^{-\alpha} - \tau_2^{-\alpha})\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)\right] \\
& - n_3\alpha S_2^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) + n_3\alpha S_2^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) = 0
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{\partial^2 \log l}{\partial \beta_0^2} = {} & -\alpha^2\sum_i\sum_{j=1}^{n_i} t_{ij}^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) - n_2\alpha^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_1 + \beta_2 S_1^2\right)\right) + n_2\alpha^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) \\
& - n_3\alpha^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right) + n_3\alpha^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right) + n_3\alpha^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_2 + \beta_2 S_2^2\right)\right) \\
& - n_3\alpha^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0 + \beta_1 S_3 + \beta_2 S_3^2\right)\right)
\end{aligned} \tag{6}
$$

$$\frac{\partial^2 \log l}{\partial \beta_1^2}=\sum_{j=1}^{n_1}\left[-\alpha^2 S_1^2 t_{1j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)\right]+\sum_{j=1}^{n_2}\left[-\alpha^2 S_2^2 t_{2j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-\alpha^2 S_1^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)\right]$$
$$+n_2\alpha^2 S_2^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)+\sum_{j=1}^{n_3}\left[-\alpha^2 S_3^2 t_{3j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)-\alpha^2 S_2^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)\right]$$
$$+n_3\alpha^2 S_3^2\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)\left[\tau_2^{-\alpha}-\tau_1^{-\alpha}\right]+\alpha^2 S_2^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right) \tag{7}$$

$$\frac{\partial^2 \log l}{\partial \beta_2^2}=\sum_{j=1}^{n_1}\left[-\alpha^2 S_1^4 t_{1j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)\right]+\sum_{j=1}^{n_2}\left[-\alpha^2 S_2^4 t_{2j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-\alpha^2 S_1^4\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)\right]$$
$$+n_2\alpha^2 S_2^4\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)+\sum_{j=1}^{n_3}\left[-\alpha^2 S_3^4 t_{3j}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)-\alpha^2 S_2^4\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)\right]$$
$$+n_3\alpha^2 S_3^4\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)\left[\tau_2^{-\alpha}-\tau_1^{-\alpha}\right]+\alpha^2 S_2^4\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right) \tag{8}$$

$$\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_0}==\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_1}-\alpha^2\sum_i\sum_{j=1}^{n_i}S_i t_{ij}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_i+\beta_2 S_i^2\right)\right)-n_2\alpha^2 S_1\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)$$
$$+n_2\alpha^2 S_2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)+n_3\alpha^2 S_3\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)$$
$$+n_3\alpha^2 S_2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right) \tag{9}$$

$$\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_0}==\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_2}-\alpha^2\sum_i\sum_{j=1}^{n_i}S_i^2 t_{ij}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_i+\beta_2 S_i^2\right)\right)-n_2\alpha^2 S_1^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)$$
$$+n_2\alpha^2 S_2^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_2^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)+n_3\alpha^2 S_3^2\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)$$
$$+n_3\alpha^2 S_2^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_3^2\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right) \tag{10}$$

$$\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_1}==\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_2}-\alpha^2\sum_i\sum_{j=1}^{n_i}S_i^{\,3} t_{ij}^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_i+\beta_2 S_i^2\right)\right)-n_2\alpha^2 S_1^3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_1+\beta_2 S_1^2\right)\right)$$
$$+n_2\alpha^2 S_2^3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_2^3\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)+n_3\alpha^2 S_3^3\tau_2^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right)$$
$$+n_3\alpha^2 S_2^3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_2+\beta_2 S_2^2\right)\right)-n_3\alpha^2 S_3^3\tau_1^{-\alpha}\exp\left(\alpha\left(\beta_0+\beta_1 S_3+\beta_2 S_3^2\right)\right) \tag{11}$$

Equation (6) - (11) is the elements of Observed Fisher Information matrix.

The Expected Fisher Information matrix is given by:

$$I=n\begin{bmatrix} E\left[-\frac{\partial^2 \log l}{\partial \beta_0^2}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_0}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_2}\right] \\ E\left[-\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_1}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_1^2}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_2}\right] \\ E\left[-\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_0}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_1}\right] & E\left[-\frac{\partial^2 \log l}{\partial \beta_2^2}\right] \end{bmatrix}$$

$$\left.\begin{aligned}
&E\left[-\frac{\partial^2 \log l}{\partial \beta_0^2}\right]=A=\alpha^2 a\left(L+p_1\right)+\alpha^2 b\left(M-2p_1+p_2\right)+\alpha^2 c\left(N+cp_1-p_2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_1^2}\right]=B=\alpha^2 aS_1^2\left(L+p_1\right)+\alpha^2 bS_2^2\left(M-2p_1+p_2\right)+\alpha^2 cS_3^2\left(N+cp_1-p_2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_2^2}\right]=D=\alpha^2 aS_1^4\left(L+p_1\right)+\alpha^2 bS_2^4\left(M-2p_1+p_2\right)+\alpha^2 cS_3^4\left(N+cp_1-p_2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_0}\right]=E\left[-\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_1}\right]=C=\alpha^2 aS_1\left(L+p_1\right)+\alpha^2 bS_2\left(M-2p_1+p_2\right)+\alpha^2 cS_3\left(N+cp_1-p_2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_0}\right]=E\left[-\frac{\partial^2 \log l}{\partial \beta_0\partial \beta_2}\right]=E=\alpha^2 aS_1^2\left(L+p_1\right)+\alpha^2 bS_2^2\left(M-2p_1+p_2\right)+\alpha^2 cS_3^2\left(N+cp_1-p_2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_2\partial \beta_1}\right]=E\left[-\frac{\partial^2 \log l}{\partial \beta_1\partial \beta_2}\right]=F=\alpha^2 aS_1^3\left(L+p_1\right)+\alpha^2 bS_2^3\left(M-2p_1+p_2\right)+\alpha^2 cS_3^3\left(N+cp_1-p_2\right)
\end{aligned}\right\} \tag{12}$$

where,

$$a=\theta_1^{\alpha}, b=\theta_2^{\alpha}, c=\theta_3^{\alpha}, p_1=\tau_1^{\alpha}, p_2=\tau_2^{\alpha}$$

$$L=(1+p_1a)\frac{\exp(-p_1a)}{a}$$

$$M=\frac{\exp(-p_1a+p_1b)}{b}\{(\exp(-p_2b)+1)(p_2b-p_1b)\}$$

$$N=\frac{\exp\{-[(p_2b-p_2c)-(p_1b-p_1c)]\}}{c}\{1-(p_2c+1)\exp(-p_2c)\}$$

Now, let us suppose that

$$\eta=\alpha^2a(L+p_1)$$

$$\psi=\alpha^2b(M-2p_1+p_2)$$

$$\xi=\alpha^2c(N+cp_1-p_2)$$

So, (12) becomes:

$$\left.\begin{aligned}
&E\left[-\frac{\partial^2\log l}{\partial\beta_0^2}\right]=A=\eta+\psi+\xi\\
&E\left[-\frac{\partial^2\log l}{\partial\beta_1^2}\right]=B=\eta S_1^2+\psi S_2^2+\xi S_3^2\\
&E\left[-\frac{\partial^2\log l}{\partial\beta_2^2}\right]=D=\eta S_1^4+\psi S_2^4+\xi S_3^4\\
&E\left[-\frac{\partial^2\log l}{\partial\beta_1\partial\beta_0}\right]=E\left[-\frac{\partial^2\log l}{\partial\beta_0\partial\beta_1}\right]=C=\eta S_1+\psi S_2+\xi S_3\\
&E\left[-\frac{\partial^2\log l}{\partial\beta_2\partial\beta_0}\right]=E\left[-\frac{\partial^2\log l}{\partial\beta_0\partial\beta_2}\right]=E=\eta S_1^2+\psi S_2^2+\xi S_3^2\\
&E\left[-\frac{\partial^2\log l}{\partial\beta_2\partial\beta_0}\right]=E\left[-\frac{\partial^2\log l}{\partial\beta_0\partial\beta_2}\right]=F=\eta S_1^3+\psi S_2^3+\xi S_3^3
\end{aligned}\right\}\qquad(13)$$

The optimum criterion here is to find the optimum stress change time $\tau_1$ and $\tau_2$. The log of the 100 $p^{th}$ percentile of the lifetime $t_p(S_0)$ at the design stress $S_0$ is given by

$$\hat{\psi}(S_0)=\log(t_p(S_0))=\beta_0+\beta_1S_0+\beta_2S_0^2+\log(\theta_i(\log p)^{-\frac{1}{\alpha}})$$

The asymptotic variance multiplied by the sample size (nAVS) at the design stress $S_0$ is then given by

$$nAV_1=\frac{BD-F^2+2S_0(EF-CD)+S_0^2\left[2\left(B^2-FC\right)+AD+CE-\left(AF+E^2\right)\right]+S_0^4\left(AB-C^2\right)+S_0^3(EC-AF)}{AED+2EF-AF^2-C^2D-E^3}\qquad(14)$$

By differentiating (14) with respect to $\tau_1$ and $\tau_2$ we find their values that minimizes $nAV_1$ and gives the optimal plan.

## 4. Optimum Linear Step-Stress Test

The Expected Fisher Information matrix is given by:

$$I=n\begin{bmatrix}E\left[-\frac{\partial^2\log l}{\partial\beta_0^2}\right] & E\left[-\frac{\partial^2\log l}{\partial\beta_1\partial\beta_0}\right]\\ E\left[-\frac{\partial^2\log l}{\partial\beta_0\partial\beta_1}\right] & E\left[-\frac{\partial^2\log l}{\partial\beta_1^2}\right]\end{bmatrix}=n\begin{bmatrix}A & C\\ C & B\end{bmatrix}$$

$$\left.\begin{aligned}
&E\left[-\frac{\partial^2 \log l}{\partial \beta_0^2}\right] = A = \alpha^2(aL + bM + cN) + \alpha^2 p_1(a - 2b + c) + \alpha^2 p_2(b - c)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_1^2}\right] = B = \alpha^2\left(aLS_1^2 + bMS_2^2 + cNS_3^2\right) + \alpha^2 p_1\left(aS_1^2 - 2bS_2^2 + cS_3^2\right) + \alpha^2 p_2\left(bS_2^2 - cS_3^2\right)\\
&E\left[-\frac{\partial^2 \log l}{\partial \beta_1 \partial \beta_0}\right] = E\left[-\frac{\partial^2 \log l}{\partial \beta_0 \partial \beta_1}\right] = C = \alpha^2(aLS_1 + bMS_2 + cNS_3) + \alpha^2 p_1(aS_1 - 2bS_2 + cS_3) + \alpha^2 p_2(bS_2 - cS_3)
\end{aligned}\right\} \quad (15)$$

where above notations are given by;

$$a = \theta_1^{\alpha}, b = \theta_2^{\alpha}, c = \theta_3^{\alpha}, p_1 = \tau_1^{\alpha}, p_2 = \tau_2^{\alpha}$$

$$L = (1 + p_1 a)\frac{\exp(-p_1 a)}{a}$$

$$M = \frac{\exp(-p_1 a + p_1 b)}{b}\{(\exp(-p_2 b) + 1)(p_2 b - p_1 b)\}$$

$$N = \frac{\exp\{-[(p_2 b - p_2 c) - (p_1 b - p_1 c)]\}}{b}\{1 - (p_2 c + 1)\exp(-p_2 c)\}$$

The log of the 100 $p^{th}$ percentile of the lifetime $t_p(S_0)$ at the design stress $S_0$ is given by

$$\hat{\psi}(S_0) = \log(t_p(S_0)) = \beta_0 + \beta_1 S_0 + \log(\theta_i(\log p)^{-\frac{1}{\alpha}})$$

The AVS is given by:

$$AV_2(\hat{\psi}(S_0)) = \log(t_p(S_0)) = AV(\hat{\beta}_0 + \hat{\beta}_1 S_0 + \log(\theta_i(\log p)^{-\frac{1}{\alpha}})) = KI^{-1}K' = K \Sigma K' \quad (16)$$

where,

$$K = \left[\frac{\partial \hat{\psi}(S_0)}{\partial \hat{\beta}_0} \; \frac{\partial \hat{\psi}(S_0)}{\partial \hat{\beta}_1}\right]$$

and $I^{-1}$ is the inverse of the expected fisher information matrix .

On solving (16) the $AV(\hat{\psi}(S_0))$ becomes,

$$nAV_2 = \frac{C - 2BS_0 + AS_0^2}{AC - B^2}$$

A, B and C is given by (15)

## 5. Result and Discussion

A numerical study was conducted in order to investigate the existence of the optimal stress change points and to evaluate them as a function of varying parameters. Simulations are conducted to investigate the performances of the MLEs through their mean square error (MSE) for both relationships. Comparison between both plans is shown by calculating efficiencies.

Table 1 present the Maximum likelihood estimates for n=20, 60, 80,100 & 120 and their respective Mean Square Error for both Quadratic and Linear relationship.

Table 2 gives the Asymptotic Variances and Covariance matrix.

Table 3 presents the results of optimal design of step-stress ALT for different sized samples and finally Table 4 Compound Linear Test-Plan Efficiencies.

**Table 1.** *The Maximum likelihood estimate and Mean Square Error for α=3.19687e⁻⁷*

| n | Parameter | Quadratic case | | Linear case | |
|---|---|---|---|---|---|
| | | Estimate | MSE | Estimate | MSE |
| 20 | $\hat{\theta}_1$ | $0.364e^{-5}$ | 0.0763 | $1.635e^{-5}$ | 0.0945 |
| | $\hat{\theta}_2$ | $1.166e^{-8}$ | 0.0063 | $3.284e^{-8}$ | 0.0592 |
| | $\hat{\theta}_3$ | $2.865e^{-8}$ | 0.0658 | - | - |
| 60 | $\hat{\theta}_1$ | $0.255e^{-5}$ | 0.0636 | $1.264e^{-5}$ | 0.0775 |
| | $\hat{\theta}_2$ | $1.643e^{-8}$ | 0.0060 | $3.846e^{-8}$ | 0.0570 |
| | $\hat{\theta}_3$ | $2.764e^{-8}$ | 0.4634 | - | - |
| 80 | $\hat{\theta}_1$ | $0.214e^{-6}$ | 0.0643 | $1.178e^{-5}$ | 0.0567 |
| | $\hat{\theta}_2$ | $1.543e^{-8}$ | 0.0072 | $4.852e^{-8}$ | 0.0529 |
| | $\hat{\theta}_3$ | $2.543e^{-8}$ | 0.0055 | - | - |
| 100 | $\hat{\theta}_1$ | $0.143e^{-6}$ | 0.0453 | $5.734e^{-6}$ | 0.0511 |
| | $\hat{\theta}_2$ | $1.654e^{-8}$ | 0.0074 | $7.836e^{-9}$ | 0.0478 |
| | $\hat{\theta}_3$ | $2.432e^{-8}$ | 0.0832 | - | - |
| 120 | $\hat{\theta}_1$ | $0.171e^{-6}$ | 0.0234 | $5.154e^{-6}$ | 0.0461 |
| | $\hat{\theta}_2$ | $1.284e^{-8}$ | 0.0031 | $7.245e^{-8}$ | 0.0359 |
| | $\hat{\theta}_3$ | $1.847e^{-8}$ | 0.0271 | - | - |
| 200 | $\hat{\theta}_1$ | $0.124e^{-6}$ | 0.0073 | $4.641e^{-6}$ | 0.0256 |
| | $\hat{\theta}_2$ | $1.133e^{-8}$ | 0.0029 | $6.735e^{-8}$ | 0.0219 |
| | $\hat{\theta}_3$ | $1.244e^{-8}$ | 0.0213 | - | - |

**Table2.** *Asymptotic Variances and Covariances of Estimates*

| n | $\beta_0$ | $\beta_1$ | $\beta_2$ | $\beta_0$ | $\beta_1$ |
|---|---|---|---|---|---|
| 20 | 2.0625 | -1.7545 | 1.6524 | 3.0862 | -2.7361 |
| | -1.7545 | 1.6524 | 1.8635 | -2.7361 | 1.9621 |
| | 1.6524 | 1.8635 | 0.8362 | - | - |
| 60 | 2.0343 | -1.5643 | 1.3413 | 3.0263 | -2.7141 |
| | -1.5643 | 1.3413 | 1.8472 | -2.7141 | 1.3856 |
| | 1.3413 | 1.8472 | 0.7356 | - | - |
| 80 | 2.0193 | -1.3425 | 1.2947 | 2.7251 | -2.3736 |
| | -1.3425 | 1.2947 | 1.8635 | -2.3736 | 1.1825 |
| | 1.2947 | 1.8635 | 0.5812 | - | - |
| 100 | 1.9753 | -1.3418 | 1.2421 | 2.1761 | -2.8362 |
| | -1.3418 | 1.2421 | 1.7623 | -2.8362 | 1.1384 |
| | 1.2421 | 1.7623 | 0.4321 | | |
| 120 | 1.9374 | -1.3261 | 1.2763 | 2.1383 | -2.1438 |
| | -1.3261 | 1.2763 | 1.5341 | -2.1438 | 1.1172 |
| | 1.2763 | 1.4826 | 0.4826 | - | - |
| 200 | 1.6443 | -1.3231 | 1.2144 | 2.0826 | -2.0836 |
| | -1.3231 | 1.2144 | 1.5135 | -2.0836 | 1.1054 |
| | 1.2144 | 1.5135 | 0.4254 | - | - |

**Table 3.** *The results of optimal design of step-stress ALT for different sized samples*

| n | $\beta_0$=0.9 $\beta_1$=1.5 | | | $\beta_0$=0.9 $\beta_1$=1.5 $\beta_2$=2.8 | | |
|---|---|---|---|---|---|---|
| | $nAV_1$ | $\tau_1$ | $\tau_2$ | $nAV_2$ | $\tau_1$ | $\tau_2$ |
| 20 | 0.190 | 2.3 | 2.8 | 0.113 | 2.2 | 2.4 |
| 60 | 0.054 | 2.3 | 2.7 | 0.056 | 1.9 | 2.4 |
| 80 | 0.037 | 1.8 | 2.7 | 0.035 | 1.4 | 2.4 |
| 100 | 0.028 | 1.7 | 27 | 0.028 | 1.4 | 2.4 |
| 120 | 0.018 | 1.5 | 2.7 | 0.014 | 1.1 | 2.4 |
| 200 | 0.009 | 1.5 | 2.7 | 0.005 | 1.1 | 2.4 |

**Table 4.** *Compound Linear Test-Plan Efficiencies*

| | | | | Efficiencies | |
|---|---|---|---|---|---|
| $S_1$ | $S_2$ | $S_3$ | n | Opt Linear | Opt Quadratic |
| 2.5 | 3.5 | 4.5 | 20 | 0.78 | 0.88 |
| | | | 60 | 0.72 | 0.88 |
| | | | 80 | 0.80 | 0.92 |
| | | | 100 | 0.85 | 0.92 |
| | | | 120 | 0.88 | 0.91 |
| | | | 200 | 0.91 | 0.94 |
| 3.5 | 4.5 | 5.5 | 20 | 0.84 | 0.84 |
| | | | 60 | 0.84 | 0.87 |
| | | | 80 | 0.86 | 0.87 |
| | | | 100 | 0.88 | 0.90 |
| | | | 120 | 0.86 | 0.92 |
| | | | 200 | 0.88 | 0.92 |
| 4.5 | 5.5 | 6.5 | 20 | 0.83 | 0.85 |
| | | | 60 | 0.84 | 0.87 |
| | | | 80 | 0.88 | 0.88 |
| | | | 100 | 0.88 | 0.92 |
| | | | 120 | 0.89 | 0.92 |
| | | | 200 | 0.87 | 0.92 |

## 6. Comparative Study

In this section, the proposed Step-Stress model have been compared with the constant accelerated life testing using geometric process by Shahab [12] in terms of maximum likelihood estimators and their respective error in Table 1. It can be seen that the error involved in step stress is more minimized as compared to constant stress and shows more stability of parameter. Hence Step-Stress ALT is efficient in comparison with constant.

Table 3 shows that the proposed model performs better when stresses are increased and more analysis are done at each step for the given data set.

## 7. Concluding Remarks

This paper deals with parameter estimation of Frechet distribution under 3-step stress ALT plan. The objective is to design a test that achieves the best reliability estimates. Two types of relationship are assumed between scale parameter and Stress. One links scale parameter linearly with stress while other have quadratic relationship. Comparison between both is shown by calculating estimates and their respective error. Efficiencies for both plans are calculated for different level of stress. Apart from that the results of optimal design of step-stress ALT for different sample size is shown.

Performance of step-stress testing plans and model assumptions are usually evaluated by the properties of the maximum likelihood estimates of model parameters. Estimates of quadratic are more stable with relatively small Mean Square error as sample size increases. Maximum likelihood estimators are consistent and asymptotically normally distributed. As the sample sizes increase the asymptotic variance and covariance of estimators decrease. In short, it is reasonable to say that the present step stress ALT plan works well and has a promising potential in the analysis of accelerated life testing.

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