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On Some Transformation and Summation Formulas for the $\overline{H}$-Function of two Variables

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Abstract: *In the present paper we establish four transformations of double infinite series involving the $\overline{H}$-function of two variables. These formulas are then used to obtain double summation formulas for the $\overline{H}$-function of two variables. Our results are quite general in character and a number of summation formulas can be deduced as particular cases.*

Keywords: $\overline{H}$ *-function of two variables, Gauss's summation theorem, Double infinite series*

(2000 Mathematics subject classification: 33C99)

1. Introduction

The $\overline{H}$-function of two variables will be defined and represented by Singh and Mandia [8] in the following manner:

$$\overline{H}[x,y]=\overline{H}\begin{bmatrix} x \\ y \end{bmatrix}=\overline{H}^{o,n_1:\, m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \tag{1.1}$$

Where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \tag{1.2}$$

$$\phi_2(\xi)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+\delta_j\xi\right)\right\}^{L_j}} \tag{1.3}$$

$$\phi_3(\eta)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+E_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+F_j\eta\right)\right\}^{S_j}} \tag{1.4}$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as

unity p_i, q_i, n_i, m_j are non-negative integers such that
$0 \le n_i \le p_i, o \le m_j \le q_j (i=1,2,3; j=2,3)$. All the
$a_j(j=1,2,...,p_1), b_j(j=1,2,...,q_1), c_j(j=1,2,...,p_2), d_j(j=1,2,...,q_2),$

$e_j(j=1,2,...,p_3), f_j(j=1,2,...,q_3)$ are complex parameters.
$\gamma_j \ge 0 (j=1,2,...,p_2), \delta_j \ge 0 (j=1,2,...,q_2)$ (not all zero simultaneously), similarly
$E_j \ge 0 (j=1,2,...,p_3), F_j \ge 0 (j=1,2,...,q_3)$ (not all zero simultaneously). The exponents
$K_j(j=1,2,...,n_3), L_j(j=m_2+1,...,q_2), R_j(j=1,2,...,n_3), S_j(j=m_3+1,...,q_3)$ can take on non-negative values.

The contour L_1 is in ξ-plane and runs from $-i\infty$ to $+i\infty$. The poles of
$\Gamma\left(d_j - \delta_j \xi\right)(j=1,2,...,m_2)$ lie to the right and the poles of
$\Gamma\left\{\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j} (j=1,2,...,n_2), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $K_j(j=1,2,...,n_2)$ not an integer, the poles of gamma functions of the numerator in (1.3) are converted to the branch points.

The contour L_2 is in η-plane and runs from $-i\infty$ to $+i\infty$. The poles of
$\Gamma\left(f_j - F_j\eta\right)(j=1,2,...,m_3)$ lie to the right and the poles of
$\Gamma\left\{\left(1-e_j+E_j\eta\right)\right\}^{R_j} (j=1,2,...,n_3), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $R_j(j=1,2,...,n_3)$ not an integer, the poles of gamma functions of the numerator in (1.4) are converted to the branch points.

The functions defined in (1.1) is an analytic function of x and y , if

$$U = \sum_{j=1}^{p_1}\alpha_j + \sum_{j=1}^{p_2}\gamma_j - \sum_{j=1}^{q_1}\beta_j - \sum_{j=1}^{q_2}\delta_j < 0 \tag{1.5}$$

$$V = \sum_{j=1}^{p_1}A_j + \sum_{j=1}^{p_3}E_j - \sum_{j=1}^{q_1}B_j - \sum_{j=1}^{q_3}F_j < 0 \tag{1.6}$$

The integral in (1.1) converges under the following set of conditions:

$$\Omega = \sum_{j=1}^{n_1}\alpha_j - \sum_{j=n_1+1}^{p_1}\alpha_j + \sum_{j=1}^{m_2}\delta_j - \sum_{j=m_2+1}^{q_2}\delta_j L_j + \sum_{j=1}^{n_2}\gamma_j K_j - \sum_{j=n_2+1}^{p_2}\gamma_j - \sum_{j=1}^{q_1}\beta_j > 0 \tag{1.7}$$

$$\Lambda = \sum_{j=1}^{n_1}A_j - \sum_{j=n_1+1}^{p_1}A_j + \sum_{j=1}^{m_2}F_j - \sum_{j=m_2+1}^{q_2}F_j S_j + \sum_{j=1}^{n_3}E_j R_j - \sum_{j=n_2+1}^{p_3}E_j - \sum_{j=1}^{q_1}B_j > 0 \tag{1.8}$$

$$|\arg x| < \frac{1}{2}\Omega\pi, \; |\arg y| < \frac{1}{2}\Lambda\pi \tag{1.9}$$

The behavior of the $\overline{H}$ -function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y] = 0(|x|^{\alpha}|y|^{\beta}), \max\left\{|x|,|y|\right\} \to 0 \tag{1.10}$$

Where

$$\alpha = \min_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right] \qquad \beta = \min_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right] \tag{1.11}$$

For large value of $|z|$,

$$\overline{H}[x,y] = 0\left\{|x|^{\alpha'}, |y|^{\beta'}\right\}, \min\left\{|x|,|y|\right\} \to 0 \tag{1.12}$$

Where

$$\alpha' = \max_{1\le j\le n_2} \mathrm{Re}\left(K_j \frac{c_j - 1}{\gamma_j} \right),\ \beta' = \max_{1\le j\le n_3} \mathrm{Re}\left(R_j \frac{e_j - 1}{E_j} \right) \tag{1.13}$$

Provided that $U < 0$ and $V < 0$.

If we take

$K_j = 1(j = 1, 2, ..., n_2), L_j = 1(j = m_2 + 1, ..., q_2), R_j = 1(j = 1, 2, ..., n_3), S_j = 1(j = m_3 + 1, ..., q_3)$

in (1.1), the $\overline{H}$ -function of two variables reduces to H -function of two variables due to [1].

2. Transformation Formulas

In this section we establish the following four transformation Formulas for the $\overline{H}$ -function of two variables:

First formula

$$\sum_{m,n=0}^{\infty} x^m y^n \overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\middle| \begin{matrix} (1-a-m,\rho;1),(1-b-n,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-m-n,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right]$$

$$= (x + y - xy)^{-1}$$

$$\sum_{s=0}^{\infty} x^{s+1} \overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\middle| \begin{matrix} (1-a-s,\rho;1),(1-b,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-s,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right]$$

$$+\sum_{t=0}^{\infty} x^{t+1} \overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\middle| \begin{matrix} (1-a,\rho;1),(1-b-t,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-t,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right]$$

(2.1)

The formula (2.1) is valid, if the following (sufficient) conditions are satisfied.

$(i)\ \rho, \sigma > 0, (ii)\ \Omega - \rho - \sigma > 0\ , |\arg u| < \frac{1}{2}(\Omega - \rho - \sigma)\pi$

$(iii) \Omega - \rho - \sigma > 0\ , |\arg v| < \frac{1}{2}(\Lambda - \rho - \sigma)\pi$

$(iv) \max\{|x|, |y|\} < 1 \quad or\ \ x = y = 1 \quad with \quad Re(a) > 1, \mathrm{Re}(b) > 1$

Second formula

$$\sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!n!} \overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\middle| \begin{matrix} (1-a-m-n,u';1),(1-b-m,v';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-c-m,w';1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (1-y)^{-a} \left(\frac{x}{1-y} \right)^k$$

$$\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\middle| \begin{matrix} (1-a-m,u';1),(1-b-m,v';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-c-m,w';1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right] \tag{2.2}$$

Provided that

$(i) u', v', \omega' > 0 \qquad (ii)\ \Omega - \omega' > 0, |\arg u'| < \frac{1}{2}(\Omega - \omega)\pi\ (iii) \Omega - \omega' > 0, |\arg v'| < \frac{1}{2}(\Lambda - \omega')\pi$

$(iv)\ |x| + |y| < 1\ and\ \ either\ \left|\frac{x}{1-y}\right| < 1\ or \left|\frac{x}{1-y}\right| = 1\ \ with\ \mathrm{Re}(c - a - b) > 0$

Third formula

$$\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!n!}\overline{H}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}^{o,n_1+2:\quad m_2,n_2:m_3,n_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\middle| \begin{matrix}(1-a-m-n,u';1),(1-b-m,v';1),(a_j,\alpha_j;A_j)_{1,p_1},(1-b'-n,\omega';1),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-a-n,u';1),(1-a-m,u';1)(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}(1-x)^{-b}(1-y)^{b'}\left(\frac{xy}{(1-x)(1-y)}\right)^k\overline{H}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}^{o,n_1+2:\quad m_2,n_2:m_3,n_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\middle| \begin{matrix}(1-b-k,v';1),(1-b'-k,v';1),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-a-k,u';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \tag{2.3}$$

Provided that

$(i)\, u',v',\omega'>0 \quad (ii)\ \Omega-\omega'>0, |\arg u'|<\frac{1}{2}(\Omega-\omega)\pi\ (iii)\Omega-\omega'>0, |\arg v'|<\frac{1}{2}(\Lambda-\omega')\pi$

$(iv)\ |x|+|y|<1$ *and either* $\left|\frac{xy}{(1-x)(1-y)}\right|<1$ *or* $\left|\frac{xy}{(1-x)(1-y)}\right|=1$ *with* $\mathrm{Re}(a-b-b')>0$

Fourth formula

$$\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!n!}\overline{H}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}^{o,n_1+2:\quad m_2,n_2:m_3,n_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\middle| \begin{matrix}(1-a-m-n,u';1),(1-b-m,v';1),(1-b'-n,\omega';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-b-b'-m-n,u'+w';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$=\sum_{k=0}^{\infty}(1-y)^{-a}\frac{1}{K!}\left(\frac{x-y}{1-y}\right)^k\overline{H}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}^{o,n_1+2:\quad m_2,n_2:m_3,n_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\middle| \begin{matrix}(1-a-k,u';1),(1-b-k,v';1),(1-b',\omega';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-b-b',u'+w';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \tag{2.4}$$

Provided that

$(i)\, u',v',\omega'>0 \quad (ii)\ \Omega-\omega'>0, |\arg u'|<\frac{1}{2}(\Omega-\omega)\pi\ (iii)\Omega-\omega'>0, |\arg v'|<\frac{1}{2}(\Lambda-\omega')\pi$

$(iv)\ \max\{|x|,|y|\}<1$, *either* $\left|\frac{x-y}{1-y}\right|<1$ *or* $\left|\frac{x-y}{1-y}\right|=1$ *with* $\mathrm{Re}(b'-a)>0$

In all the aforementioned formulas Ω, Λ are given by (1.7), (1.8) respectively.

Derivation of the first formula: Using Mellin-Barnes type of contour integral (1.1) for the $\overline{H}$-function of two variables occurring on the L.H.S. of (2.1) and changing the order of integration and summation, we find that L.H.S. of (2.1).

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)\frac{\Gamma(a+\rho\xi)\Gamma(b+\sigma\xi)}{\Gamma(a+b+(\sigma+\rho)\xi)}$$

$$F_2\left[a+\rho(\xi+\eta),b+\sigma(\xi+\eta),1,1,;a+b+(\sigma+\rho)\xi+\eta;u,v\right]u^{\xi}v^{\eta}d\xi d\eta \tag{2.5}$$

Now appealing to a known result due to Srivastava ([9], p.1259, eq. (2.2))

$$F_2\left[a,b,1,1;a+b;x,y\right]=(x+y-xy)^{-1}\left\{x,\ {}_2F_1\left[a,1;a+b;x\right]+y\ {}_2F_1\left[b,1;a+b;x\right]\right\} \tag{2.6}$$

in (2.6) , we get L.H.S. of (2.1)

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)\frac{\Gamma(a+\rho\xi)\Gamma(b+\sigma\xi)}{\Gamma(a+b+(\sigma+\rho)\xi)}(u+v-uv)^{-1}$$
$$\left\{u\ {}_2F_1\left[a+\rho(\xi+\eta),1;a+b+(\rho+\sigma)\xi+\eta;u\right]\right.$$
$$\left.+v\ {}_2F_1\left[b+\sigma(\xi+\eta),1;a+b+(\rho+\sigma)\xi+\eta;v\right]\right\}d\xi d\eta \tag{2.7}$$

Now expressing the ${}_2F_1$ functions in terms of their series and changing the order of integration and summation, and interpreting the result so obtained with the help of (1.1), we arrived at the formula (2.1).

Derivation of the formulas (2.2) to (2.4) : The summation formulas (2.2), (2.3) and (2.4) can be developed on the lines similar to the formula (2.1) except that, in place of(2.6) , here we use the following known results ([2],p.238,eq.(2),eq.(3) and eq.(1) respectively):

$$F_2\left[\alpha,\beta,\beta';\gamma,\beta;x,y\right]=(1-y)^{-\alpha}\ {}_2F_1\left[\alpha,\beta;\gamma;\frac{x}{(1-y)}\right] \tag{2.8}$$

$$F_2\left[\alpha,\beta,\beta';\alpha,\alpha;x,y\right]=(1-x)^{-\beta}(1-y)^{-\beta}\ {}_2F_1\left[\beta,\beta';\alpha;\frac{xy}{(1-x)(1-y)}\right] \tag{2.9}$$

$$F_2\left[\alpha,\beta,\beta';\beta+\beta';x,y\right]=(1-y)^{-\alpha}\ {}_2F_1\left[\alpha,\beta;\beta+\beta';\frac{x-y}{(1-y)}\right] \tag{2.10}$$

3. SUMMATION FORMULAS

If we take x = y = 1 in (2.1) and use the well known Gauss's summation theorem, we arrived at the result

$$\sum_{m,n=0}^{\infty}x^m y^n\,\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l} u & (1-a-m,\rho;1),(1-b-n,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v & (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-m-n,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right]$$

$$\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l} u & (1-a-m,\rho;1),(2-b,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v & (b_j,\beta_j;B_j)_{1,q_1},(2-a-b,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right]$$
$$+\sum_{t=0}^{\infty}x^{t+1}\,\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l} u & (2-a,\rho;1),(1-b,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v & (b_j,\beta_j;B_j)_{1,q_1},(2-a-b,\sigma+\rho;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right] \tag{3.1}$$

Valid under the conditions of (2.1) .

Again if we put u= v = $\frac{1}{2}$ in (2.2), v= 1-u in (2.3) and x = 1 in (2.4) and make use of well known Gauss's summation theorem therein, we shall arrive at the following results

$$\sum_{m,n=0}^{\infty}\frac{\left(\tfrac{1}{2}\right)^{m+n}}{m!n!}\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1+1:p_2,q_2;p_2,q_2}$$
$$\left[\begin{array}{c|l} u(1-x)^{-u'} & (1-a-m-n,u';1),(1-b-m,v';1),(1-b'-n,\omega';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v(1-y)^{-v'} & (1-c-m,u'+w';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right]$$

$$= 2^{a}\,\overline{H}{}^{o,n_1+3:\ m_2,n_2:m_3,n_2}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\left| \begin{matrix} (1-a,u';1),(1-b,v';1),(1-c+a+b,\omega'-u'-v';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-c+a,w'-u';1),(1-c+b,w'-v';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \quad (3.2)$$

Where $\omega'-u'-v'>0, \omega'\neq u' \ or \ \omega'\neq v'$

And valid under the conditions of (2.1)

$$\sum_{m,n=0}^{\infty}\frac{x^m(1-x)^n}{m!n!}\overline{H}{}^{o,n_1+3:\ m_2,n_2:m_3,n_2}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\left| \begin{matrix} (1-a-m-n,u';1),(1-b-m,v';1),(1-b'-n,\omega';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-a-n,u';1),(1-a-m,u';1)(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= u^{-b'}(1-u)^{-b}\,\overline{H}{}^{o,n_1+3:\ m_2,n_2:m_3,n_2}_{p_1+3,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix}\left| \begin{matrix} (1-b,v';1),(1-b',v';1),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-a+b,u'-v';1),(1-ab',u'-w';1),(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \quad (3.3)$$

Where $u'-v'-\omega'>0, \ u'\neq\omega', \ u'\neq v'$

And valid under the conditions of (2.3).

$$\sum_{m,n=0}^{\infty}\frac{y^n}{m!n!}\overline{H}{}^{o,n_1+3:\ m_2,n_2:m_3,n_2}_{p_1+3,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\left| \begin{matrix} (1-a-m-n,u';1),(1-b-m,v';1),(1-b'-n,w';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-b-b'-m-n,w'+v';1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= (1-y)^{-a}\,\overline{H}{}^{o,n_1+3:\ m_2,n_2:m_3,n_2}_{p_1+3,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\left| \begin{matrix} (1-a,u';1),(1-b,v';1),(1-b'+a,w'-u';1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-b-b'+a,w'+v';1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \quad (3.4)$$

Where $v'\neq\omega', v'\neq u'$ and valid under the conditions (2.4).

4. SPECIAL CASES

(i) In (2.1), taking $_s$and $m_2 = m_3 = 1, q_2 = q_3 = 2$ the $\overline{H}$ -function of two variables reduced to the multiplivation of two generalized Wright's function [8] and we get

$$\sum_{m,n=0}^{\infty} x^m y^n \frac{\Gamma(a+m+\rho(\xi+\eta))\Gamma(b+n+\sigma(\xi+\eta))}{\Gamma(a+b+m+n+(\sigma+\rho)(\xi+\eta))}\overline{J}{}_{\lambda}^{\nu,\mu}[u]\overline{J}{}_{\lambda}^{\nu,\mu}[v] = (x+y-xy)^{-1}$$

$$\left\{\sum_{s=0}^{\infty} x^{s+1}\overline{H}{}^{0,0:1,0:1,0}_{0,0:0,2,0,2}\left[\begin{matrix} u \\ v \end{matrix}\left| \begin{matrix} (1-a-s,\rho;1),(1-b,\sigma;1) \\ (0,1),(-\lambda,\nu;\mu),(1-a-b-s,\sigma+\rho;1) \end{matrix}\right.\right] + \sum_{t=0}^{\infty} y^{t+1}\overline{H}{}^{0,0:1,0:1,0}_{0,0:0,2,0,2}\left[\begin{matrix} u \\ v \end{matrix}\left| \begin{matrix} (1-a,\rho;1),(1-b-t,\sigma;1) \\ (0,1),(-\lambda,\nu;\mu),(1-a-b-t,\sigma+\rho;1) \end{matrix}\right.\right]\right\} \quad (4.1)$$

where $(1-\nu)>0, (1+\nu)\geq 0, |\arg u| < \frac{1}{2}(1-\nu-\rho-\sigma)\pi, |\arg v| < \frac{1}{2}(1-\nu-\rho-\sigma)\pi$ and the conditions (i) and (iii) given with (2.1) also satisfied.

Further if we take $\rho \to 0$ in (2.1), we get the following new transformation formula:

$$\sum_{m,n=0}^{\infty} x^m y^n (a)_m \overline{H}{}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\left| \begin{matrix} (1-b-n,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-m-n,\sigma;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= (x+y-xy)^{-1}$$
$$\sum_{s=0}^{\infty} x^{s+1} \overline{H}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l} u & (1-b,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v & (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-m,\sigma;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{array}\right]$$
$$+\sum_{t=0}^{\infty} x^{t+1} \overline{H}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l} u & (1-b-n,\sigma;1),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ v & (b_j,\beta_j;B_j)_{1,q_1},(1-a-b-n,\sigma;1),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{array}\right]$$
(4.2)

Valid under the conditions of (2.1).

A similar type of result can be obtained by taking $\sigma \to 0$ in (2.1).

5. Conclusion

We establish four transformations of double infinite series involving the $\overline{H}$ -function of two variables. These formulas are used to obtain double summation formulas for the $\overline{H}$ - function of two variables. Special cases are also derived.

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