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Laplacian Polynomial and Laplacian Energy of Some Cluster Graphs

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Abstract: *The graphs with large number of edges are referred as graph representation of inorganic clusters, so called cluster graphs. H . B. Walikar and H .S. Ramane introduced class of graph obtained from complete graph by deleting edges. In this paper, the Laplacian polynomial and Laplacian energy of this class of graph is obtained.*

Keywords: *Laplacian polynomial and Laplacian energy of graph, cluster graphs.*

AMS Subject Classification: 05C50.

1. Introduction

The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories.

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, ---v_n\}$ where n is the number of vertices of G. The adjacency matrix of a graph G is $A(G) = [a_{ij}]$, where a_{ij}=1 if v_i is adjacent to v_j and a_{ij}=0, otherwise. The characteristic polynomial of a graph G is defined as

$\phi(G:\lambda) = det(\lambda I - A(G))$ where I is the identity matrix of order n.

The degree matrix of a graph G is the diagonal matrix $D(G) = diag[d_i]$ where $d_i = d_G(v_i)$. The matrix $C(G) = D(G) - A(G)$ is called Laplacian matrix. It is also called as matrix of admittance due to its role in electrical theory [1]. The Laplacian polynomial of graph G is defined as $C(G:\mu) = det(\mu I - C(G))$ where I is the identity matrix of order n. The roots $\mu_1, \mu_2, ---\mu_n$ of $C(G:\mu)$ are called the Laplacian eigenvalues of G, where $\mu_1 \geq \mu_2 \geq --- \geq \mu_n$.

Laplacian energy is defined as $CE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

Let K_n denote the complete graph on n vertices. The class of graphs defined [2] is as follows.

2. Some Cluster Graphs

I.Gutman and L. Pavlovic [2] introduced four classes of graphs obtained from complete graph by deleting edges and analyzed their energies. For completeness we produce these here.

DEFINITION 1: Let v be a vertex of a complete graph K_n , $n \geq 3$ and let e_i, $i = 1,2, ---k$, $1 \leq k \leq n-1$, be its distinct edges, all being incident to v. The graph $Ka_n(k)$ is obtained by deleting e_i, $i = 1,2, ---k$ from K_n. In addition $Ka_n(0) \cong K_n$.

DEFINITION 2: Let f_i, $i = 1,2, ---k$, $1 \leq k \leq \lfloor n/2 \rfloor$ be independent edges of the complete graph K_n, $n \geq 3$. The graph $Kb_n(k)$ is obtained by deleting f_i, $i = 1,2, ---k$ from K_n. In addition $Kb_p(0) \cong K_n$.

DEFINITION 3: Let V_k be a k-element subset of the vertex set of complete graph K_n, $2 \le k \le n$, $n \ge 3$. The graph $Kc_n(k)$ is obtained by deleting from K_n all the edges connecting pairs of vertices from V_k. In addition $Kc_n(0) \cong Kc_n(1) \cong K_n$.

DEFINITION 4: Let $3 \le k \le n$, $n \ge 3$. The graph $Kd_n(k)$ is obtained from K_n, the edges belonging to a k-membered cycle.

H.S. Ramane and H.B.Walikar [3] has introduced another class of graph obtained from K_n and is denoted by $Ka_n(p,k)$ which is as follows.

DEFINITION 5: Let $(K_p)_i$, $i = 1,2,---k$, $1 \le k \le \lfloor n/p \rfloor$, $1 \le p \le n$, be independent complete graphs with p vertices of the complete graph K_n, $n \ge 3$. The graph $Ka_n(p,k)$ is obtained from K_n by deleting all edges of $(K_p)_i$, $i = 1,2,---k$. In addition

$Ka_n(p,0) \cong Ka_n(0,k) \cong Ka_n(0,0) \cong K_n$.

In this paper Laplacian polynomial and energy of $Ka_n(p,k)$ is obtained.

Note that the Laplacian polynomial and Laplacian energy of $Kb_n(k)$ and $Kc_n(k)$ [4] are particular cases of the graph $Ka_n(p,k)$.

Theorem 1: For $n \ge 3$, $1 \le k \le \lfloor n/p \rfloor$, $1 \le p \le n$,

$$C\big(Ka_n(p,k)\big) = \mu(\mu-n)^{n-k(p-1)-1}\,(\mu-n+p)^{k(p-1)} \tag{1}$$

Proof: Without loss of generality we assume that the vertices of $(K_p)_i$ are

$v_{m(i-1)+1}, v_{m(i-1)+2}, ---v_{m(i-1)+m}$, $i = 1,2,---k$.

In order to make the following result more compact, the auxiliary quantity X is introduced

$X = \mu - n + p$.

Then the Laplacian polynomial of $Ka_n(p,k)$ is equal to the determinant

$$\begin{vmatrix}
X & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & X & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & 0 & X & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & X & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & X & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & X & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & X & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & X & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & X & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & X & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & X & \cdots & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & X & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & X-p+1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & X-p+1
\end{vmatrix} \tag{2}$$

Subtract first column from 2,3,…n columns of (2) to obtain (3)

$$\begin{vmatrix} X & -X & \cdots & -X & 1-X & 1-X & \cdots & 1-X & 1-X & \cdots & 1-X & 1-X & \cdots & 1-X \\ 0 & X & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & X-1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & -1 & X-1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & X-1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X-1 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 & \cdots & X-1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & X-p & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & X-p \end{vmatrix} \quad (3)$$

Add 2,3,……n rows to first row of (3) to obtain (4)

$$\begin{vmatrix} X+n-p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & X & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & X-1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & -1 & X-1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & X-1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X-1 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 & \cdots & X-1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & X-p & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & X-p \end{vmatrix}$$

(4)

Evidently , expression (4) is equal to (5)

$$(X+n-p)\,(X-p)^{n-pk}\,X^{\,p-1}\begin{vmatrix} X-1 & -1 & -1 & \cdots & -1 \\ -1 & X-1 & -1 & \cdots & -1 \\ -1 & -1 & X-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & X-1 \end{vmatrix}^{k-1} \quad (5)$$

Subtract first column from 2,3,….p columns of (5) to obtain (6)

$$(X+n-p)\,(X-p)^{n-pk}\,X^{\,p-1}\begin{vmatrix} X-1 & -X & -X & \cdots & -X \\ -1 & X & 0 & \cdots & 0 \\ -1 & 0 & X & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & X \end{vmatrix}^{k-1} \quad (6)$$

Add 2,3,…p rows to first row of (6) to obtain (7)

$$(X+n-p)\,(X-p)^{n-pk}\,X^{\,p-1}\begin{vmatrix} X-p & 0 & 0 & \cdots & 0 \\ -1 & X & 0 & \cdots & 0 \\ -1 & 0 & X & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & X \end{vmatrix}^{k-1} \quad (7)$$

Expression (7) is equal to

$$(X+n-p)\,(X-p)^{n-pk}\,X^{\,p-1}\,(X-p)^{k-1}(X^{p-1})^{k-1} \quad (8)$$

On simplification expression (8) reduces to (9)

$$(X+n-p)\,(X-p)^{n-pk+k-1}\,X^{\,k(p-1)} \tag{9}$$

This leads to the expression (1).

This completes the proof.

3. Laplacian Spectra and Laplacian Energy of $Ka_n(p,k)$

Corollary 2:

For $1 \le k \le \lfloor n/p \rfloor$, $1 \le p \le n$, the spectrum of $C\big(Ka_n(p,k)\big)$ consists of 0, *n* {(*n*-*k*(*p*-1)-1) times} and *n*-*p* {*k*(*p*-1) times}

Theorem 3:

For $1 \le k \le \lfloor n/p \rfloor$, $1 \le p \le n$,

$$CE(Ka_n(p,k) = \frac{2}{n}[n(n-1) - pk(p-1) + k(n-pk)(p-1)^2] \tag{10}$$

Proof: The Laplacian energy is given by

$$CE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

$$CE(Ka_n(p,k) = \left|0 - \frac{2m}{n}\right| + (n-k(p-1)-1)\left|n - \frac{2m}{n}\right| + k(p-1)\left|n - p - \frac{2m}{n}\right| \tag{11}$$

Substituting the value of $2m$ in the equation (11) we straightforwardly obtain (10).

Remarks:

1. If $k=0$ equation (1) reduces to

   $C\big(Ka_n(p,0)\big) = \mu(\mu-n)^{n-1}$ which is a Laplacian polynomial of K_n .
2. If $p=1$, then equation (1) reduces to

   $C\big(Ka_n(1,k)\big) = \mu(\mu-n)^{n-1}$ which is a Laplacian polynomial of K_n .
3. If $p=n$ and $k=1$ then equation (1) reduces to

   $C\big(Ka_n(n,1)\big) = \mu^n$ which is a Laplacian polynomial of $\overline{K}_p$, the complement of K_p .
4. If $p=2$, $k=n/2$ then equation (1) reduces to

   $C\big(Ka_n(2,n/2)\big) = \mu(\mu-n)^{\frac{n}{2}-1}(\mu-n+2)^{n/2}$ which is a Laplacian polynomial of Cocktail party graph
5. If $n=pk$ then equation (1) reduces to

   $C\big(Ka_n(p,k)\big) = \mu(\mu-pk)^k\big(\mu-p(k-1)\big)^{k(p-1)}$ which is a Laplacian polynomial of complete multipartite graph $K_{n_1,n_2,\ldots\ldots\ldots n_k}$ where $n_1 = n_2 = -----= n_k = n/k$
6. If $p=2$ and $k=1$ then equation (1) reduces to

   $C\big(Ka_n(2,1)\big) = \mu(\mu-n)^{n-2}(\mu-n+2)$ which is a Laplacian polynomial of $Ka_n(1)$ [4]
7. If $p=2$ then equation (1) reduces to

   $C\big(Ka_n(2,k)\big) = \mu(\mu-n)^{n-k-1}(\mu-n+2)^k$ which is a Laplacian polynomial of $Kb_n(k)$ [4].
8. If $k=1$ then equation (1) reduces to

$C\big(Ka_n(p,1)\big) = \mu(\mu-n)^{n-p}(\mu-n+p)^{p-1}$ which is a Laplacian polynomial of $Kc_n(k.)$ [4].

9. If $k = 1$ and $p = 3$ then equation (1) reduces to

$C\big(Kd_n(3,1)\big) = \mu(\mu-n)^{n-3}(\mu-n+3)^2$ which is a Laplacian polynomial of $Kd_n(3)$.

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