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Application of Fixed point Theorem in Game Theory

Geeta Modi
Professor & Head
Department of Mathematics
Govt. M.V.M. Bhopal,
MP, India.
modi.geeta@gmail.com

Arvind Gupta
Professor
Department of Mathematics
Govt. M.V.M. Bhopal,
MP, India.
arvind533@yahoo.com

Sushma Duraphe
Assistant Professor
Department of Mathematics
Govt. M.V.M. Bhopal,
MP, India.
duraphe_sus65@rediffmail.com

Varun Singh
Research Scholar
Department of Mathematics
Govt. M.V.M. Bhopal,
MP, India.
varunsinghhh@gmail.com

Abstract: *In this paper, we generalized the Kakuntani's fixed point theorem in Hausdroff topological space, and showed it's an application in game theory.*

Keyword: *fixed point, convex set, closed graph,*

1. INTRODUCTION

It is known that the theory of correspondences has very widely developed and produced many applications, especially during the last few decades. Most of these applications concern fixed point theory and game theory. The fixed point theorems are closely connected with convexity.

In 1928, John von Neumann found his celebrated minimax theorem [5] and, in 1937, his intersection lemma [4], which was intended to establish his minimax theorem and his theorem on optimal balanced growth paths. In 1941, Kakutani [9] obtained a fixed point theorem, from which von Neumann's minimax theorem and intersection lemma are easily deduced. In 1951, John Nash [3] established his celebrated equilibrium theorem. In 1952, Fan [7] and Glicksberg [2] extended Kakutani's theorem to locally convex Hausdorff topological vector spaces, and Fan generalized the von Neumann intersection lemma by applying his own fixed point theorem. In 1964, Fan [6] obtained another intersection theorem for a finite family of sets having convex sections. This was extended, by Ma [18] in 1969, to infinite families by using Fan's generalization of the von Neumann intersection lemma. Ma applied his result to an analytic formulation of Fan type and to Nash's theorem for arbitrary families. Note that all of the above results are extended in our recent works [17,16,14,12,13,10,15,11,1] in several directions. In fact, those results are mainly concerned with convex subsets of (Hausdorff) topological vector spaces.

We showed an application of fixed point theorem in game theory with convex subsets of Hausdorff topological spaces.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, we shall use the following notations and definitions:

2.1 Hausdorff Topological Space

A topological space $X = (X;T)$ is ***Hausdorff*** (or possesses the Hausdorff property) if for every two points $a, b \in X$ such that $a \neq b$ there are open neighborhoods U_a and U_b of a and b respectively such that $U_a \cap U_b = \emptyset$.

2.2 Compact set

Let X be a topological space . A subset $K \subset X$ is said to be ***compact set*** in X, if it has the finite open cover property:

(f.o.c.) Whenever $\{D_i\}_{i\in I}$ is a collection of open sets such that $K \subset \bigcup_{i\in I} D_i$, there exists a finite sub collection $D_{i_1}, D_{i_2}, \dots \dots \dots \dots \dots . . D_{i_n}$ such that $K \subset D_{i_1} \cup D_{i_2} \cup \dots \dots . . \cup D_{i_n}$

An equivalent description is the finite intersection property:

(f.i.p.) If $\{F_i\}_{i\in I}$ is a collection of closed sets such that for any finite sub collection

$F_{i_1}, F_{i_2}, \dots \dots \dots \dots \dots . . F_{i_n}$ we have $K \cap F_{i_1} \cap F_{i_2} \dots \dots \dots \dots \dots . . \cap F_{i_n} \neq \emptyset$

"*A topological space (X, T) is called compact if X itself is a compact set*"

2.3 Convex Set

Let S be a vector space over the real numbers. This includes Euclidean spaces. A set C in S is said to be ***convex*** if, for all x and y in C and all t in the interval [0, 1], the point $(1 - t)x + ty$ also belongs to C

2.4 Graph

For any function $T : X \rightarrow Y$, we define the *graph* of T to be the set

$$\{(x, y) \in \mathrm{X} \times \mathrm{Y} : \mathrm{Tx} = \mathrm{y}\}$$

2.5 Upper and Lower semi continuous

Let X, Y be topological spaces and $T : X \rightarrow 2^Y$ be a correspondence

I. T is said to be ***upper semi continuous*** if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that T(y) ⊂ V for each y ∈ U.

II. T is said to be ***lower semi continuous*** if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each y ∈ U.

III. T is said to have open lower sections if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

2.6 Lemma

If X_1 and X_2 are compact Hausdroff topological space and $T\colon \mathrm{X}_1 \times \mathrm{X}_2 \rightarrow \mathrm{R}$ is continuous, then the functions $f(x) = \min T(x, \mathrm{X}_2)$, $\mathrm{x} \in \mathrm{X}_1$; $g(x) = \max T(\mathrm{X}_1, \mathrm{y})$, $\mathrm{y} \in \mathrm{X}_2$ are continuous too.

2.7 Proposition

Let X, Y be topological spaces and $F : X \rightarrow Y$ a set-valued mapping.

I. If Y is ***regular***, F is ***usc*** and for every $x \in X$ the set $F(x)$ is nonempty and closed, then F has ***closed graph***.

II. Conversely, if the space Y is ***compact Hausdorff*** and F is with closed graph, then F is usc.

2.8 Kakutani's Fixed Point Theorem

Let X be a convex closed bounded body in R^n $\forall x \in X$, $f(x)$ is a nonempty sub set of X, $\{\langle \mathrm{x}, f(x)\rangle\}$ is closed. Then $\exists\, \mathrm{x}^*$ such that $\mathrm{x}^* \in f(x^*)$.

2.9 Theorem

(S.Cobzas 2006) [8] States and prove the Kakutani theorem in the locally convex case.

An element $x \in X$ is called a fixed point of a set-valued mapping $F : X \rightarrow Y$ if $x \in X$. If F is single valued then the usual notion of fixed point.

2.10 Proposition

If $M_x = \{\, y \in X_2 : T(x,y) = f(x)\}$ and $N_y = \{\, x \in X_1 : T(x,y) = g(y)\}$ for all $x \in X_1$ and $y \in X_2$, then M_x and N_y are nonempty and closed for all $(x,y) \in X_1 \times X_2$

3. MAIN RESULT

3.1 Theorem

Let X_1 and X_2 are compact Hausdroff topological space and $X = X_1 \times X_2$ define $F: X \to X$ by $F(x,y) = N_y \times M_x \ \forall (x,y) \in X$, $F(x,y)$ is a non empty convex subset of X, $\{\langle (x,y), F(x,y)\rangle\}$ is closed, Then $\exists\ x^* = (x_0, y_0)$ such that $x^* \in F(x^*)$.

Proof

Let X_1 and X_2 are compact Hausdroff topological space and $X = X_1 \times X_2$ define $F: X \to X$ by

$F(x,y) = N_y \times M_x \quad \forall (x,y) \in X,$

Since N_y and M_x are non empty and closed sets for every$(x,y) \in X_1 \times X_2$, it follows that $F(x,y)$ is non empty and closed subset of X.

Consider $G_F = \{\langle (x,y), F(x,y)\rangle \in X \times X\}$ be a graph for F.

Since $\{\langle (x,y), F(x,y)\rangle\}$ is closed, it follows that F has a closed graph.

By section 2.7 F is usc, So that by theorem (2.9) F has a fixed point $x^* = (x_0, y_0)$

i.e. $(x_0, y_0) \in F(x_0, y_0) \Rightarrow x^* \in F(x^*)$

4. APPLICATION OF THEOREM

The proof of Nash makes use of theorem 3.1.

Suppose there are 3 players, A, B, and C. Let $\bar{p}$, $\bar{q}$, and $\bar{r}$ be their probability distributions over the action sets. And α β and γ are their payoff functions. $P(\bar{q}, \bar{r})$ denotes the set of best-play $\bar{p}'$ s. Not hard to see $P(\bar{q}, \bar{r})$ is a convex closed set. Similarly, we define $Q(\bar{r}, \bar{p})$ and $R(\bar{p}, \bar{q})$. Define function $F: \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} \to \begin{pmatrix} P(\bar{q}, \bar{r}) \\ Q(\bar{r}, \bar{p}) \\ R(\bar{p}, \bar{q}) \end{pmatrix}$

Then if we could apply theorem 3.1, we have

$$\begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} \in F \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix}$$

which shows the existence of Nash equilibrium. Now what we need to do is just verify our setup satisfies the conditions of theorem 3.1. The first two conditions are trivially satisfied.

We only need to show that $\{\langle (x,y), F(x,y)\rangle\}$ is closed, which means we need to show if $\langle (x,y), F(x,y)\rangle \to \langle x^*, y^*\rangle$.

Let $x = (\bar{p}, \bar{q}, \bar{r})$ and $y = (\bar{u}, \bar{v}, \bar{w})$.

$$\langle x^*, y^*\rangle \in F(x^*) \Leftrightarrow y^* \in F(x^*) \Leftrightarrow \begin{matrix} \alpha(\bar{u}^*, \bar{q}^*, \bar{r}^*) \geq \alpha(\bar{p}', \bar{q}^*, \bar{r}^*) \ \forall\, \bar{p}' \\ \beta(\bar{p}^*, \bar{v}^*, \bar{r}^*) \geq \beta(\bar{p}^*, \bar{q}', \bar{r}^*) \ \forall\, \bar{q}' \\ \gamma(\bar{p}^*, \bar{q}^*, \bar{w}^*) \geq \gamma(\bar{p}^*, \bar{q}^*, \bar{r}') \ \forall\, \bar{r}' \end{matrix}$$

Which shows $\{\langle x, F(x)\rangle\}$ is closed. And the above argument can easily apply to any finite number players.

5. CONCLUSION

In this paper, we proved a fixed point theorem in compact Hausdroff topological space, which generalization of Kakutani's fixed point theorem. As an application of our result, we showed every finite game has a mixed strategy Nash equilibrium.

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AUTHORS' BIOGRAPHY

Dr. Geeta Modi has been awarded Ph.D., in 1990. She has 31 years of teaching experience. She is currently working as a Professor & Head of Department of Mathematics, Govt. MVM Bhopal. She is presently The Chairman of Board of Studies (Mathematics) Barkatullah University Bhopal, and associated with the Member of central Board of Studies (Mathematics) Government of Madhya Pradesh. She is V.C. nominee member of Board of Studies IEHE Bhopal Madhya Pradesh, India. She has published more than 45 articles in national and international journals. 08 Research scholar awarded Ph.D. under her supervision and 06 research scholar registered.

Dr. Arvind Gupta has been awarded Ph.D., in 1996. He has 30 years of teaching experience. He is currently working as a Professor of department of Mathematics, Govt. MVM Bhopal, Madhya Pradesh, India. He has published more than 12 articles in national and international journals. 01 Research scholar awarded Ph.D. under his supervision and 05 research scholar registered.

Dr. Sushma Duraphe has passed M.Sc.(Statistics) in 1984 and M.Sc. (Mathematics) in 1986. She has been awarded Ph.D.(Mathematics) , in 2010. She has 27 years of teaching experience. She is currently working as Assistant Professor of Department of Mathematics Govt. MVM Bhopal, Madhya Pradesh, India. She is associated with the member of editorial board (for 2013-2015) of the journal of the Indian Academy of mathematics and member of executive committee. She has published more than 06 articles in national and international journals.

Mr. Varun Singh is a research scholar in the Department of Mathematics, Govt. MVM Bhopal Madhya Pradesh, India. He has published 02 research papers in International Journals. His field of research is Fixed point Theory and Game Theory.