International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 4, April 2014, PP 361-365
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

On Left Quasi Noetherian Rings

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Abstract: *In this paper we prove; If R is a left quasi-Noetherian ring ,then every nil subring is nilpotent) . Next we show that a commutative semi-prime quasi-Noetherian ring is Noetherian. Then we study the relationship between left Quasi-Noetherian and left Quasi-Artinian, in particular we prove that If R is a non-nilpotent left Quasi-Artinian ring. Then any left R-module is left Quasi-Artinian if and only if it is left Quasi-Noetherian. Finally we show that a commutative ring R is Quasi-Artinian if and only ifR is Quasi-Noetherian and every proper prime ideal of R is maximal.*

Keywords: *Noetherian and Artinian , Left Quasi-Noetherian and Left Quasi-Artinian Rings.*

1. Introduction

By a ring we mean an associative ring that need not have an identity. Following [1] we saythat a left R-Module M is *left quasi-Noetherian* if for every ascending chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of R-submodules of M,there exists $m \in \mathbb{Z}^+$ such that$R^m(\bigcup_n N_n) \subseteq N_m$. We say that the ring R is a left quasi-Noetherian ring if ${}_RR$is quasi-Noetherian. Note that any left Noetherian ring or module is a left quasi-Noetherian.Also any nilpotent ring is a left quasi-Noetherian, however $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & 0 \end{bmatrix}$ is a non-nilpotent ring which is a left quasi-Noetherian but not Noetherian.

Note that : if M is a left quasi-Noetherian module and N is a submodule of M, then M/N is a left quasi-Noetherian[1. Proposition 1.3]

Proposition 1.1:

Let R be a left quasi-Noetherian, $I \lhd R$. Then I is a left quasi-Noetherian.

Proof:

Let $J_1 \subseteq J_2 \subseteq \cdots$ be any ascending chain of left ideals of I, then $IJ_1 \subseteq IJ_2 \subseteq \cdots$ is an ascending chain of left ideals of R.But R is a left quasi-Noetherian so there exists $m \in \mathbb{Z}^+$ such that $I^m(\bigcup_n IJ_n) = I^{m+1}(\bigcup_n J_n) \subseteq J_m \subseteq J_{m+1}$. Hence I is a left quasi-Noetherian.

Now : If $I \lhd R$ and $I\,, R/I$ are left quasi-Noetherian then R need not be a left quasi-Noetherian ring, as the following example shows: Let $= \begin{bmatrix} \mathbb{Z}\mathbb{Q} \\ 0 \quad \mathbb{Q} \end{bmatrix}$, $I = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} \lhd$R , hence I and $R/I = \mathbb{Z} \oplus \mathbb{Q}$ are left quasi-Noetherian but R is not, however we can prove the following :

Let $I \lhd R$,then$R$ is a left quasi-Noetherian if one of the following holds:

(a) $R/I$is a left quasi-Noetherian and if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots \subseteq I$ and $I_i \lhd R$ then there exists $m \in \mathbb{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$.

(b) R/I is a left quasi-Noetherian and I is a left Noetherian.

Proof:

(a) Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be any ascending chain of left ideals of R .Then $I_1 \cap I \subseteq I_2 \cap I \subseteq \cdots \subseteq I_n \cap I \subseteq \cdots \subseteq I$ then there exist $m \in \mathbb{Z}^+$ such that $R^m(\bigcup_n I_n \cap I) \subseteq I_m \cap I$. Also $\frac{I_1+I}{I} \subseteq \frac{I_2+I}{I} \subseteq \cdots \subseteq \frac{I_n+I}{I} \subseteq \cdots$ is an ascending chain of left ideals of R/I. But R/I is a left quasi-

Noetherian ring so there exists$m \in \mathbb{Z}^+$ such that $(R/I)^m(\bigcup_n \frac{I_n+I}{I}) \subseteq \frac{I_m+I}{I}$ which implies that $R^m(\bigcup_n I_n + I) \subseteq I_m + I$. Now $R^m(\bigcup_n I_n) \subseteq R^m(\bigcup_n I_n + I) \cap (\bigcup_n I_n) \subseteq (I_m + I) \cap (\bigcup_n I_n) = (\bigcup_n I_n \cap I) + I_m$ so $R^{2m}(\bigcup_n I_n) \subseteq R^m\big((\bigcup_n I_n \cap I) + I_m\big) = R^m(\bigcup_n I_n \cap I) + R^m I_m \subseteq (I_m \cap I) + I_m = I_m$Therefore $R^{2m}(\bigcup_n I_n) \subseteq I_m \subseteq I_{2m}$. Hence is a left quasi-Noetherian.

(b) Can be prove by the same way.

Proposition 1.3:

A finite direct sum of left quasi-Noetherian rings is a left quasi-Noetherian.

Proof:

By induction, it is enough to prove the result for t = 2.So let $= R_1 \oplus R_2$, R_1, R_2 are left Quasi-Noetherian. Now let$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$be any ascending chain of left ideals of R . Then $R_1 I_1 \subseteq R_1 I_2 \subseteq \cdots \subseteq R_1 I_n \subseteq \cdots$ is an ascending chain of left ideals of R_1 and $R_2 I_1 \subseteq R_2 I_2 \subseteq \cdots \subseteq R_2 I_n \subseteq \cdots$ is an ascending chain of left ideals of R_2.But R_1 and R_2 are leftQuasi-Noetherianrings , therefore there exists$m \in \mathbb{Z}^+$such that ${R_1}^m(\bigcup_n R_1 I_n) \subseteq R_1 I_m \subseteq I_m$ and ${R_2}^m(\bigcup_n R_2 I_n) \subseteq R_2 I_m \subseteq I_m$ .Hence $R^{m+1}(\bigcup_n I_n) \subseteq {R_1}^m(\bigcup_n R_1 I_n) + {R_2}^m(\bigcup_n R_2 I_n) \subseteq I_m \subseteq I_{m+1}$ .Therefore R is a left Quasi-Noetherian ring

An ideal Q in a ring R is said to be a *semi-prime ideal* if and only if $A^2 \subseteq Q$, $A \lhd R$, then $A \subseteq Q$, it follows easily by induction that if Q is a semi-prime ideal in R and $A^n \subseteq Q$ for an arbitrary positive integer n, then $A \subseteq Q$[15,P.67]

A ring R is said to be *regular* if for each element $a \in R$ there exist some $a' \in R$ such that $aa'a = a$. Note that a commutative ring R is regular if and only if every ideal of R is semiprime [5, P.186].

By the nil radical N=N(R) of a ring R we mean the sum of all nilpotent ideals of R, which is a nil ideal. It is well known [10. P.28. Theorem 2],that$N$ is the sum of all nilpotent left ideals of R and it is the sum of all nilpotent right ideals of R.

A ring R is said to be a *left Goldie ring* if:

(a)R satisfies the a.c.c on left annihilator ideals. (b) R has no infinite direct sum on left ideals. We can prove the following:

Proposition 1.4:

If R is a left quasi-Noetherian ring and $r(R) = 0$, then R is a left Goldie ring.

Proof:

First we show that any ascending chain of left annihilator idealsterminates. Let$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$be any ascending chain of left annihilator ideals of R . Suppose

that$J_i = l(I_i)$for all i. Since$R$ is a left quasi-Noetherian ring then there exists$m \in \mathbb{Z}^+$such that$R^m(\bigcup_n J_n) \subseteq J_m = l(I_m)$, therefore $R^m(\bigcup_n J_n)I_m = 0$, and $R(R^{m-1}(\bigcup_n J_n)I_m) = 0$But $r(R) = 0$,hence$R^{m-1}(\bigcup_n J_n)I_m = 0$.Continuing in this way we have $R(\bigcup_n J_n)I_m = 0$, therefore $(\bigcup_n J_n)I_m = 0$, and $\bigcup_n J_n \subseteq l(I_m) = J_m$. Hence $J_m = J_{m+1} = \cdots$and the chain terminates.

Nowlet $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be any ascending chain of complement left ideals of R . Since R is a left Quasi-Noetherian ring then there exists $m \in \mathbb{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$.Now suppose that I_m is a complement of$J_m$then$(R^m(\bigcup_n I_n)) \cap J_m \subseteq I_m \cap J_m = 0$. But $(\bigcup_n I_n) \cap J_m \subseteq \bigcup_n I_n$and $(\bigcup_n I_n) \cap J_m \subseteq J_m$, hence $R^m((\bigcup_n I_n) \cap J_m) \subseteq R^m(\bigcup_n I_n)$ and $R^m(\bigcup_n I_n) \cap J_m) \subseteq (\bigcup_n I_n) \cap J_m \subseteq J_m$.Therefore

$R^m\big((\bigcup_n I_n) \cap J_m\big) \subseteq \big(R^m(\bigcup_n I_n)\big) \cap J_m = 0$ and $R(R^{m-1}\big((\bigcup_n I_n) \cap J_m\big)) = 0$. But $r(R) = 0$ hence $R^{m-1}\big((\bigcup_n I_n) \cap J_m\big) = 0$.Continuing in this way we have$(\bigcup_n I_n) \cap J_m = 0$, and bymaximalityof$I_m$ we have $\bigcup_n I_n = I_m$. Hence $I_m = I_{m+1} = \cdots$. Therefore R is a left Goldie ring.

Following [2] we say thata left R-Module M is *left quasi-Artinian* if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ of R-submodules of M ,there exists $m \in \mathbb{Z}^+$ such that$R^m N_m \subseteq N_n$for all n. we say that the ring R is a left quasi-Artinian ring if ${}_R R$is quasi-Artinian. Now we proof the following:

Proposition 1.5:

Any semi-prime left quasi-Artinian ringis a semi-simple left Artinian

Proof:

By[2, Theorem 2.4] every non-zero left ideal of R is generated by a non-zero idempotent e, say.But we know that e acts as right identity for the left ideal $I = Re$, and since R is itself an ideal, hence R has an identity element. Therefore R is left Artinian. Now, $J(R)$ is nilpotent, and R is a semi-prime ring, implies that $J(R) = 0$. Hence R is a semi-simple.

2.

In this section we prove the following

Theorem 2.1:

Let R be a left quasi-Noetherian ring. Then every nil subring of $R$is nilpotent.

Proof:

Since $R \supseteq R^2 \supseteq \cdots \supseteq R^n \supseteq \cdots$, it follows that $r(R) \subseteq r(R^2) \subseteq \cdots \subseteq r(R^n) \subseteq \cdots$ is an ascending chain of ideals of R. But R is a left quasi-Noetherian ring hence there exists $m \in \mathbb{Z}^+$ such that $R^m(r(R^t)) \subseteq r(R^m)$ for all t .Therefore $R^{2m}\left(r(R^t)\right) \subseteq R^m r(R^m) = 0$, and $r(R^t) \subseteq r(R^{2m})$ so that $r(R/r(R^{2m})) = 0$. But $\bar{R} = R/r(R^{2m})$ is a left quasi-Noetherian hence $\bar{R}$ is a left Goldie ring.By Laniski Theorem [14] any nil subring $\bar{S}$ of $\bar{R}$ is nilpotent so there exists $n \in \mathbb{Z}^+$ such that $\bar{S}^n = \bar{o}$ and then $S^n \subseteq r(R^{2m})$so $S^{n+2m} = 0$. Hence S is nilpotent subring of R

An immediate consequence we have the following:

Corollary 2.2:

Let Rbe a left quasi-Noetherian ring , then $N(R)$ is nilpotent

Theorem 2.3:

If R is a left quasi-Noetherian ring. Then R satisfies the ascending chain condition on semi-prime ideals.

Proof:

Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be any ascending chain of semi-prime ideals of R .Then there exists $m \in \mathbb{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$. But $\bigcup_n I_n \lhd R$, hence $(\bigcup_n I_n)^m \subseteq R^m$ and$(\bigcup_n I_n)^{m+1} = (\bigcup_n I_n)^m(\bigcup_n I_n) \subseteq R^m(\bigcup_n I_n) \subseteq I_m$. But I_m is a semi-prime ideal, hence $(\bigcup_n I_n) \subseteq I_m$ so $I_m = I_{m+1} = \cdots$.

Corollary 2.4:

If R is a commutative regular quasi-Noetherian ring. Then R is Noetherian.

Proof:

Since R is a commutative regular ring it follows that every ideal of R is semi-prime. But R is quasi-Noetherian hence by (Theorem 2.3) R is Noetherian ring.

Theorem 2.5:

Let R be a commutative semi-prime quasi-Noetherian ring. Then R is Noetherian.

To prove this we need the following lemma

Lemma 2.6:

If R is a left quasi-Noetherian ring so R has a finite number of minimal prime ideals of R.

Proof:

By [1, Corollary 3.8] There exists a finite number of prime ideals $P_1, P_2, \dots, P_n$ of R such that $\prod_{i=1}^{n} P_i = 0$. Now let P be any minimal prime ideal of R so $\prod_{i=1}^{n} P_i \subseteq P$ therefore $P_i \subseteq P$ for some $i$but P is minimal so $P = P_i$ hence there exists a finite number of minimal prime ideals of R.

Proof of theorem 2.5:

LetP_1 be a minimal prime ideal in$R$, P_2 is a minimal prime ideal of P_1(isolated prime of P_1) so $P_1 \subseteq P_2$(P_1 is maximal prime ideal in P_2) continuing in this way we have $P_1 \subseteq P_2 \subseteq \cdots$ (*) is an ascending chain of prime ideals of R. But$R$ is a quasi-Noetherian ring so (*) terminates(by Theorem 2.3) , therefore there exists $t \in \mathbb{Z}^+$ such that $P_n = P_t$ for all $n \geq t$, so P_t is a maximal prime ideal in R. Now we can write P_2 as $P_2 = P_1 \oplus P_2/P_1$, since $P_2 \lhd R$ then P_2 is a quasi-Noetherian(by proposition 1.1) also since P_1 is a maximal prime ideal in P_2 so P_2/P_1 contains no non-zero prime ideal, therefore every factor ofP_2/P_1(Otherwise if $T = P_2/P_1$ and $\bar{T} = T/I$, $I \lhd T$, has a non-zero prime ideal say $\bar{J}$ so $\pi^{-1}(\bar{J})$ is a prime ideal in T where $\pi: T \to T/I$ is a natural homomorphism, which mean that $\pi^{-1}(\bar{J}) = 0$ then $\bar{J} = \pi(\pi^{-1}(\bar{J})) = \pi(0) = \bar{0}$. Hence T has no non-zero prime ideal). Therefore every factor of P_2/P_1 is a semi-prime quasi-Noetherain. Hence by Proposition 1.4every factor of P_2/P_1 is a Goldie ring and by Camilo's Theorem P_2/P_1 is a Noetherian ring

Now $R = P_t \oplus R/P_t$, $P_t = P_{t-1} \oplus P_t/P_{t-1}$,P_{t-1} is maximal prime ideal in P_t and so on.

Therefore $R = P_1 \oplus P_2/P_1 \oplus P_3/P_2 \oplus \dots \oplus P_t/P_{t-1} \oplus R/P_t$ and R/P_t , P_i/P_{i-1} for all $i = 1, \dots, t$ are Noetherian. Hence $R/P_1 \cong P_2/P_1 \oplus P_3/P_2 \oplus \dots \oplus P_t/P_{t-1} \oplus R/P_t$ is a finite direct sum of Noetherian rings so it is Noetherian. By Lemma 2.6 R has a finitenumber of minimal prime ideals therefore$N(R) = \bigcap_{i=1}^{n} P_i$,P_i minimal prime ideal in R ,but R is a semi-prime ring hence $N(R) = 0$ and $R \cong R/N(R) \cong \oplus_{i=1}^{n} R/P_i$ is Noetherian.

3.

In this section we study the relationship between left quasi-Noetherian and left quasi-Artinian. In particular we prove the following :

Theorem 3.1:

If R is a non-nilpotent left quasi-Artinian ring. Then any left R-module is a left quasi-Artinian if and only if it is a left quasi-Noetherian.

Proof:

Since R is a non-nilpotent it follows that $R \neq N(R)$ .But R is a left quasi-Artinian, hence the nil radical $N(R)$ =N is nilpotent.Therefore $N^t = 0$for some t. Now let $_RM$be any left quasi-Artinian left R-module. This has a chain of submodules $M \supseteq NM \supseteq N^2M \supseteq \cdots \supseteq N^tM = 0$ which factor modules $F_k = N^{k-1}M/N^kM$, k = 1,…,t .Now F_k is annihilated by N hence maybe regarded as an R/N-module. Since R is a left quasi-Artinian ring so R/N is a semi-prime left quasi-Artinian and by [2, Theorem] R/N is a semi-simple Artinian so by[15, proposition 2,pg 68] R/N is completely reducible, hence F_k is completely reducible as an R/N-module and therefore also as an R-module so since F_k is a unital left quasi-Artinian R/N-module so F_k is a left Artinian as an R/N-module then F_k is the direct sum of finite number of irreducible R-modules, hence F_k is Noetherian and then left quasi-Noetherian. Thus $F_t = N^{t-1}M/N^tM = N^{t-1}M$ and $F_{t-1} = N^{t-2}M/N^{t-1}M$ are left quasi-Noetherian , hence so is $N^{t-2}M$ .Continuing in this way we have M is a left quasi-Noetherian R-module.

To prove the converse replace Noetherian instead of Artinian

Theorem 3.2:

Let R be a commutative ring. Then R is quasi-Artinian if and only if R is quasi-Noetherian and every proper prime ideal of R is maximal

Proof:

Since R is commutative so $(R) = rad(R) = \bigcap_i P_i$, P_i is minimal prime ideal of R , $rad(R)$ denoted the prime radical of R. Let R be a quasi-Noetherian ring so by (Lemma 2.6) R has a finite number of minimal prime ideals of R so $N(R) = \bigcap_{i=1}^{n} P_i$.

Now $R \cong N(R) \oplus R/N(R)$ but $N(R/N(R)) = \bar{0}$ so $\bar{R} \cong \bar{R}/N(\bar{R}) \cong \oplus_{i=1}^{n} \bar{R}/\bar{P_i}$ and $\bar{R}/\bar{P_i}$ prime ring. Since every prime ideal of R is maximal so also in $\bar{R} = R/N(R)$ then each of $\bar{P_i}$ is maximal ideal in $\bar{R}$ therefore $\bar{R}/\bar{P_i}$ simple rings so quasi-Artinian and hence $\bar{R} = R/N(R)$ is a quasi-Artinian ring. Since $N(R)$ is nil ideal of R and R is a quasi-Notherian ring so $N(R)$ is nilpotent then quasi-Artinian hence R is quasi-Artinian ring.

To prove the converse let R be a quasi-Artinian ring. $R \cong N(R) \oplus R/N(R)$, $N(R)$ is nilpotent ring so quasi-Noetherian ring and $R/N(R)$ is a semi-prime quasi-Artinian ring so it is a semi-simple Artinian ring therefore quasi-Noetherian and hence R is a quasi-Noetherian ring.

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