International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 3, March 2014, PP 262-268
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

On Transitivity and Mixing of G-maps

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Abstract: *Let X be a $G-$space, $f: X \to X$ be a $G-$map. In this paper, it is shown if f is weakly $G-$mixing then f_n is $G-$transitive for all integers $n \geq 1$, and two sufficient and necessary conditions for f to be weakly $G-$mixing are given.*

Keywords: *$G-$space, $G-$map, $G-$transitivity, weakly $G-$mixing.*

1. Introduction

Let (X, d) be a metric space, $f: X \to X$ be a continuous map, and the iterates f^n are defined inductively by

$$f^1 = f, f^{n+1} = f(f^n) \quad (n \geq 1).$$

We also take f^0 to be the identity map, defined by $f^0(x) = x$ for each $x \in$ X. Evidently, f^n is also a continuous map of X into itself. The map $f \times \cdots \times f$ (with $n$times f) on the product space X^n is denoted by f_n.

By a topological transformation group we mean a triple (G, X, θ), where G is a topological group, X is a Hausdorff topological space and$\theta: G \times X \to X$ is a map such that:

(i)$\theta\big(g, \theta(h, x)\big) = \theta(gh, x)$ for all g, $h \in G$, and $x \in X$;

(ii) $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G.

The mapθ is called an action of Gon X. Ifθ is an open map, then G is called an open transformation. The space X together with a given actionθ of G is called a $G-$space ([1]). A continuous map $h: X \to X$ between two $G-$spaces is called a $G-$map, if $h(\theta(g, x)) = \theta(g, \mathrm{h(x)})$ for each $g \in G$ and each $x \in X$.

The dynamical properties of G-maps have been studied by several authors in recent years (see [2-7]). In [2], Ruchi Das and Tarun Das defined the transitivity on $G-$spaces and gave an example to show that a $G-$transitive map need not to be transitive. In [3], Ruchi Das defined the

G −transitive subset of X and proved that the following three statements are equivalent: (i) A is a G −transitive subset of (X,f), (ii) If V_A is a non-empty open subset of A and U is a non-empty open subset of X with $U \cap A \neq \emptyset$, then there exist $n \in N$ and $g \in G$ such that $V_A \cap \theta(g, f^{-n}(U) \neq \emptyset$, (iii) If U is a non-empty open subset of X with $U \cap A \neq \emptyset$ then $\bigcup\{\theta(g, f^{-n}(U)|n \in N, g \in G\}$ is dense in A. In [4], Ruchi Das introduced the notion of G −expansiveness and gave the sufficient and necessary condition for f to be G −expansive. In [5], T.Choi and J.Kim proved the decomposition theorem on G −spaces. Let $(X,d),(Y,\tilde{d})$ be two G −spaces and$F = \{f_k\}_{k=1}^{\infty}$, $H = \{h_k\}_{k=1}^{\infty}$ be two sequences of maps on X, Y respectively. If there exists an equivariant uniform homeomorphism $t: X \to Y$ such that F and H are t −conjugate, then F is G −chaotic implies H is G −chaotic ([6]).

We say that f is transitive if for every pair of non-empty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. f is said to be weakly mixing if f_2 is transitive. In [8], Liao Gongfu proved that the following three statements are equivalent: (i) f is weakly mixing, (ii) For any non-empty open subsets U and V there is a $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ and $f^n(V) \cap V \neq \emptyset$. (iii) For any non-empty open subsetsU, V and W there is a integer $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$and $f^n(V) \cap W \neq \emptyset$.

In this paper, we introduce the definitions of weakly G −mixing and G −mixing, and provethat f is weakly G −mixing if and only if for any non-empty open sets U,V and W of X,there exist $n \in N$ and $g \in G$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$ and $\theta(g, f^n(U)) \cap W \neq \emptyset$.

It is well known that f is topological mixing implies f is topological weakly mixing. In this paper, an example is given to show that f is G −mixing doesn't imply f is weakly G −mixing.

2. G −TRANSITIVITY AND G −MIXING OF G −MAPS

Definition 2.1 ([2] Definition 3.1.) Let X be a metric $G-$ space and $f: X \to X$ be a continuousmap. $f$is called G −transitive if for every pair of non-empty open subsets $U$and V of X,there exist $n \in N$ and $g \in G$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$.

Definition 2.2 Let X be a G −space and $f: X \to X$ be a continuous map. $f$is called weakly G −mixing if f_2 is G −transitive.

Definition 2.3 Let$X$ be a G −space and $f: X \to X$ be a continuous map. $f$is called G −mixing if for every pair of non-empty open subsets$U$ and V of X, there exist $N \geq 0$ and $g \in G$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$ for all $n \geq N$.

Proposition 2.4 Let X be a G-space and $f: X \to X$ be a G-map. The following are equivalent:

(i) f is G −transitive.
(ii) For every pair of non-empty open subsets$U$ and $V$of X, there exist $n \in N$ and $g \in G$ such that $U \cap \theta(g, f^{-n}(V)) \neq \emptyset$.
(iii) For any non-empty open subset $U$of X, $\bigcup\{\theta(g, f^{-n}(U)): n \in N, g \in G\}$ is dense in$X$.

Proof. The proof is similar to that of the Theorem 3.4 in [3], and is omitted. ■

Proposition 2.5 Let X be a $G-$ space and $f: \mathrm{X} \to \mathrm{X}$ be a $G-$ map, where G is an open transformation. If f is weakly G −mixing, then f_n is G −transitive for all integers $n \geq 1$.

Proof. We prove this proposition by induction on k.

By the definition of weakly G −mixing, f_2 is G −transitive. Thus, f_1 is G −transitive.

Assume that for $k \geq 2$, f_k is G −transitive. Let $U_1, \cdots, U_k, U_{k+1}, V_1, \cdots, V_k, V_{k+1}$ be any $2(k+1)$

non-empty open subsets of X. Since f_2 is $G-$transitive, by Proposition 2.4, there exist $l \in N$ and $g' \in G$ such that

$U_k \cap \theta(g', f^{-l}(U_{k+1})) \neq \emptyset$and $V_k \cap \theta(g', f^{-l}(V_{k+1})) \neq \emptyset$.

Let $U_0 = U_k \cap \theta(g', f^{-l}(U_{k+1}))$and $V_0 = V_k \cap \theta(g', f^{-l}(V_{k+1}))$. Since f is a $G-$map and G is an open transf- ormation, both U_0 and V_0 are non-empty open subsets of X. By the induction hypothesis, there exist $n \in N$ and $g \in G$ such that

$\theta(g, f^n(U_i)) \cap V_i \neq \emptyset$, for each $i = 0,1,2,\cdots, k-1$.

Noting that

$\theta(g, f^n(U_k)) \cap V_k \supset \theta(g, f^n(U)) \cap V \neq \emptyset$,

we have

$\theta(g, f^n(U_{k+1})) \cap V_{k+1} \supset \theta(g, f^n(\theta((g')^{-1}, f^l(U_0)))) \cap \theta((g')^{-1}, f^l(V_0))$

$\supset \theta((g')^{-1}, \theta(g, f^{n+l}(U_0)) \cap f^l(V_0))$

$\supset \theta((g')^{-1}, f^l(\theta(g, f^n(U_0)) \cap V_0))$

$\neq \emptyset$.

It follows that f_{k+1} is $G-$transitive. Thus, f_n is $G-$transitive for all integers $n \geq 1$.■

Proposition 2.6 Let X be a $G-$space and $f: X \to X$ be a $G-$map, where G is an open transformation. If f is weakly $G-$mixing, thenf^n is weakly $G-$mixing for all integers $n \geq 1$.

Proof. Put $n \geq 1$ and let U, V, U', V' be non-empty open subsets of X. Since f is continuous,$U, f^{-1}(U), \cdots, f^{-(n-1)}(U), V, f^{-1}(V), \cdots, f^{-(n-1)}(V)$are non-empty open subsets of$X$. It follows from Proposition 2.5 that f_n is $G-$transitive. Therefore, there exist $k \in N$ and $g \in G$ such that

$U' \cap \theta(g, f^{-(k+i)}(U)) \neq \emptyset$and$V' \cap \theta(g, f^{-(k+i)}(V)) \neq \emptyset$ for all $0 \leq i \leq n-1$.

This means that there is i_0, $1 \leq i_0 \leq n-1$, such that $k + i_0$ is multiple of n. Assume $k + i_0 = np$, we have

$U' \cap \theta(g, f^{-np}(U)) \neq \emptyset$and $V' \cap \theta(g, f^{-np}(V)) \neq \emptyset$.

Hence,f^n is weakly $G-$mixing. ■

Theorem 2.7 Let X be a $G-$space and $f: X \to X$ be a $G-$map, where G is an open transformation. Then f is weakly $G-$mixing if and only if for any non-empty open subsets U and $V$of X, there exist $n \in N$ and $g \in G$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$ and $\theta(g, f^n(V)) \cap V \neq \emptyset$.

Proof. The necessity is obvious, so it is enough to prove the sufficiency.

Let U_1, V_1, U_2, V_2be any non-empty open subsets of X. Since f is $G-$transitive, there exist $n_1 \in N$ and $g_1 \in G$ such that

$A = V_1 \cap \theta\left(g_1, f^{-n_1}(V_2)\right) \neq \emptyset$,

and there exist $n_2 \in N$ and $g_2 \in G$ such that

$B = \theta(g_1, f^{-n_1}(U_2)) \cap \theta(g_2, f^{-n_2}(A)) \neq \emptyset$.

Hence, there exist $n_3 \in N$ and $g_3 \in G$ such that

$\theta(g_3, f^{n_3}(B)) \cap B \neq \emptyset$and $\theta(g_3, f^{n_3}(U_1)) \cap B \neq \emptyset$.

Putting $n = n_2 + n_3$and $g = {g_2}^{-1} g_3$, we have

$\theta(g, f^n(U_1)) \cap V_1 = \theta({g_2}^{-1} g_3, f^{n_2+n_3}(U_1)) \cap V_1$

$\supset \theta({g_2}^{-1} g_3, f^{n_2+n_3}(U_1)) \cap A$

$= \theta({g_2}^{-1} g_3, f^{n_2+n_3}(U_1)) \cap \theta({g_2}^{-1} g_2, f^{n_2}(f^{-n_2}(A)))$

$\supset \theta({g_2}^{-1} g_3, f^{n_2+n_3}(U_1)) \cap \theta({g_2}^{-1}, f^{n_2}(\theta(g_2, f^{-n_2}(A))))$

$\supset \theta({g_2}^{-1}, f^{n_2}(\theta(g_3, f^{n_3}(U_1)) \cap \theta(g_2, f^{-n_2}(A))))$

$\supset \theta({g_2}^{-1}, f^{n_2}(\theta(g_3, f^{n_3}(U_1)) \cap B))$

$\neq \emptyset$.

Noting that$\theta(g_3, f^{n_3}(B)) \cap B \neq \emptyset$, we have

$\emptyset \neq \theta({g_2}^{-1} {g_1}^{-1}, f^{n_2+n_1}(\theta(g_3, f^{n_3}(B)) \cap B))$

$\subset \theta({g_2}^{-1} {g_1}^{-1}, f^{n_2+n_1}(\theta(g_3, f^{n_3}(B)))) \cap \theta({g_2}^{-1} {g_1}^{-1}, f^{n_2+n_1}(B))$

$\subset \theta({g_2}^{-1} {g_1}^{-1}, f^{n_2+n_1}(\theta(g_3, f^{n_3}(\theta(g_1, f^{-n_1}(U_2))))))$
$\cap \ \theta({g_2}^{-1} {g_1}^{-1}, f^{n_2+n_1}(\theta(g_2, f^{-n_2}(A))))$
$= \theta(g, f^n(U_2)) \cap \theta({g_1}^{-1}, f^{-n_1}(A))$

$\subset \ \theta(g, f^n(U_2)) \cap V_2$.

Hence, f is weakly G −mixing. ■

Theorem 2.8 Let X be aG −space and $f: X \to X$ be a G −map, where $G$is open transformation. Then f is weakly G −mixing if and only if for any non-empty open subsets U, V and W of X, there exist $n \in N$ and $g \in G$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$ and $\theta(g, f^n(U)) \cap W \neq \emptyset$.

Proof. The necessity is obvious, so it is enough to prove the sufficiency.

Let U_1, V_1, U_2, V_2be any non-empty open subsets of X. By the hypothesis, there exist $k \in N$ and $g_1 \in G$ such that
$U' = \theta(g_1, f^k(U_1)) \cap U_2 \neq \emptyset$and $V' = \theta(g_1, f^k(U_1)) \cap V_2 \neq \emptyset$,

so

$U = U_1 \cap \theta({g_1}^{-1}, f^{-k}(U_2)) \neq \emptyset$and $V = U_1 \cap \theta({g_1}^{-1}, f^{-k}(V_2)) \neq \emptyset$.

Hence, there exist $n \in N$ and $g \in G$ such that

$\theta(g, f^n(U)) \cap V \neq \emptyset$and$\theta(g, f^n(U)) \cap V_1 \neq \emptyset$.

It follows that

$\theta(g, f^n(U_1)) \cap V_1 \supset \theta(g, f^n(U)) \cap V_1 \neq \emptyset$.

Noting that $\theta(g, f^n(U)) \cap V \neq \emptyset$, we have

$\emptyset \neq \theta(g_1, f^k(\theta(g, f^n(U)) \cap V))$

$\subset \ \theta(g_1, f^k(\theta(g, f^n(U)))) \cap \theta(g_1, f^k(V))$

$\subset \ \theta(g, f^n(\theta(g_1, f^k(U)))) \cap V_2$

$\subset \theta(g, f^n(U_2)) \cap V_2$.

Hence, f is weakly G −mixing. ■

3. Examples

It is obvious that f is transitive implies f is G −transitive.

Under trivial action of G on X, G − transitivity coincides with transitivity. However, under non-trivial action of G on X, the G − transitivity does not imply the transitivity. In [2], the following example was given to show that the G −transitivity does not imply the transitivity.

Example 3.1 ([2] Example 3.3) Let $X = \left\{\pm\frac{1}{n}, \pm\left(1-\frac{1}{n}\right) | n \in N\right\}$ under usual metric. Consider action of Z_2, additive group of integers mod 2, on X given by $\theta(0,t) = t$ and $\theta(1,t) = -t,\ t \in X$. Map $f: X \to X$ defined by

$$f(x) = \begin{cases} x_+ & \text{if } x \in \left\{\frac{1}{n}, 1-\frac{1}{n} \middle| n \neq 1, n \in N\right\}, \\ x_- & \text{if } x \in \left\{-\frac{1}{n}, -\left(1-\frac{1}{n}\right) \middle| n \neq 1, n \in N\right\}, \\ x & \text{if } x \in \{-1,0,1\}. \end{cases}$$

Where x_+ denotes element of X immediate to right of x, x_- denotes element of X immediate to left of x.

Remark. Example 3.1 can't be used to show that the G −transitivity does not imply the transitivity. In fact, let $U = (\frac{1}{4}, \frac{1}{2}) \cap X$ and $V = (\frac{1}{9}, \frac{1}{7}) \cap X$. For every $n \geq 1, \theta(1, f^n(U)) \cap V = \emptyset$ and $\theta(0, f^n(U)) \cap V = f^n(U) \cap V = \emptyset$. Hence, f is neither Z_2-transitive nor transitive.

Inspired by the Example 3.1, we give the following example to show that the G −transitivity does not imply the transitivity.

Example 3.2 Let $f: [-1,1] \to [-1,1]$ be defined by

$$f(x) = \begin{cases} -2x-2 & \text{if } -1 \leq x \leq -\frac{1}{2}, \\ 2x & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ -2x+2 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Consider action of Z_2, additive group of integers mod 2, on $[-1,1]$ given by $\theta(0,t) = t$ and $\theta(1,t) = -t, t \in [-1,1]$. Then f is Z_2-transitive, but f is not transitive.

Proof. It follows by $f([0,1]) = [0,1]$ that f is not transitive. It is easy to see that $h_1 = f|_{[0,1]}$ and $h_2 = f|_{[-1,0]}$ are transitive.

Let U, V be any non-empty open subsets of [-1,1] and let $U_1 = U \cap (0,1)$, $V_1 = V \cap (0,1)$, $U_2 = U \cap (-1,0)$, $V_2 = V \cap (-1,0)$.

Case 1. $U_1 \neq \emptyset$ and $V_1 \neq \emptyset$. Since h_1 is transitive, there exists $n \in N$ such that ${h_1}^n(U_1) \cap V_1 \neq \emptyset$. Thus $\theta(0, f^n(U)) \cap V \supset \theta(0, {h_1}^n(U_1) \cap V_1) \neq \emptyset$.

Case 2. $U_2 \neq \emptyset$ and $V_2 \neq \emptyset$. Since h_2 is transitive, there exists $n \in N$ such that ${h_2}^n(U_2) \cap V_2 \neq \emptyset$. Thus $\theta(0, f^n(U)) \cap V \supset \theta(0, {h_2}^n(U_2) \cap V_2) \neq \emptyset$.

Case 3. $U_1 \neq \emptyset$ and $V_2 \neq \emptyset$. Since h_1 is transitive, there exists $n \in N$ such that ${h_1}^n(U_1) \cap \theta(1, V_2) \neq \emptyset$. Thus $\theta(1, f^n(U)) \cap V \supset \theta(1, f^n(U) \cap \theta(1, V)) \supset \theta(1, {h_1}^n(U_1) \cap \theta(1, V_2)) \neq \emptyset$.

Case 4. $U_2 \neq \emptyset$ and $V_1 \neq \emptyset$. Since h_2 is transitive, there exists $n \in N$ such that ${h_2}^n(U_2) \cap \theta(1, V_1) \neq \emptyset$. Thus $\theta(1, f^n(U)) \cap V \supset \theta(1, f^n(U) \cap \theta(1, V)) \supset \theta(1, {h_2}^n(U_2) \cap \theta(1, V_1)) \neq \emptyset$.

It follows by Case 1-4 that there exist $n \in N$ and $g \in Z_2$ such that $\theta(g, f^n(U)) \cap V \neq \emptyset$. Thus f is Z_2-transitive, but f is not transitive. ■

Remark. Under trivial action of G on X, if f is G −mixing then f is weakly G −mixing. But the following example shows that the G −mixing doesn't imply the weakly G −mixing.

Example 3.3 Let f and Z_2 be defined as Example 3.2. Then f is Z_2 −mixing but not weakly Z_2 −mixing.

Proof. The proof of Z_2 −mixing is similar to that of Z_2 −transitivity, and is omitted. The following we will show that f is not weakly Z_2 −mixing.

Let $U_1 = (-1,1)$, $V_1 = (0, \frac{1}{3})$, $U_2 = (\frac{1}{3}, \frac{2}{3})$, $V_2 = (\frac{2}{3}, 1)$. Since $\theta(0, f^n(U_1)) \subset [-1,0]$ and

$\theta(1, f^n(U_2)) \subset [-1,0]$ for all integers $n \geq 0$, then

$$\theta(0, f^n(U_1)) \cap V_1 = \emptyset \text{ and } \theta(1, f^n(U_2)) \cap V_2 = \emptyset$$

for all integers $n \geq 0$. Hence f is not weakly Z_2-mixing. ■

4. Conclusion

In this paper we study some properties and two equivalent conditions of G −weakly mixing.

Acknowledgements

This work is supported by NSFC (11161029), NSFGX (2013GXNSFBA 019020) and Education Science Foundation of Guangxi (2013ZD061).

References

[1] G. Bredon. Introduction to compact transformation Groups.Academic, 1972.

[2] R. Das, T. Das. Topological Transitivity of Uniform Limit Functions on G −spaces. Int. Journal of Math. Analysis, 30:1491-1499,2012.

[3] R.Das. Transitive Subsets on G-spaces.Applied Mathematical Sciences,92:4561-4568,2012.

[4] R.Das. Expansive self-homeomorphisms on G − spaces.PeriodicaMathematicaHungarica, 31(2):123-130,1995.

[5] T.Choi and J.Kim. Decomposition Theorem on G − spances.Osaka J. Math., 46:87-104,2009.

[6] R.Das. Chaos of a Sequence of Maps in a Metric G −space. Applied Mathematical Sciences, 6(136):6769-6775,2012.

[7] D.Thakkar and R.Das. Topological stability of a sequence of maps on a compact metric space. Bull. Math. Sci., 4:99-111,2014.

[8] G.F.Liao, L.D.Wang and Y.C.Zhang. Transitivity,mixing and chaos for a class of set-valued. Science in China Series A Mathematies,49 :1-8,2006.

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