International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 3, March 2014, PP 249-255
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Establishing the Galois Group of a Polynomial Equation the Roots of which are not Known

Andrei Nicolaide
Professor, D.Sc.
Faculty of Electrical Engineering and Computer Science
Department of Electrical Engineering and Applied Physics
Transilvania University of Brasov
Brasov, Romania
andrei.nicolaide@gmail.com

Abstract: *A procedure for establishing the Galois group of an equation the roots of which are not known will be developed and applied. The procedure is based on the Galois transform of the given equation, namely a polynomial equation. The steps are as follows: the roots of the transform equation are expressed in terms of those of the given equation using the Galois procedure. The values of the roots of the transform equation being not known should be used in relations in which only the values of the symmetric functions of the given equation will intervene. This step not found in the known works has been accomplished in the present work. Then the binomial product representing in this form the transform equation will be obtained. Solving this equation, factors of a degree smaller that those of the transform equation, and sometimes of the given equation permits to construct the resolvent and Galois group. The use of a symbolic programming language, like Maple 12 software, simplifies very much the construction of the program.*

Keywords: *Galois Theory, Transform equation, Factorization, Galois group, Resolvent.*

1. INTRODUCTION

From the several methods for establishing the group in the case of an equation the roots of which are not known; in the present work, a procedure starting from a method proposed and used by Gustave Verriest, will be developed. Although, the procedure is presented in his book [1, p. 176] cannot be easily used. The reason is that the author introduced some symbols for the not known roots, and the values of the same number of free constants, the values of which will be determined so that no multiple roots exist. At the same time, in the transformed equation, the free constants intervene, but the supposed roots do not appear, what is not justified. A possibility could be, the author had introduced for the roots the values of the starting equation, but in this case, the assumed condition that the roots of the starting equation are not known is no more fulfilled. This is the principal reason that we aim to analyze and solve the considered problem. Moreover, an advantage of the present procedure is the use of symbolic language, namely Maple 12 software, which permitted to simplify the computation and shorten the working time, not possible at the time when the method was proposed.

2. THE EXAMINED PROBLEM

Many works have been devoted to this subject, among which [1]-[15]. For the sake of clarity, we shall consider an example from [1], namely, the following equation [1, p. 176], with rational coefficients:

$$f(x) = x^3 - 3x^2 + 1 = 0 . \qquad (1)$$

We shall denote the not known roots of this equation, by ξ_1, ξ_2, ξ_3, and we shall use the Galois transform [4, p. 319], [8, p. 38], the roots of them will be denoted by η_i, $i \in [1, N]$.

This transform is obtained by replacing the given equation, by another one, in the form of a binomial product, by replacing the roots of the given equation, by a rational linear function, obtaining now $N = n!$ roots, if the given equation has n roots. In the present case, we have $N = 3! = 6$. The relation between the roots of the given equation and the Galois transform, in matrix form, may be written as:

$$[\eta] = [\lambda]\,[\xi]. \tag{2}$$

In expanded form, like in [1], there follows the results of Table I:

Table 1. *Relations between the roots of an equation and its Galois transform.*

$\eta_1 = \lambda_1\,\xi_1 + \lambda_2\,\xi_2 + \lambda_3\,\xi_3$	$\eta_2 = \lambda_1\,\xi_2 + \lambda_2\,\xi_3 + \lambda_3\,\xi_1$	$\eta_3 = \lambda_1\,\xi_3 + \lambda_2\,\xi_1 + \lambda_3\,\xi_2$
$\eta_4 = \lambda_1\,\xi_1 + \lambda_2\,\xi_3 + \lambda_3\,\xi_2$	$\eta_5 = \lambda_1\,\xi_2 + \lambda_2\,\xi_1 + \lambda_3\,\xi_3$	$\eta_6 = \lambda_1\,\xi_3 + \lambda_2\,\xi_2 + \lambda_3\,\xi_1$

The Galois transform will be of the form:

$$F = \prod_{i=1}^{N} \left(w - \eta_i\right), \tag{3}$$

where the current variable is w. Replacing in (3), we get:

$$\begin{aligned} &F(w) = \text{expand}\left(\left(w-\eta_1\right)\cdot\left(w-\eta_2\right)\cdot\left(w-\eta_3\right)\cdot\left(w-\eta_4\right)\cdot\left(w-\eta_5\right)\cdot\left(w-\eta_6\right)\right).\\ &W = F(w)\,. \end{aligned} \tag{4}$$

3. The Coefficients of the Transformed Equation

For the quantities of Table I, a problem occurs. We do not know the values of the roots of the given equation, but it may be possible to use the expressions that intervene in terms of the coefficients of the given equation without using its roots. At the same time, the indetermined introduced coefficients have to be chosen, one condition is to avoid the Galois transform have multiple roots, and the other is to facilitate the solving of the transform equation, which already has a greater number of roots. We shall adopt $\lambda_1 = 1; \lambda_2 = -1; \lambda_3 = 0$. Under these conditions, there results that the transform equation will have only even terms.

3.1. The Coefficient of the Term w^0

We want to calculate the value of the coefficient of w^0. For this purpose we shall divide all the terms of this coefficient, into several partial sums of like terms, each of them containing the factors in increasing order. First, we write all terms of this coefficient,

$$\begin{aligned} &-2\,\xi_1\xi_3^3\xi_2^2 - 2\,\xi_2^3\xi_3\xi_1^2 - 2\,\xi_2^3\xi_3^2\xi_1 - \xi_1^4\xi_2^2 + 2\,\xi_2^4\xi_3\xi_1 + 2\,\xi_1^4\xi_2\xi_3 + 2\,\xi_1\xi_3^4\xi_2 - 2\,\xi_1^3\xi_2^2\xi_3\\ &+2\,\xi_2^3\xi_3^3 + 6\,\xi_2^2\xi_3^2\xi_1^2 - \xi_2^4\xi_1^2 - 2\,\xi_2\xi_3^3\xi_1^2 - \xi_2^2\xi_3^4 + 2\,\xi_1^3\xi_3^3 - \xi_1^2\xi_3^4 + 2\,\xi_1^3\xi_2^3 - \xi_1^4\xi_3^2 - \xi_2^4\xi_3^2\\ &-2\,\xi_1^3\xi_3^2\xi_2\,. \end{aligned} \tag{5 a}$$

We shall sort in like elements of the form:

$$-\xi_1^3\xi_2^2\xi_3\,;\; +\xi_1\xi_2^4\xi_3\,;\; -\xi_1^4\xi_3^2\,;\;\; \xi_1^4\xi_2\xi_3 + \xi_1\xi_2^4\xi_3 + \xi_1\xi_2\xi_3^4\,;\; -\xi_1^2\xi_3^4 - \xi_1^4\xi_3^2 \ldots,$$

therefore:

$$
\begin{aligned}
T = &-2\left(\xi_1^2\xi_2^3\xi_3 + \xi_1^3\xi_2^2\xi_3 + \xi_1\xi_2^2\xi_3^3 + \xi_1\xi_2^3\xi_3^2 + \xi_1^2\xi_2\xi_3^3 + \xi_1^3\xi_2\xi_3^2\right) \\
&+ 2\left(\xi_1^4\xi_2\xi_3 + \xi_1\xi_2^4\xi_3 + \xi_1\xi_2\xi_3^4\right) + 2\left(\xi_1^3\xi_2^3 + \xi_1^3\xi_3^3 + \xi_2^3\xi_3^3\right) + 6\,\xi_1^2\xi_2^2\xi_3^2 \\
&-\left(\xi_1^2\xi_3^4 + \xi_1^4\xi_3^2 + \xi_1^2\xi_2^4 + \xi_1^4\xi_2^2 + \xi_2^2\xi_3^4 + \xi_2^4\xi_3^2\right).
\end{aligned} \tag{5 b}
$$

For calculating the sums in closed form of these symmetrical functions, we must have in view that the present case is only few related to the general theory of symmetric functions, not found in known books or in [1], and must be directly considered. For this reason, we shall separate the sums into independent partial sums. Successively, we have:

$$T1 = -2\,\xi_1\xi_2\xi_3\left(\xi_1\xi_2^2 + \xi_1^2\xi_2 + \xi_2\xi_3^2 + \xi_2^2\xi_3 + \xi_1\xi_3^2 + \xi_1^2\xi_3\right), \tag{6}$$

$$T1 = -2\,\xi_1\xi_2\xi_3\left((\xi_1 + \xi_2 + \xi_3)(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) - 3\,\xi_1\xi_2\xi_3\right), \tag{7}$$

$$T2 = 2\left(\xi_1^4\xi_2\xi_3 + \xi_1\xi_2^4\xi_3 + \xi_1\xi_2\xi_3^4\right) + 6\,\xi_1^2\xi_2^2\xi_3^2 = 2\,\xi_1\xi_2\xi_3\left(\xi_1^3 + \xi_2^3 + \xi_3^3\right) + 6\,\xi_1^2\xi_2^2\xi_3^2. \tag{8}$$

In the case, in which the terms of the partial sums are relatively complicate, we shall use a notation without indices, and at the end, in the established summing formula, we shall replace the new symbols in terms of the previously indexed symbols. For the parentheses with cubic quantities, we put: $\xi_1 = a;\quad \xi_2 = b;\quad \xi_3 = c$, and we have:

$$
\begin{aligned}
(a+b+c)^3 &= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3 \\
&= 3(ab+ac+bc)(a+b+c) + a^3 + b^3 + c^3 - 3abc,
\end{aligned} \tag{9}
$$

$$\xi_1^3 + \xi_2^3 + \xi_3^3 = (\xi_1 + \xi_2 + \xi_3)^3 - 3(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)(\xi_1 + \xi_2 + \xi_3) + 3\,\xi_1\xi_2\xi_3.$$

$$T3 = -\left(\xi_1^2\xi_2^4 + \xi_1^4\xi_2^2 + \xi_1^2\xi_3^4 + \xi_1^4\xi_3^2 + \xi_2^2\xi_3^4 + \xi_2^4\xi_3^2\right), \tag{10}$$

$$T3 = -\left(ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c\right),$$

$$T3 = -\left(ab(a+b+c) + ac(a+b+c) + bc(a+b+c) - 3abc\right), \tag{11}$$

$$T3 = -(ab+ac+bc)(a+b+c) + 3abc,$$

$$T3 = -\left(\left(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2\right)\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) - 3\,\xi_1^2\xi_2^2\xi_3^3\right), \tag{12}$$

$$
\begin{aligned}
&\left(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2\right) = (\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)^2 - 2\,\xi_1\xi_2\xi_3(\xi_1 + \xi_2 + \xi_3), \\
&\xi_1^2 + \xi_2^2 + \xi_3^2 = (\xi_1 + \xi_2 + \xi_3)^2 - 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3).
\end{aligned} \tag{13 a, b}
$$

$$T4 = 2\left(\xi_1^3\xi_2^3 + \xi_1^3\xi_2^3 + \xi_2^3\xi_3^3\right). \tag{14}$$

We put: $\xi_1 = a;\quad \xi_2 = b;\quad \xi_3 = c$.

We shall expand $F = (ab+ac+bc)^3 = (\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)^3$, its value being considered, because, in this way, we can obtain an expression close to $T4$, and at the same time, we change the variables by using the Maple 12 commands: expand, exponent and multivariate functions, in the following steps:

$$F = a^3b^3 + 3a^3b^2c + 3a^2b^3c + 3a^3bc^2 + 6a^2b^2c^2 + 3ab^3c^2 + a^3c^3 + 3a^2bc^3 + 3ab^2c^3 + b^3c^3, \tag{15}$$

$$F1 = 3a^3b^2c + 3a^2b^3c + 3a^3bc^2 + 3ab^3c^2 + 3a^2bc^3 + 3ab^2c^3 + 6a^2b^2c^2. \tag{16}$$

$$F1 = 3abc\left(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2\right) + 6a^2b^2c^2. \tag{17}$$

$$F1(a,b,c) = 3abc\left((ab + ac + bc)(a + b + c) - 3abc\right) + 6a^2b^2c^2. \tag{18}$$

In the present case, we have the following given data:

$$\xi_1 + \xi_2 + \xi_3 = 3; \quad \xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 = 0; \quad \xi_1\xi_2\xi_3 = -1. \tag{19}$$

Therefore $F1 = -3; \quad F = 0$.

We obtain the following results: $T1 = 6$; $T2 = -42$; $T3 = -6 \cdot 9 + 3 = -51$; $T4 = 6$; and the coefficient of w^0 will be the amount of the partial sums above, hence -81.

3.2. The Coefficient of the Term w^2

We want to calculate the value of the coefficient of w^2. For this purpose we shall divide all the terms of this coefficient, into several partial sums of like terms, each of them containing the factors in increasing order. First, we write all terms:

$$(-2\xi_1\xi_3^3 - 2\xi_2^3\xi_1 - 2\xi_2\xi_3^3 - 2\xi_1^3\xi_2 + \xi_1^4 + 3\xi_2^2\xi_3^2 - 2\xi_2^3\xi_3 + 3\xi_1^2\xi_3^2 - 2\xi_1^3\xi_3 + 3\xi_2^2\xi_1^2 + \xi_3^4 + \xi_2^4) \cdot w^2. \tag{20}$$

Now, we shall sort in like elements:

$$F = -2\left(\xi_1\xi_2^3 + \xi_1^3\xi_2 + \xi_1\xi_3^3 + \xi_1^3\xi_3 + \xi_2\xi_3^3 + \xi_2^3\xi_3\right) + 3\left(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2\right) + \xi_1^4 + \xi_2^4 + \xi_3^4. \tag{21}$$

We denote:

$$\xi_1 = a; \quad \xi_2 = b; \quad \xi_3 = c. \tag{22}$$

$$F = -2\left(ab^3 + a^3b + ac^3 + a^3c + bc^3 + b^3c\right) + 3\left(a^2b^2 + a^2c^2 + b^2c^2\right) + \left(a^4 + b^4 + c^4\right). \tag{23}$$

$$F = -2\left(a^3 + b^3 + c^3\right)(a + b + c) + 3\left(a^4 + b^4 + c^4\right) + 3\left(a^2b^2 + a^2c^2 + b^2c^2\right). \tag{24}$$

$$A1 = a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc).$$
$$A2 = \left(a^2b^2 + a^2c^2 + b^2c^2\right) = (ab + ac + bc)^2 - 2abc(a + b + c). \tag{25}$$

$$A3 = a^3 + b^3 + c^3 = (a + b + c)^3 - 3(ab + ac + bc) \cdot (a + b + c) + 3abc.$$
$$A4 = a^4 + b^4 + c^4 = \left(a^2 + b^2 + c^2\right)^2 - 2\left(a^2b^2 + a^2c^2 + b^2c^2\right). \tag{26}$$

Using formula (24) and expressions $A1 - A4$, we obtain:

$$F = -2\left(a^3 + b^3 + c^3\right)(a + b + c) + 3\left(a^4 + b^4 + c^4\right) + 3\left(a^2b^2 + a^2c^2 + b^2c^2\right). \tag{27}$$

$$F = F(\xi_1, \xi_2, \xi_3). \tag{28}$$

Replacing the known values, we find:

$$\begin{aligned}
&a^3 + b^3 + c^3 = 24;\\
&a + b + c = 3;\\
&a^4 + b^4 + c^4 = 81 - 12;\\
&a^2b^2 + a^2c^2 + b^2c^2 = 6;\\
&F = -2 \cdot 24 \cdot 3 + 3 \cdot 69 + 3 \cdot 6.
\end{aligned} \tag{29}$$

The value of the coefficient of w^2 is 81.

3.3. The Coefficient of the Term w^4

We want to calculate the value of the coefficient of w^4. For this purpose we shall divide all the terms of this coefficient, into several partial sums of like terms, each of them containing the factors in increasing order. First, we write all terms:

Now, we shall sort in like elements:

$$\begin{aligned}
&2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) - 2(\xi_1^2 + \xi_2^2 + \xi_3^2) = 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)\\
&-2\left((\xi_1 + \xi_2 + \xi_3)^2 - 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)\right).
\end{aligned} \tag{30}$$

There follows:

$$2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) - 2(\xi_1^2 + \xi_2^2 + \xi_3^2) = 6(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) - 2(\xi_1 + \xi_2 + \xi_3)^2. \tag{31}$$

Replacing the known data, we find the value of the coefficient w^4 as -18.

4. Solving the Galois Transform Equation

Considering equations (4), its expansion in terms of w, and the coefficients calculated in the three preceding paragraphs, we get:

$$w^6 - 18w^4 + 81w^2 - 81 = 0. \tag{32}$$

We shall prove if the obtained equation can be split into irreducible components, using Maple procedure. The factorization and solutions are given below:

$$\text{factor}(w^6 - 18w^4 + 81w^2 - 81); \quad (w^3 - 9w - 9)(w^3 - 9w + 9) \tag{33}$$

$$9.770700706, \quad -3.393328744, \quad -6.377371962 \tag{34}$$

$$1.184792531, \quad 2.226681597, \quad -3.411474128 \tag{35}$$

The solutions, having in view their permutations, are presented in Table II.

Therefore, we obtained two equations of third degree, irreducible in number fields $\boldsymbol{Q}$ and $\boldsymbol{R}$. The number of roots of the Galois transform is 6, whereas that of the given equation is 3. It is interesting to see to which of the two equations does correspond with the solution of the given equation. An analysis of this subject has been carried out in [1], but we consider that the best is to make further on, some simple examination.

5. Galois Group

We can now write the group of the given equation, according to Galois meaning: **Table 2.** *The order of the three roots of the given equation*

ξ_1	ξ_2	ξ_3
ξ_1	ξ_2	ξ_3

ξ_1	ξ_2	ξ_3
ξ_2	ξ_3	ξ_1

ξ_1	ξ_2	ξ_3
ξ_3	ξ_1	ξ_2

Since the solution of the equation (4) is, according to Table I, of the form $\eta_1 = \xi_1 - \xi_2$, there follows that if $\xi_1 > \xi_2$, we have $\eta_1 > 0$. If all η_i have same sign, their product will be positive, and the last term of the equation should be negative, hence it is the first equation, of the two irreducible factors. Let the roots of the given equation in ξ be: $3; 2; -1$. There follows the values of η: $(3-2); (2-(-1)); \quad ((-1)-3)$, hence $:1; 3; -4$. It is obvious that this situation does not correspond to the first factor, but to the second factor, with the roots product of the Galois transform negative, and last term positive, and for the given equation with the roots product negative and the last term positive. Therefore, this is the resolvent of the given equation. In [3], we found the following Galois group, for the equation (1):

$$"3T1",\{"A(3)"\},"+",3,\{"(123)"\} \tag{36}$$

hence another form of the Galois group of the equation.

The meaning of the notation is the following: a. “3T1” first group in the list of degree 3, transitive group. b. A set of strings giving the description of the group. c. A string indicating the parity (signature) of the group, here even group. d. Here, order 3 of the group. e. Set of generators (of permutations) in disjoint cycle notation.

6. Conclusion

A procedure for the calculation of the resolvent and the Galois group of an equation, the roots of which are not known, are developed. For this purpose, the Galois transform is used. At the difference relatively to other methods, the case of equations, the roots of which are not known, is deeply analyzed. The usage of a symbolic language, namely Maple 12 facilitates the programming activity and shorten the working time.

References

[1] G. Verriest, Leçons sur la Théorie des Équations selon Galois, précédées d’une Introduction à la Théorie des Groupes. Gauthier-Villars, Imprimeur-Éditeur, Paris, 1939.
[2] D.A. Cox, Galois Theory, Second Edition, John Wiley & Sons, 2012
[3] ***Maple - 12 Handbook.
[4] Th. Anghelutza, Curs de Algebră superioară (Cours of higher Algebra), Vol. II, Editura Universității din Cluj, 1945.
[5] D. Cox, Évariste Galois, Solvable permutation groups, Amherst College, Bilbao, May, 2012, p. 41.
[6] J.-P. Tignol, Galois Theory of Algebraic Equations, World Scientific Publishing Co. Pte. Ltd., 2011.
[7] E. Artin, supplemented by A.N. Milgram, Galois Theory, Second Edition.
[8] H. Edwards, Galois Theory, New York, Berlin, Heidelberg, 1984.
[9] É. Galois, Mémoire sur les conditions de résolubilité des équations par radicaux, Auteur:: Évariste Galois (1811-1832). Publication: Mémoire manuscrit de 1830, publication dans le Journal de mathématiques pures et appliquées, pp. 417-433. Année de publication:1830. Nombre de pages:18.
[10] C. Erhardt, Le mémoire d’Évariste Galois sur les conditions de résolubilité par radicaux (1831).

[11] E. Picard, Traité d'Analyse, Tome III, Troisième édition, Gauthier-Villars et C-ie, Paris, 1928.
[12] D. Grieser, Grundideen der Galoistheorie, Oldenburg, 2007.
[13] D. Kalman, J.E. White, Polynomial equations, Circulant Matrices, Mathematical Association of America, Monthly 108, November 2001, pp. 821-840.
[14] A. Baker, An Introduction to Galois Theory, School of Mathematics and Statistics, University of Glascow, 2013.
[15] H.U. Besche, B. Eick, E. O'Brien, The groups of order at most 2000, Electron. Res. Announc. Amer. Math. Soc., 7 (2001).

AUTHOR'S BIOGRAPHY

Andrei Costin Nicolaide was born on the 1st of September 1933 in Bucharest. He received the degree of Electrical Engineer with honours, from Technical Institute of Craiova, Faculty of Electrotechnics (1956); Doctor of Engineering and Doctor of Sciences (Polytechnic Institute of Bucharest, in 1962 and 1974, respectively). Full professor at the Transilvania University of Brasov (1969-2003), consulting professor since 2004. His scientific activity includes field computation by conformal transformation and numerical methods, and Special and General Theory of Relativity. He is a regular member of the Academy of Technical Sciences of Romania.