International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 2, February 2014, PP 180-185
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Effect on the Flow of Bile Due to Radially Varying Viscosity in the Cystic Duct

Suchi Agarwal
Department of Mathematics
Dayalbagh Educational Institute
Agra, India
suchissrai@gmail.com

S.P.Singh
Department of Mathematics
Dayalbagh Educational Institute
Agra, India

Abstract: *The flow of bile in the elastic cystic duct during the emptying phase is studied. The bile behaves as a non-Newtonian fluid and power law fluid model is taken in account to study the flow of bile in the duct. The work has been done to know the impact of various parameters on the flow pattern of bile. It is observed that the flow of bile increases with the increase in flow index parameter and with the increase in radius of the duct. However, with the increase in viscosity of gallbladder bile the flow of bile in the duct decreases.*

Keywords: *Bile, Gallbladder, Viscosity, non-Newtonian fluid.*

1. Introduction

The presence of gallstone is one of the most common human biliary disease which is also known as Cholelithiasis. Clinical treatments for the diseases often involve the surgical removal of the gallbladder, known as cholecystectomy, which is the most often performed abdominal operation in the West. In order to understand the cause of biliary diseases, it is important to understand the mechanical behavior of the human biliary system. The role of human biliary system is to create, store, transport and release bile into the duodenum to aid digestion of fats. It consists of an organ and duct system. Bile, the liquid that flow in the biliary system is a yellow-brown fluid produced by the liver and is composed of three main components – cholesterol, bile salts and bilirubin. When the gallbladder is not functioning properly, components of bile are supersaturated leading to the formation of solid crystals called gallstones. Most common biliary diseases are: Cholelithiasis - the presence of gallstones and Cholecystitis – the inflammation of gallbladder.

Jungst et al [1] suggested that the supersaturation and rapid nucleation of cholesterol in bile were of key importance in the pathogenesis of cholesterol gallstones. Ooi et al[2] presented a numerical study of steady flow in human cystic duct models. Idealized models were constructed, first with staggered baffles in a channel to represent the valves of Heister and lumen. Al-Atabi et al[3] presented a mathematical model accounting for the effects of geometry of the baffle configurations .It predicted the pressure drop in circular pipe fitted with segmental baffles. Bird et al[4] described the variation in geometry of the cystic duct, obtained from acrylic resin casts of the neck and first part of the cystic duct in gallbladders removed for gallstones disease and obtained from patients undergoing partial hepatectomy for metastatic disease. Li et al[5] worked on two one-dimensional models to estimate the pressure drop in the normal human biliary system for Reynolds number up to 20. Excessive pressure drop during bile emptying and refilling may result in incomplete bile emptying, leading to bile stasis and subsequent gallstones formation. The effects of biliary system geometry, elastic property of cystic duct and bile viscosity on pressure drop was studied. It was found that the maximum pressure drop occurs during bile emptying immediately after a meal, and was greatly influenced by the viscosity of bile and the geometric configuration of the cystic duct. Luo et al[6] researched to understand the physiological and pathological functions of the biliary system. It was believed that the mechanical factors play an important role in the mechanism of the gallstone formation and biliary diseases. Li et al[7] extended their previous study of the human biliary system to include two new factors :the non–Newtonian properties of bile, and elastic deformation of the cystic duct. A one – dimensional (1D) model was analyzed and compared with three– dimensional fluid structure interaction simulations. It was found that non – Newtonian bile raises resistance to the flow of bile, which can be increased enormously by the elastic deformation(collapse) of the cystic duct.

2. Formulation and Solution of the problem

The bile behaves as a non-Newtonian fluid [7] and power law fluid model is taken in account to study the flow of bile in the duct. It is assumed that the viscosity of bile is varying radially. The bile under observation is incompressible and the flow is steady and laminar. It is assumed that the bile enters at the inlet pressure and leaves it with lower pressure, while pressure outside the duct is P_0. As the consequence of the pressure difference the duct may expand or contract, and hence the shape of its cross-section may deform due to the elastic property of the wall. The duct is long and the pressure is a function of z alone. As the gallbladder bile is layered so there is a change in viscosity along the radius. It was found that the bile in the gallbladder is in the form of layers [5].The effect of flow index parameter, viscosity and other parameters on flow pattern of bile in the duct is studied. The equation governing the flow of bile is given by [8]

$$-\frac{dP}{dz} = \frac{1}{r}\frac{d(r\tau)}{dr} \tag{1}$$

The constitutive equation is given by [8]

$$-\frac{du_z}{dr} = \left(\frac{\tau}{\mu}\right)^{1/n} = f(\tau) \tag{2}$$

In equation (1) and (2), z is the co-ordinate along the axis of the cystic duct in the flow direction, r is the co-ordinate in the radial direction and perpendicular to bile flow, τ represents the shear stress of bile taken as power-law fluid and P is the pressure at any point, u_z stands for the axial velocity of bile, $-\frac{dP}{dz}$ is the pressure gradient, n is the flow behavior index of the fluid, μ is the viscosity. Also, the viscosity μ is the function of r/R_0, where R_0 is the radius of the cystic duct. Therefore, viscosity is given by

$$\mu = \mu(r) = \mu_1\left(1 - \frac{kr}{R_0}\right) \tag{3}$$

We are assuming that the viscosity variation along the radius direction is linear, where, μ_1 is the viscosity of fluid at $r = 0$ and k is a constant parameter which is very less then unity ($k << 1$).

2.1. Boundary Conditions

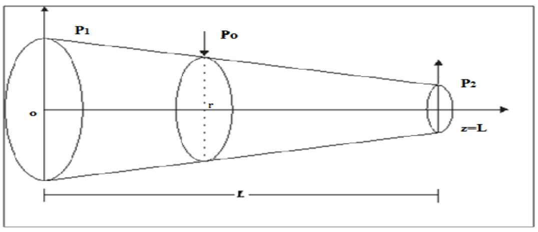
The equation (1) to (3) are solved using these boundary conditions–

a) τ is finite at $r = 0$ (4)

b) $u_z = 0$ at $r = R_0$ (5)

c) $P(z) - P_0 = P_1 - P_0$ at $z = 0$ and $P(z) - P_0 = P_2$ at $z = L$ (6)

The equations (4) and (5) are called as regularity and no-slip condition,



where R_0 is the radius of cystic duct, P_0 is the external pressure on the duct. P_1 and P_2 are the pressure at the inlet and outlet of duct, L is the length of duct. The expression $P(z) - P_0$ is called transmural pressure. The pressure $P(z)$ is a function of z alone such that $P(z)$ (at $z = 0$) = P_1 and $P(z)$ (at $z = L$) = P_2.The model of cystic duct is shown with the help of Fig. 1.

Fig1. *The flow through cystic duct (variation in radius due to pressure).*

The equations (1) – (3) are solved using boundary condition (4)–(6).Thus, the axial velocity is given by

$$u_z = \left(\frac{1}{2\mu} p\right)^{1/n} \frac{1}{\left(\frac{1}{n}+1\right)}\left(R_0^{(1/n)+1} - r^{(1/n)+1}\right) \text{, where } p=-dP/dz \qquad (7)$$

The flux is given by [8]

$$Q = \int_0^{R_0} u_z \cdot 2\pi r\, dr$$

$$Q = \left(\frac{p}{2\mu}\right)^{1/n} \left(\frac{n\pi}{3n+1}\right) R_0^{(1/n)+3} \qquad (8)$$

Now, let us denote the tension per unit length of duct and per unit thickness as T. Consequently, as a good approximation we can treat the wall as thin elastic membrane) and is balanced by net upward force per unit length which is given by

$$Th = r(P - P_0) \qquad (9)$$

Also, according to Hook's law, we have

$$\text{Tension} = \frac{E(r - R_0)}{R_0} = kx = T \qquad (10)$$

Also k is the constant factor and x is the extension produced by the force, where r is R_0 in equilibrium position when the tension $T = 0$

From equation (9) and using (10) we get

$$P(z) - P_0 = \frac{Eh}{R_0}\left(1 - \frac{R_0}{r}\right) \qquad (11)$$

From equation (11) we get

$$r = \frac{R_0 Eh}{Eh\left(1 - \frac{R_0 P'}{Eh}\right)} = \frac{R_0}{\left(1 - \frac{R_0 P'}{Eh}\right)} \qquad (12)$$

Where $P' = P(z) - P_0$ and $r = f(P - P_0) = r(P - P_0) = r(P')$.

The equation (8) becomes

$$Q = \frac{M}{b}\frac{1}{\mu^{1/n}} \int_{P_2 - P_0}^{P_1 - P_0} r^{3+1/n}\, dP' \qquad (13)$$

Where $M = \frac{n\pi}{3n+1}$ and $b = \frac{1}{(2L)^{1/n}}$

Considering that the viscosity variation along the radius is linear we have

$$\mu = \mu(r) = \mu_1\left(1 - k\frac{r}{R_0}\right) \qquad (14)$$

Where $k << 1$

Substituting the value from equation (14) in equation (13) we get

$$Q = \frac{n\pi}{(3n+1)(2L)^{1/n}\mu_1^{1/n}} R_0^{3+1/n}\left(1+\frac{k}{n}\right)(P_1 - P_2) \tag{15}$$

The equation (15) gives the volume flux of bile in the elastic cystic duct due to linearly varying viscosity. The expression shows the dependence of flow on the flow index parameter n and on k. However, if we assume that there is no radial variation of viscosity i.e. $\mu = \mu_1$ (and k=0) and also the flow index parameter is n=1, we get

$$Q = \frac{\pi}{8\mu L} R_0^4 (P_1 - P_2)$$

which is in accordance with the Poiseuille's law.

3. Results and Discussion

All The graphs obtained show the variation in flow with changing pressure difference. Fig.2 (a) illustrates the effect of pressure difference with changes in flow index parameter n. It is clearly observed that the flow is increasing with the increasing pressure difference and the flow is maximum for maximum value of n i.e. n =1.5. As the flow index parameter decreases the flow of bile in the duct decreases sharply.

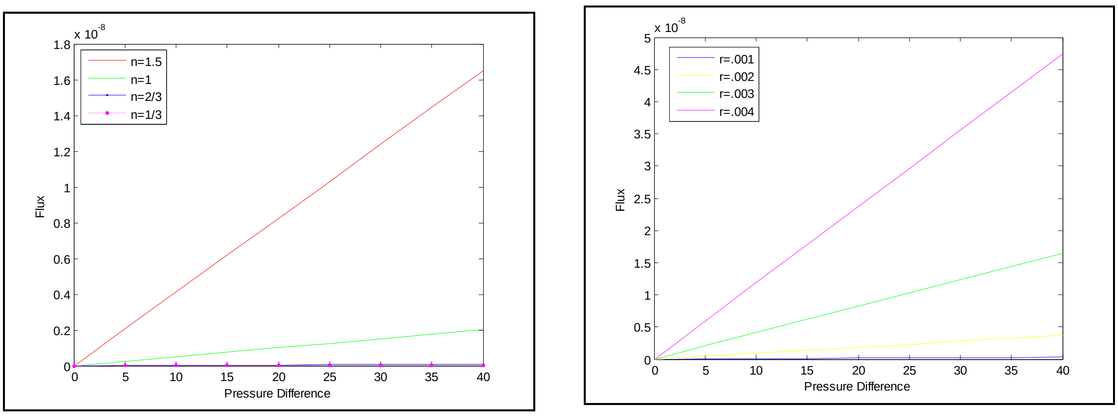
Fig.2 (b) depicts the effect on the flow of bile in the cystic duct due to changing radius. With the increase in radius of the duct the flow of bile increases and is maximum at the highest radius.

Fig.3 shows the effect of pressure difference on the flow of bile with change in parameter k. The change in value of k causes no effect on the flow pattern of bile in the duct. The flow of bile is same for all the four values of k = 0.001, 002, .003 and .004. Fig.4 gives the variation in flow with changing pressure difference for different values of viscosity. It is clearly seen that the flow attains maximum value for least value of viscosity. As the viscosity of bile increases the flow of bile in the duct decreases.

Fig.2. *The variation of the flow with changing pressure difference for different values of*

(a) flow index parameter n

(b) for different values of radius r.



(a) (b)

Fig.3. the variation of the flow with changing pressure difference for different values of k.

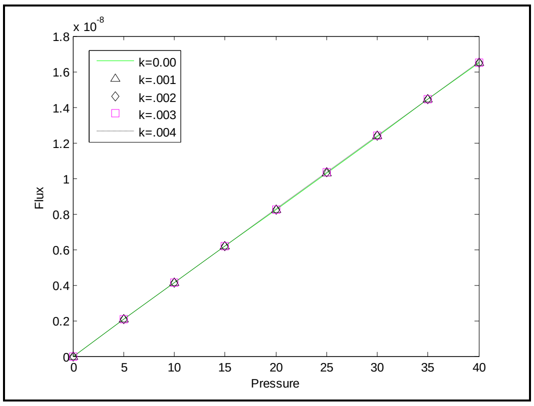
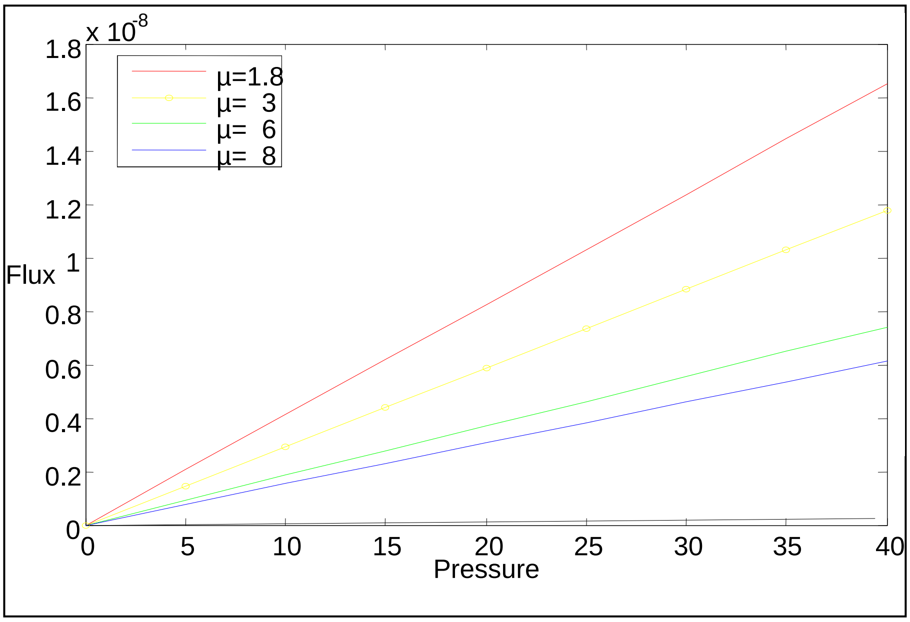


Fig.4: The variation of the flow with changing pressure difference for different values of viscosity.



4. Conclusion

The study under consideration throws light on the flow of bile in the cystic duct during the emptying phase. The rheology of bile is non-newtonian and fluid behaves as the power law fluid. It is assumed that the viscosity of bile is varying radially. A thorough investigation has been done to investigate the impact of various parameters on the flow pattern of bile. As the gallbladder bile is layered so there is a change in viscosity along the radius. It is observed that the flow of bile increases with the increase in flow index parameter n.The increase in radius of the duct increases the flow of bile in the cystic duct. The flow patterns remain the same for different values of k. Moreover, the increase in viscosity of gallbladder bile decreases the flow of bile. The deductions emphasizes on the fact that the decreasing flow rate of bile can serve as an indication of gallbladder malfunction. With the study of factors affecting the flow behavior of bile in the biliary system, the preventive measures could be prescribed to patients at risk to gallstones.

References

[1] Jungst, D., Lang, T., Huber, P., Lang, V. and Paumgartner, G., "Effect of phospholipids and bile acids on cholesterol nucleation time and vesicular/micellar cholesterol in gallbladder bile of patients with cholesterol stones", Journal of lipid research Vol.34, 1993.

[2] Ooi, R. C., Luo, X. Y., Chin, S., Johnson, A. and Bird, N. C., "The flow of bile in the human cystic duct", Journal of biomechanics 37: 1913-1922, 2004.

[3] Al-Atabi, M. A., Chin, S. B. and Luo, X. Y., "Pressure drop in laminar and turbulent flows in circular pipe with baffles – An experimental and analytical study", International journal of fluid mechanics research ,Vol. 33,No. 4 , 2006.

[4] Bird, N. C., Ooi, R., Chin, S. B. and Johnson, A., “Investigation of the functional three-dimensional anatomy of the human cystic duct: A single helix”, Clinical anatomy 19; 528-534, 2006.
[5] Li, W., Chin, S. B., Johnson, A. and Bird, N. C., “One dimensional models of the human biliary system”, Vol.129, 2007.
[6] Luo, X. Y., Li, W., Bird, N., Chin, S. B. and Johnson, A., “On the mechanical behavior of the human biliary system”, World J. Gastroentero March 7; 13(9):1384- 1392, 2007.
[7] Li, W., Luo, X. Y., Chin, S. B., Hill, N. A., Johnson, A. and Bird, N. C., “Non- Newtonian bile flow in elastic cystic duct: one and three dimensional modeling”, Annals of Biomedical Engineering, Vol.36, No.11, 2008.
[8] Kapur, J.N. Mathematical Models in Biology and Medicine. Affiliated East-West Press LTD.

AUTHORS’ BIOGRAPHY

Suchi Agarwal has completed her M.Sc in Mathematics from Chaudhary Charan Singh University (Meerut) and M.phil from Dayalbagh Educational Institute, Deemed University, Agra. She is presently pursuing her Ph.D in the area of Bio-mathematics.

S.P.Singh has been working as a Professor of mathematics in the Dayalbagh Educational Institute, Deemed University for more than fifteen years. His specialization is in the area of fluid dynamics. He was appointed as editorial board member for good number of journals. He is presently guiding several students in the areas of blood flow and gallbladder malfunctioning.