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Some Elementary Problems from the Note Books of Srinivasa Ramanujan-Part (III)

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Problem 27: **NBSR Vol II p 390**

Sum and product of primes.

$1+2=3$; $2+3=5$; $1+5=2.3$; $3+7=2.5$;

$1+2.7=3.5$; $2.5+11=3.7$; $3.5+7=2.11$; $2+3.11=5.7$;

$2.3.7+13=5.11$; $3.5.11+17=2.7.13$ *and* $1+2.3.7.17=5.11.13$

Note:

1. Each of these above results can easily be verified.
2. Ramanujan, in the above jotting, represented certain products of distinct primes as a sum of a prime (1 is regarded as a prime) and another product of distinct primes.
3. There are several gaps.
   An example: $3+11=2.7$, $(2,3,7,11 \text{ are primes})$ but not consecutive.
4. Ramanujan's Mind:
   Perhaps, Ramanujan is conjecturing that every product of distinct primes can be represented as a sum of a prime (including 1) and another product of distinct primes.

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Ramanujan Constant (R)

The Irrational constant: $R \equiv e^{\pi\sqrt{163}} = 262537412640768743.99999999999925 \ldots$ …. is very close to an integer.

Note: A few rather spectacular examples of this type are given by S.R including the above one and can be generated by employing some amazing properties of the j – function.

Ref: Ramanujan.S.: Modular Equations and Approximations to π –

Quart.J. Pure and Applied Mathematics Vol 45 (1913 – 1914) pp 350 – 352.

A Comment: Martin Gardener (April 1975) played an April-Fool's joke on the readers of Scientific American by claiming that this number was exactly an INTEGER. However, he himself admitted the hoax a few months later (July 1975).

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Problem 28: **NBSR Vol II (Book No. 3) p 390**

$\frac{1}{4}+2=\left(1\frac{1}{2}\right)^2$

$$\frac{1}{4} + 2.3 = \left(2\frac{1}{2}\right)^2$$

$$\frac{1}{4} + 2.3.5 = \left(5\frac{1}{2}\right)^2$$

$$\frac{1}{4} + 2.3.5.7 = \left(14\frac{1}{2}\right)^2$$

……………………

$$\frac{1}{4} + 2.3.5.7.11.13.17 = \left(714\frac{1}{2}\right)^2$$

Note:

1. Each of the above results can be easily verified.
2. Ramanujan's Conjecture, product of successive primes, starting from 2, increased by one quarter $\left(=\frac{1}{4}\right)$ is the square of a number which is an integer increased by $\frac{1}{2}$, Counter example for this equation

   $$\frac{1}{4} + 2.3.5.7.11.13.17.19 \neq \left(an\ integer + \frac{1}{2}\right)^2$$

3. There are two omissions between the last two equalities, stated above.

   $$\frac{1}{4} + 2.3.5.7 \quad and \quad \frac{1}{4} + 2.3.5.7.11.13.17$$

   We note that

   $$2256\tfrac{1}{4} = \left(47\tfrac{1}{2}\right)^2 < \tfrac{1}{4} + 2.3.5.7.11 = 2310\tfrac{1}{4} < \left(48\tfrac{1}{2}\right)^2 = 2352\tfrac{1}{4} \text{ and}$$

   $$29756\frac{1}{4} = \left(172\frac{1}{2}\right)^2 < \frac{1}{4} + 2.3.5.7.11.13 = 30030\frac{1}{4} < \left(173\frac{1}{2}\right)^2 = 30102\frac{1}{4}$$

4. Do there exist infinitely many positive integers {n}such that

   $$\frac{1}{4} + \prod_{j=1}^{n} Pj = \left(\frac{N}{2}\right)^2$$

   where N is a positive odd integer, $Pj = j^{th}$ prime

   $$\therefore \prod_{j=1}^{n} Pj = \left(\frac{N}{2}\right)^2 - \frac{1}{4} = \frac{(N-1)(N+1)}{4} = 2\frac{(N-1)(N+1)}{8}$$

   If $N = 2k + 1$, then

   $$\frac{N^2 - 1}{8} = \frac{k(k+1)}{2} = k^{th}\ triangular\ number\ (T_k)$$

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Problem 29: **NBSR Vol I p 329**

$3 = 2^2(2^2 - 1)/4$

$18 = 3^2(3^2 - 1)/4$

$60 = 4^2(4^2 - 1)/4$ & c.

A few additions

$$150 = 5^2(5^2 - 1)/4$$

$$315 = 6^2(6^2 - 1)/4$$

$$588 = 7^2(7^2 - 1)/4$$

Remark:

1. Ramanujan recognizes the fact that the product of a square number (n^2) and its predecessor $(n^2 - 1)$ is divisible by 4 and quotes a few example in this jotting.
2. In fact $n^2(n^2 - 1)$ for *n>2* is divisible by 24.
3. Probably S.R, while making this entry, may be aware that $\frac{n(n+1)}{2}$ is an integer and might not be sure that $n(n+1)(n+2)$ is divisible by 6. He might have been very young to recognize this fact.

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Problem 30: **NBSR Vol II Top of p 309 and also at the bottom of p 36**

$$\sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)} = \left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)$$

Proof:

$$L.H.S = \sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)}$$

$$= \sqrt{2\frac{3^2-1}{3^2}\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{19}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)}$$

$$= \sqrt{2\left(\frac{8}{3^2}\right)\left(\frac{6}{7}\right)\left(\frac{10}{11}\right)\left(\frac{18}{19}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)}$$

$$= \sqrt{\left(\frac{8}{7}\right)\left(\frac{12}{11}\right)\left(\frac{20}{19}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)}$$

$$= \sqrt{\left(1+\frac{1}{7}\right)^2\left(1+\frac{1}{11}\right)^2\left(1+\frac{1}{19}\right)^2}$$

$$= \left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right) = R.H.S$$

Note: This is a (curious!) simple identity that can be verified by straight forward computation. An arithmetic question would arise. Is this an isolated (such) result or are there other identities of this type; an open problem.

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Problem 31: **NBSR Vol II (Note Book 2) p-346**

$$A + B + C + \ \ldots\ldots\ldots.\&\, c = P + Q + R + \ \ldots\ldots\ldots.\&\, c \qquad (1)$$

$$A^A.B^B.C^C \ldots\ldots\ldots.\&\, c = P^P.Q^Q.R^R \ldots\ldots\ldots.\&\, c \qquad (2)$$

From (2) above, find the quantities first.

Eq. $2^2.6^2 = 3^3.3^3.4^4$

and multiply the result by as many 1 as to satisfy (1)

Eq

$$1^1.1^1.2^2.6^6 = 3^3.3^3.4^3$$

$$1^1.1^1.2^2.3^3.4^4.5^5.6^6 = 3^3.3^3.3^3.4^4.4^4.5^5$$

$$1^1.8^8.9^9 = 3^3.3^3.12^{12}$$

$$1^1.3^3.12^{12}.20^{20} = 5^5.15^{15}.16^{16}$$

$$1^1.4^4.20^{20}.30^{30} = 6^6.24^{24}.25^{25}$$

Note:

1. This is a typical pattern of S.R Familiarity with the laws of indices:
   (i) $a^m.a^n = a^{m+n}$
   (ii) $(a^m)^n = a^{mn}$
   (iii) $(ab)^n = a^n b^n$ and the skill/ability to use these laws are very much needed.

To write the numbers having the curious pattern in the equation (2)

2. The product of the numbers in L.H.S = The product of the numbers in R.H.S
3. The sum of the bases in L.H.S = The sum of the bases in R.H.S
4. Sum of the indices in L.H.S = Sum of the indices in R.H.S

An illustration

$$\begin{aligned} 8^8.9^9 &= (4\times 2)^8 \times (3\times 3)^9 \\ &= 4^8 \times 2^8 \times 3^9 \times 3^9 \\ &= 4^8 \times 3^8 \times 3^1 \times (2^2)^4.3^3.3^6 \\ &= 12^8 \times 4^4 \times 3^4.3^3.3^3 \\ &= 12^8 \times 12^4 \times .3^3.3^3 \\ &= 12^{12} \times 3^6 \end{aligned}$$

L.H.S	R.H.S	Deficiency of L.H.S over R.H.D
Sum of the bases 8 + 9 = 17	12 + 6 = 18	1
Sum of the indices 8 + 9 = 17	18	1

The deficiency is covered by the inclusion of sufficient number (in this case only once) 1.

By this inclusion of 1 once on the L.H.S, we now get

$$1^1.8^8.9^9 = 12^{12} \times 3^6 = 204049226851434$$

Sum of the bases: 1+8+9 =18

Sum of the indices: 12+6 = 18

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Problem 32: **NBSR Vol II p 31**

$If \quad e^{A_1x+A_2\frac{x^2}{2}+A_3\frac{x^3}{3}+ \,\ldots\ldots\&c} = P_0 + P_1x + P_2x^2 + P_3x^3 + \,\ldots\ldots\&c$ then

$P_0 = 1$ and $nP_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \,\ldots\ldots\ldots\&c$ to n terms

Note: The factor "n" on the L.H.S of the result is missing in the manuscript of S.R.

Solution: It is given that

$$e^{A_1x+A_2\frac{x^2}{2}+A_3\frac{x^3}{3}+\,\ldots\ldots} = P_0 + P_1x + P_2x^2 + P_3x^3 + \,\ldots\ldots \tag{1}$$

When $x = 0$, equation (1) reduces to $e^0 = P_0$, $\therefore\ P_0 = 1$ (2)

On taking logarithms to the base e on both the sides of (1), we have

$$A_1 + A_2\frac{x^2}{2} + A_3\frac{x^3}{3} + \,\ldots\ldots = log_e(P_0 + P_2x^2 + P_3x^3 + \,\ldots\ldots)$$

Differentiating both sides with respect to x,

$$A_1 + A_2x + A_3x^2 + \,\ldots\ldots + A_nx^n + \,\ldots\ldots = \frac{P_1 + 2P_2x + 3P_3x^2 + \,\ldots\ldots}{P_0 + P_1x + P_2x^2 + P_3x^3 + \,\ldots\ldots}$$

which on cross multiplying can be written as

$$(P_1 + 2P_2x + 3P_3x^2 + \,\ldots) = (A_1 + A_2x + A_3x^2 + \,\ldots)(P_0 + P_1x + P_2x^2 + P_3x^3 + \,\ldots)$$

(3)

Equating the coefficients of like powers of x in (3), we get

Coefficient of $x^0(= constant\ term)$: $P = A_1P_0 (= A_1)$

Coefficient of x: $2P_2 = A_1P_1 + A_2P_0$

Coefficient of x^2: $3P_3 = A_1P_2 + A_2P_1 + A_3P_0$

Coefficient of x^3: $4P_4 = A_1P_3 + A_2P_2 + A_3P_1 + A_4P_0$ .... and so on.

Coefficient of x^{n-1}: $nP_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \,\ldots\ldots + A_nP_0$

Note: The Coefficient of $= nP_n = \sum_{k=1}^{n} A_k\,P_{n-k}$

The sum given on R.H.S above is the convolution product of the two expressions

$$A = A_1 + A_2 + A_3 + \,\ldots\ldots\ldots + A_n \quad \text{and} \quad P = P_0 + P_1 + P_2 + \,\ldots\ldots\ldots + P_n$$

$$n.P_n = (A * P) = \sum_{k=1}^{n} A_kP_{n-k}$$

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Problem 33: **NBSR Vol II p 369**

$If\ for\ n \geq 0, a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ then $a_m a_n = a_{m+n} + a_{m-n}$

Proof:

Let $Let\ \ \alpha = 2+\sqrt{3}, then\ \beta = 2-\sqrt{3} = \frac{1}{\alpha}$ (1)

Further $a_n = \alpha^n + \frac{1}{\alpha^n}$

$$\therefore\ a_m a_n = \left(\alpha^m + \frac{1}{\alpha^m}\right)\left(\alpha^n + \frac{1}{\alpha^n}\right)$$

$$= \alpha^{m+n} + \alpha^{m-n} + \alpha^{n-m} + \frac{1}{\alpha^{m+n}}$$

$$= \left(\alpha^{m+n} + \frac{1}{\alpha^{m+n}}\right) + \left(\alpha^{m-n} + \frac{1}{\alpha^{m-n}}\right)$$

$$= a^{m+n} + a^{m-n}$$

Hence the result: $a_m a_n = a_{m+n} + a_{m-n}$

Present author's generalization:

1. If $a_n = \alpha^n + \beta^n\ \ with\ \alpha\beta = 1, then\ a_m a_n = a_{m+n} + a_{m-n}$
2. Recurrence formula to compute a_n:

   $a_{m+n} = a_m a_n - a_{m-n}$

From 32.1

$a_0 = (2+\sqrt{3})^0 + (2-\sqrt{3})^0 = 2$

$a_1 = (2+\sqrt{3})^1 + (2-\sqrt{3})^1 = 4$

$a_2 = a_1.a_1 - a_0 = 4 \times 4 - 2 = 14$

$a_3 = a_{2+1} = a_2.a_1 - a_{2-1} = 14 \times 4 - 4 = 52$

$a_4 = a_2^2 - a_0 = 196 - 2 = 194$

Table of value of $\underline{a}$

The recurrence relation $a_m a_n = a_{m+n} + a_{m-n}$ can be used for fast computation of a_n.

n	$a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$	Value of a_n
0	$a_0 = (2+\sqrt{3})^0 + (2-\sqrt{3})^0$	2
1	$a_1 = (2+\sqrt{3})^1 + (2-\sqrt{3})^1$	4
2	$a_2 = a_1.a_1 - a_0$	14
3	$a_3 = a_2.a_1 - a_1$	52

4	$a_4 = a_2^2 - a_0$	194
5	$a_5 = a_3 . a_2 - a_1$	724
6	$a_6 = a_3^2 - a_0$	2702
7	$a_7 = a_4 . a_3 - a_1$	10084
8	$a_8 = a_7 . a_1 - a_6$	37084
9	$a_9 = a_8 . a_1 - a_7$	140452
10	$a_{10} = a_9 . a_1 - a_8$	524174
11	$a_{11} = a_{10} . a_1 - a_9$	1956244
12	$a_{12} = a_{11} . a_1 - a_{10}$	2059604

and so on.

Note: a_n is even for all values of n. This is evident from the definition of a_n.

Since $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

$$= 2\{2^n + n_{C_2} 2^{n-2}(3)^1 + n_{C_4} 2^{n-4}(3)^2 + \ ......\}$$

is a multiple of 2. The last term in the bracket of R.H.S = $3^{n/2}$ or $2n3^{(n-1)/2}$ according as *n* is even or odd.

Note: The next two entries involve double power series with Bernoulli numbers as coefficients and these are beyond the scope of school going pupils.

Notation given on p 56 in Vol II NBSR

$I(p)$: the greatest integer $\leq p)$ i.e., $I(p) =$ integral part of p

Examples: $I(7) = 7$; $I\left(\frac{22}{3}\right) = 7$ and so on. $I(p)$ is also denoted by $[p]$ or $\lfloor p \rfloor$.

$N(p)$: the integer nearest to p.

Examples: $N(7) = 7; N\left(\frac{32}{5} = 6\frac{2}{5}\right) = 6; N\left(\frac{34}{5} = 6\frac{4}{5}\right) = 7; N\left(\frac{50}{7} = 7\frac{1}{7}\right) = 7; N\left(\frac{55}{7} = 7\frac{6}{7}\right)$
$= 8$ and so on.

Note: This definition for $N(p)$ of S.R is ambiguous when $p + 1/2$ is an integer. In such a situation, it is obvious from the sequel that $N(p) = p + \frac{1}{2}$ instead of $-\frac{1}{2}$.

$G(p)$: the least integer $\geq p$

$G(7) = 7$; $G\left(\frac{50}{7} = 7\frac{1}{7}\right) = 8$ and so on.

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Scribbling of S.R **NBSR Vol II p 267**

$$N(p) = I\left(p + \frac{1}{2}\right) \qquad *$$

Problem 34: For each real number p

$$N(p) = I\left(p + \frac{1}{2}\right)$$

Solution:

Let $p = [p] + \{p\}$

(1)

where $\{p\}$ is the fractional part of p. Evidently $0 < \{p\} < 1$

If $\{p\} < \frac{1}{2}$, then $N(p) = [p]$ and $I\left(p + \frac{1}{2}\right) = [p]$.

Further, if $\{p\} > \frac{1}{2}$, then $N(p) = [p] + 1$ and $I\left(p + \frac{1}{2}\right) = [p] + 1$

In either case, $N(p) = I\left(p + \frac{1}{2}\right)$

The result * is established for

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Scribbling of S.R

$I\left(\frac{n}{p}\right)$ is the coefficient of x^n in the expansion of $\frac{x^p}{(1-x)(1-x^p)}$

Problem 35: **NBSR Vol II p 267**

If p and n are positive integers, then the coefficient of x^n in the series expansion (in positive powers

of x) of $\frac{x^p}{(1-x)(1-x^p)}$ is $I\left(\frac{n}{p}\right)$.

Solution:

$$\frac{x^p}{(1-x)(1-x^p)} = x^p(1-x)^{-1}(1-x^p)^{-1}$$

$$= x^p\{1 + x + x^2 + x^3 + \dots\dots + x^i + \dots\dots\}$$

$$\times \{1 + x^p + x^{2p} + x^{3p} + x^{4p} + \dots\dots\dots + x^{jp} + \dots\dots\}$$

$$= \{1 + x + x^2 + x^3 + \dots\dots + x^i + \dots\dots\}$$

$$\times \{x^p + x^{2p} + x^{3p} + x^{4p} + \dots\dots\dots + x^{(j+1)p} + \dots\dots\} \quad (1)$$

Estimation of the coefficient of x^n in the above product (1):

The coefficients (of powers of x) in both the factors in (1) are the same and each is equal to 1.

$\therefore$ coefficient of x^n in the product (1) i.e., the series expansion of the L.H.S of (1) is equal to the number of times x^n gets repeated by the term by term multiplication of the two (factor) series of the product (1).

The indices of the terms in the first factor are all equal to one and those of the terms in the second factor are multiples of p $(i.e.\, p, 2p, 3p, 4p \dots\dots)$

The terms with x^n in the product (1) would be got by the multiplication of term pairs contained in the set:

$$\{(1,x^n);\ (x,x^{n-1});\ (x^2,x^{n-2})\ \dots\dots\ (x^k,x^{n-k})\ \dots\dots\ (x^{n-p},x^p)\}$$

$$=\left|\left\{x^k,(x^p)^{\frac{n-k}{p}}\right\},\qquad k=0,1,\dots\dots n-p\right|$$

The index of x^p in the second member of a typical pair in the set must be an integer.

$\therefore$ The number of such acceptable pairs $=I\left(\frac{n}{p}\right)$

$\therefore$ The coefficient of x^n in the expansion of $\frac{x^p}{(1-x)(1-x^p)}$ is $I\left(\frac{n}{p}\right)$.

Examples:

1. The coefficient of x^{10} in the expansion $\frac{x^3}{(1-x)(1-x^3)}$ is $I\left(\frac{10}{3}\right)=3$

$$\frac{x^3}{(1-x)(1-x^3)}=x^3\{1+x+x^2+x^3+\ \dots\dots\}\{1+x^3+x^6+x^9+\ \dots\dots\}$$

$$=\{1+x+x^2+x^3+\ \dots\dots\}\{x^3+x^6+x^9+x^{12}+\ \dots\dots\}$$

Terms with x^{10} can be got by multiplying the term pairs $(x,\ x^9)$; (x^4,x^6) and (x^7,x^3) (no. of pairs = 3)

$\therefore$ The coefficient of $x^{10}=3=I\left(\frac{10}{3}\right)$

2. The coefficient of x^{20} in the expansion of $\frac{x^5}{(1-x)(1-x^3)}$

$$\frac{x^5}{(1-x)(1-x^3)}=x^2.\frac{x^3}{(1-x)(1-x^3)}$$

$\therefore$ The coefficient of x^{20} in the expansion $\frac{x^5}{(1-x)(1-x^3)}$

$$=\text{coefficient of } x^{18} \text{ in the expansion of} \frac{x^3}{(1-x)(1-x^3)}=I\left(\frac{18}{3}\right)=6$$

Corollary:

The coefficient of x^n in the expansion of $\frac{x^q}{(1-x)(1-x^p)}$:

$$\frac{x^q}{(1-x)(1-x^p)}=x^{q-p}.\frac{x^p}{(1-x)(1-x^p)}$$

$\therefore$ The coefficient of x^n in the expansion $\frac{x^q}{(1-x)(1-x^p)}$

$$=\text{coefficient of }\{x^{n-(q-p)}\}\text{ in the expansion of }\frac{x^p}{(1-x)(1-x^p)}$$

$$=\left[\frac{n-(q-p)}{p}\right]=\frac{n+p-q}{p}$$

Note: The coefficient is zero if $n\le(q-p)$

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Problem 36: **NBSR Vol II p 361**

Scribbling of S.R

The coefficient of x^{100} in $\frac{x^7}{(1-x^2)(1-x^3)}$

$$= \text{coefficient of } x^{95} \text{ in } \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$$

$$= I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right)$$

$$= 16$$

*

Problem: The coefficients of x^{100} and x^{95} in the power series expansions of

$\frac{x^7}{(1-x^2)(1-x^3)}$ and $\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$ respectively are each equal to

$\left[\frac{95}{2}\right] - \left[\frac{95}{3}\right] = 16$

Solution: It can be noted that

$$\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = \frac{x^2(1-x^3) - x^3(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2 - x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2(1-x)}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2}{(1-x^2)(1-x^3)}$$

$\therefore$ coefficient of x^{100} in $\frac{x^7}{(1-x^2)(1-x^3)}$ = coefficient of x^{95} in $\left\{\frac{x^2}{(1-x^2)(1-x^3)}\right\}$

$$= \text{coefficient of } x^{95} \text{ in } \left\{\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}\right\}$$

$$= \text{coefficient of } x^{95} \text{ in } \left\{\frac{x^2}{(1-x)(1-x^2)}\right\} -$$

$$\text{coefficient of } x^{95} \text{ in } -\frac{x^3}{(1-x)(1-x^3)}$$

$$= I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right) = 47 - 31 = 16$$

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Scribbling of S.R:

$$I\left(\frac{n+4}{6}\right) - I\left(\frac{n-3}{6}\right) + I\left(\frac{n+2}{6}\right)$$

$$= I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$$

*

Problem 37: **NBSR Vol II p.361**

For each positive integral value for n ,

$$I\left(\frac{n+4}{6}\right)-I\left(\frac{n-3}{6}\right)+I\left(\frac{n+2}{6}\right)=I\left(\frac{n}{2}\right)-I\left(\frac{n}{3}\right)$$

Solution: It can be noticed that

$$\frac{x^2}{(1-x)(1-x^2)}-\frac{x^3}{(1-x)(1-x^3)}=\frac{x^2(1-x^3)-x^3(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$$

$$=\frac{x^2-x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$=\frac{x^2}{(1-x^2)(1-x^3)}$$

$$=\frac{x^2(1+x^3)}{(1-x^2)(1-x^6)}$$

$$=\frac{x^2(1-x+x^2)}{(1-x)(1-x^6)}$$

Thus

$$\frac{x^2}{(1-x)(1-x^2)}-\frac{x^3}{(1-x)(1-x^3)}$$

$$=\frac{x^2}{(1-x)(1-x^6)}-\frac{x^3}{(1-x)(1-x^6)}+\frac{x^4}{(1-x)(1-x^6)} \quad (1)$$

The coefficient of x^n in the expansion of $L.H.S$ of $*$

$$=I\left(\frac{n}{2}\right)-I\left(\frac{n}{3}\right) \quad (2)$$

The coefficient of x^n in the expansion of $R.H.S$ of $*$

$$=I\left(\frac{n+6-2}{6}\right)-I\left(\frac{n+6-3}{6}\right)+I\left(\frac{n+6-4}{6}\right)$$

$$=I\left(\frac{n+4}{6}\right)-I\left(\frac{n+3}{6}\right)+I\left(\frac{n+2}{6}\right) \quad (3)$$

Hence equating (2)and (3) we get the result:

$$I\left(\frac{n+4}{6}\right)-I\left(\frac{n+3}{6}\right)+I\left(\frac{n+2}{6}\right)=I\left(\frac{n}{2}\right)-I\left(\frac{n}{3}\right)$$

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Entry 30 (p 369)Vol II

Let α be as above. Then

$$\emptyset(x) = \sum_{k=0}^{6} Sin\left(\alpha^k x\right)$$

$$= 7\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{14n+7}}{(14n+7)!} \qquad (1)$$

and

$$\Psi(x) = \sum_{k=0}^{6} Cos\left(\alpha^k x\right)$$

$$= 7\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{14n}}{(14n)!} \qquad (2)$$

Proof: α is a 7th root of unity. ∴ α^7 and $\alpha^{7p} = 1$ (p is an integer) (3)

$$\text{Now } \emptyset(x) = \sum_{k=0}^{6} Sin\left(\alpha^k x\right) = \sum_{m=0}^{\infty}(-1)^m\frac{\left(\alpha^k x\right)^{2m+1}}{(2m+1)!}$$

$$= \sum_{k=0}^{6}\ \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}\,\alpha^{(2m+1)k}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{k=0}^{6}\alpha^{(2m+1)k} \qquad (4)$$

$$\text{The interior sum on the R. H. S of (4)} = \sum_{k=0}^{6}\alpha^{(2m+1)k}$$

where $2m+1$ is a multiple of 7 i.e., when $2m+1 = (2n+1)7$,

$$\alpha^{2m+1} = \alpha^{(2n+1)7} = (\alpha^7)^{2n+1} = 1 \quad (\because\ \ \alpha^7 = 1)$$

i.e., each term in the interior sum is equal to 1. Hence the interior sum $= 7$.

If $2m+1$ is not a multiple of 7, $\alpha^{2m+1} \neq 1$ and

$$\text{the interior sum} = \frac{1-(\alpha^{2m+1})^7}{1-\alpha^{2m+1}} = \frac{1-(\alpha^7)^{2m+1}}{1-\alpha^{2m+1}} = 0$$

$$\therefore\ \text{The interior sum} = \begin{matrix} 7 & \text{if } 2m+1 \text{ is a multiple of 7} \\ 0 & \text{otherwise} \end{matrix}$$

$$\therefore \qquad \emptyset(x) = 7\sum_{m=0}^{\infty}\frac{(-1)^n x^{14n+7}}{(14n+7)!} \qquad (1)$$

Further

$$\Psi(x)=\sum_{k=0}^{6} Cos\left(\alpha^k x\right)=\sum_{k=0}^{6}\quad\sum_{m=0}^{\infty}(-1)^m\frac{\left(\alpha^k x\right)^{2m}}{(2m)!}$$

$$=\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}\sum_{k=0}^{6}\left(\alpha^k\right)^{2m} \qquad (5)$$

The interior sum on the $R.H.S$ of $(5)=\sum_{k=0}^{6}(\alpha^m)^{2k}$

When m is a multiple of 7, $\alpha^m=1$ and so the interior sum =7 (since each term =1).

When m is not a multiple of 7, $\alpha^m\neq 1$

The sum $=\dfrac{1-(\alpha^{2m})^7}{1-\alpha^{2m}}=\dfrac{1-(\alpha^7)^{2m}}{1-\alpha^{2m}}=0\ (\because\ \alpha^7=1)$

$\therefore$ The interior sum $=\begin{matrix}7, \text{if } m \text{ is a multiple of 7 i.e.,} m=7k\\ 0 \qquad\qquad\qquad\qquad \text{otherwise}\end{matrix}$

$\therefore$ $\Psi(x)=$
$7\sum_{n=0}^{\infty}(-1)^n\frac{x^{14n}}{(14n)!}$ (2)

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Problem 39: **Lost NBSR p 344**

$$\frac{\left(\sqrt{a^2+ab+b^2}-a\right)\left(\sqrt{a^2+ab+b^2}-b\right)}{a+b-\sqrt{a^2+ab+b^2}}=a+b \qquad *$$

Solution:

$$L.H.S\ of\ *=\frac{(a^2+ab+b^2)-(a+b)\sqrt{a^2+ab+b^2}+ab}{(a+b)-\sqrt{a^2+ab+b^2}}$$

$$=\frac{(a+b)^2-(a+b)\sqrt{a^2+ab+b^2}}{(a+b)-\sqrt{a^2+ab+b^2}}$$

$$=\frac{(a+b)\left\{(a+b)-\sqrt{a^2+ab+b^2}\right\}}{(a+b)-\sqrt{a^2+ab+b^2}}=a+b$$

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Problem 40:

$$\left\{\sqrt[3]{(a+b)(a^2+b^2)}-a\right\}\left\{\sqrt[3]{(a+b)(a^2+b^2)}-b\right\}$$

$$=\frac{\sqrt[3]{(a+b)^2}-\sqrt[3]{a^2+b^2}}{\sqrt[3]{(a+b)^2}+\sqrt[3]{a^2+b^2}}.(a^2+ab+b^2) \qquad *$$

Solution:

$$L.H.S\ of\ *=\left\{\sqrt[3]{(a+b)(a^2+b^2)}-a\right\}\left\{\sqrt[3]{(a+b)(a^2+b^2)}-b\right\}$$

$$=(a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}}-(a+b)^{\frac{1}{3}}(a^2+b^2)^{\frac{1}{3}}(a+b)+ab$$

$$= (a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab$$

$\therefore$ *L.H.S of * × Denominator of R.H.S of **

$$= \left\{(a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab\right\} . \left\{(a+b)^{\frac{2}{3}} + (a^2+b^2)^{\frac{1}{3}}\right\}$$

$$= (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^2(a^2+b^2)^{\frac{1}{3}} + ab(a+b)^{\frac{2}{3}}$$

$$+(a+b)^{\frac{2}{3}}(a^2+b^2) - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab(a^2+b^2)^{\frac{1}{3}}$$

$$= (a^2+b^2)^{\frac{1}{3}}\{ab - (a+b)^2\} + (a+b)^{\frac{2}{3}}(a^2+b^2+ab)$$

$$= \left\{(a+b)^{\frac{2}{3}} - (a^2+b^2)^{\frac{1}{3}}\right\}(a^2+ab+b^2)$$

$= Numerator\ of\ R.H.S\ of\ *$

This establishes *.

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"Numbers and Letters are Twin Eyes of Mankind".

------------------ (To be continued ……)

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