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Hybrid Laplace Transformation

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Abstract: *The hybrid Laplace transformation, defined by the composition between the usual Laplace and Z transformations, was introduced, studied and applied in a series of papers by the second author. It applies to functions of both continuous and discrete variables and can be used to solve hybrid equations, which can have both algebraic, differential and integral terms. Here we present some of the properties of the hybrid transformation, going them proofs based on the corresponding properties of the usual Laplace and Z transformations. Finally the hybrid Laplace transformation is used to solve an algebraic-integral equation previously solved by other methods by the first author. The simplicity of the proposed approach proves the great advantage of the hybrid Laplace transformation.*

Keywords: *Usual Laplace and Z transformations, hybrid Laplace transformation, hybrid equations, hybrid convolution, algebraic-integral recurrence equations, algebraic-differential recurrence equations.*

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1. INTRODUCTION

Many problems of solving hybrid equations have been imposed in the last decade (see [1], [3], [4], [5], [14], [15]), as well as the study of the hybrid multidimensional control systems, which are governed by several hybrid equations (see [8]-[12], [14]-[20]). The unknowns of the hybrid equations and the states, inputs and outputs of the hybrid physical systems are functions of both continuous and discrete variables, These hybrid equations and control systems have many applications in various domains such as seismology, geophysics, image processing, computer tomography, probabilities, statistics, queuing theory and other.

A fundamental method of solving continuous-time or discrete-time physical systems is to use the Laplace transformation or Z-transformation, respectively. The hybrid Laplace transformation was introduced in [14] and it was used to solve hybrid equations or control systems, simultaneously in respect with the continuous and discrete variables. The main properties of the hybrid Laplace transformation were presented in the quoted paper, their proofs being obtained on the basis of the definitions. Also, the method of using this transformation was indicated for solving different types of hybrid equations and for the study of some control or physical systems described by such equations (see [14]-[16], [18], [19]).

The present paper emphasizes some properties of the hybrid Laplace transformation, which are actually derived from the corresponding properties of the usual Laplace and Z transformations. This approach is useful especially in studies which present and employ all these three transformations, since in this case it avoids the repetitions of some ideas and techniques in their proofs. The proofs given here of the theorems which concern the hybrid Laplace transformation are simpler, if the justification of the corresponding properties of the usual Laplace and Z transformations are assumed to be known. Finally, using the hybrid Laplace transformation, an integral-algebraic recurrence equation with hybrid auto-convolution is solved. This is one of the equations studied in [4]. There, it was reduced to a hybrid differential-algebraic equation by using the usual Laplace transformation. The obtained equation was then solved using the methods given in an earlier paper [5]. The comparison of these solutions shows the advantage of applying the hybrid Laplace transformation method, to solve such hybrid equations.

2. THE USUAL LAPLACE AND Z TRANSFORMATIONS

2.1. The Laplace Transformation

A function $f(t)$ of real continuous-variable t having complex values is called *original function* for the Laplace transformation if it is piecewise smooth, $f(t)=0$ if $t<0$ and there exist $M(f)>0$ and $\sigma(f)>0$, such that $\left|f(t)\right| \le M(f)e^{\sigma(f)t}$, $\forall t \ge 0$. The function of a complex variable s having complex values, defined by the formula

$$L(f(t))(s) = L_t(f(t))(s) = \int_0^\infty f(t)e^{-st}dt \text{, for } \mathrm{Re}(s) > \sigma(f),$$

is called *Laplace transform* or the *image* by the Laplace transformation of the original function $f(t)$. The linear operator L which associate to the original function $f(t)$ its Laplace transform $L(f(t))(s)$ is called *Laplace transformation* and has the following properties (see [7] and [22]):

$$L\left(f^{(m)}(t)\right)(s) = s^m L(f(t))(s) - \sum_{k=0}^{m-1} f^{(k)}(0+)s^{m-k-1}, \quad m = 1,2,\dots, \tag{1}$$

$$L\left(\int_0^t f(x)dx\right)(s) = \frac{1}{s}L(f(t))(s), \tag{2}$$

$$L(f(at))(s) = \frac{1}{a}L(f(t))\left(\frac{s}{a}\right), \tag{3}$$

Where $a > 0$,

$$L(f(t-a))(s) = e^{-as}L(f(t))(s), \tag{4}$$

where $a > 0$,

$$L\left(e^{at}f(t)\right)(s) = L(f(t))(s-a), \tag{5}$$

for $\mathrm{Re}(s) > \mathrm{Re}(a)$, where a is a complex number,

$$L\left(t^m f(t)\right)(s) = (-1)^m \frac{d^m L(f(t))(s)}{ds^m}, \quad m = 1,2,\dots, \tag{6}$$

$$\lim_{s\to\infty} sL(f(t))(s) = f(0+), \tag{7}$$

$$\lim_{s\to 0} sL(f(t))(s) = \lim_{t\to\infty} f(t). \tag{8}$$

when both limits exist.

If $g(t)$ is a second original function and

$$(f*g)(t) = \int_0^t f(x)g(t-x)dx$$

denotes the causal continuous-variable convolution product of the functions f and g, then we have

$$L((f*g)(t))(s) = L(f(t))(s)L(g(t))(s). \tag{9}$$

2.2. The Z Transformation

A function $f(n)$ of an integer discrete-variable n having complex values is called *original function* for the Z transformation if $f(n) = 0$, for $n < 0$ and there exist

$M(f) > 0$ and $R(f) > 0$, such that $|f(n)| \le M(f)R(f)^n$, $\forall n \ge 0$. The function of a complex variable z having complex values, defined by the formula

$$Z(f(n))(z) = Z_n(f(n))(z) = \sum_{n=0}^{\infty} f(n)z^{-n}, \text{ for } |z| > R(f),$$

is called the *Z transform* or the *image* by Z transformation of the original function $f(n)$. The linear operator which associates to the original function $f(n)$ its Z transform $Z(f(n))(z)$ is called *Z transformation* and has the following properties (see [21]):

$$Z(f(n-m))(z) = z^{-m}Z(f(n))(z),\ m = 1,2,\dots, \tag{10}$$

$$Z(f(n+m))(z) = z^{m}Z(f(n))(z) - \sum_{n=0}^{m-1} f(n)z^{m-n},\ m = 1,2,\dots, \tag{11}$$

$$Z(a^n f(n))(z) = Z(f(n))\left(\frac{z}{a}\right), \tag{12}$$

for all complex numbers z with $|z| > |a|R(f)$, where $a \ne 0$ is a complex number,

$$Z(n(n+1)\cdots(n+m-1)f(n))(z) = (-1)^m z^m \frac{d^m Z(f(n))(z)}{dz^m},\ m = 1,2,\dots. \tag{13}$$

If $f(n) = 0$, $\forall n \ne mk$, $k = 0,1,2,\dots$, then

$$Z(f(mn))(z) = Z(f(n))\left(z^{\frac{1}{m}}\right), m = 1,2,\dots. \tag{14}$$

$$\lim_{z\to\infty} Z(f(n))(z) = f(0), \tag{15}$$

$$\lim_{z\to 1+,\,\mathrm{Im}(z)=0} (z-1)Z(f(n))(z) = \lim_{n\to\infty} f(n). \tag{16}$$

when both limits exist.

If $g(n)$ is a second original function for the Z transformation and

$$(f * g)(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

denotes the *causal discrete-variable convolution product* of the functions f and g, then we have

$$Z((f * g)(n))(z) = Z(f(n))(z)Z(g(n))(z). \tag{17}$$

3. THE HYBRID LAPLACE TRANSFORMATION

A function $f(t,n)$ of a real continuous-variable t and an integer discrete-variable n having complex values is called *original function* for the hybrid Laplace transformation or *hybrid Laplace transformable function* if it is piecewise smooth in t for every integer number n, we have $f(t,n) = 0$, if $t < 0$ or $n < 0$ and there exist $M(f) > 0$, $\sigma(f) > 0$ and $R(f) > 0$, such that

$\left| f(t,n) \right| \le M(f) e^{\sigma(f)t} R(f)^n$, $\forall t \ge 0$, $\forall n \ge 0$.

The function of complex variables s and z having complex values, given by the formula

$$L_2(f(t,n))(s,z) = \int_0^\infty \sum_{k=0}^\infty f(t,k) z^{-k} e^{-st} dt = L_t(Z_n(f(t,n))(z))(s), \tag{18}$$

for $\mathrm{Re}(s) > \sigma(f)$ and $|z| > R(f)$, is called the *hybrid Laplace transform* or the *image* by the hybrid Laplace transformation of the original function $f(t,n)$. The linear operator which associate to the original function $f(t,n)$ its hybrid Laplace transform $L_2(f(t,n))(s,z)$ is called the *hybrid Laplace transformation*. We note that the improper integral and series in formula (18) are absolutely convergent. They are uniformly convergent if $\mathrm{Re}(s) \ge \sigma > \sigma(f)$ and $|z| \ge R > R(f)$. In this last case, we have

$$L_2(f(t,n))(s,z) = Z_n(L_t(f(t,n))(s))(z). \tag{19}$$

By using the properties of the usual Laplace and Z transformations, presented in Section 2 and the definitions (18) or (19), we will derive some properties of the hybrid Laplace transformation L_2, which were independently proved in [14], based only on its definition. In all following results, $f(t,n)$ is an original for the hybrid Laplace transformation. Unless other conditions are imposed, in the theorems below one assumes that $\mathrm{Re}(s) > \sigma(f)$ and $|z| > R(f)$.

Theorem 1. (Homothety) *If* $a > 0$, *for* $\mathrm{Re}(s) \ge a\sigma > a\sigma(f)$ *and* $|z| \ge R > R(f)$, *we have*

$$L_2(f(at,n))(s,z) = \frac{1}{a} L_2(f(t,n))\left(\frac{s}{a}, z\right). \tag{20}$$

For $m = 1,2,\ldots$, *if* $f(t,n) = 0$, *when* $n \ne mk$, $k = 0,1,2,\ldots$, *then*

$$L_2(f(t,mn))(s,z) = L_2(f(t,n))\left(s, z^{\frac{1}{m}}\right). \tag{21}$$

Proof. Using (3) and (19), we have $L_2(f(at,n))(s,z) = Z_n(L_t(f(at,n))(s))(z) =$

$\frac{1}{a} Z_n\left(L_t(f(t,n))\left(\frac{s}{a}\right)\right)(z) = \frac{1}{a} L_2(f(t,n))\left(\frac{s}{a}, z\right)$. Using (14) and (18), one obtains

$$L_2(t(t,mn))(s,z) = L_t(Z_n(f(t,mn))(z))(s) = L_t\left(Z_n(f(t,n))\left(z^{\frac{1}{m}}\right)\right)(s) = L_2(f(t,n))\left(s, z^{\frac{1}{m}}\right).$$

Theorem 2. (Translation) If a and $b \ne 0$ are complex numbers, then for $\mathrm{Re}(s) > \mathrm{Re}(a) + \sigma(f)$ and $|z| > |b| R(f)$, we have

$$L_2\left(e^{at} b^n f(t,n)\right)(s,z) = L_2(f(t,n))\left(s - a, \frac{z}{b}\right). \tag{22}$$

Proof. Using formulas (5), (12) and (18), one obtains

$$L_2\left(e^{at} b^n f(t,n)\right)(s,z) = L_t\left(e^{at} Z_n\left(b^n f(t,n)\right)(z)\right)(s) = L_t\left(e^{at} Z_n(f(t,n))\left(\frac{z}{b}\right)\right)(s) =$$

$$L_t\left(Z_n(f(t,n))\left(\frac{z}{b}\right)\right)(s-a)=L_2(f(t,n))\left(s-a,\frac{z}{b}\right).$$

Theorem 3. (Time delay) For a real number $a>0$ and a natural number m, we have

$$L_2(f(t-a,n-m))(s,z)=e^{-as}z^{-m}L_2(f(t,n))(s,z). \tag{23}$$

Proof. Using formulas (4), (10) and (18), one obtains

$$L_2(f(t-a,n-m))(s,z)=L_t(Z_n(f(t-a,n-m)))(s,z)=z^{-m}L_t(Z_n(f(t-a,n))(z))(s)=$$

$$e^{-as}z^{-m}L_s(Z_n(f(t,n))(z))(s)=e^{-as}z^{-m}L_2(f(t,n))(s,z).$$

Theorem 4. (Second discrete-time delay) *For a natural number* $m\neq 0$, *we have*

$$L_2(f(t,n+m))(s,z)=z^mL_2(f(t,n))(s,z)-\sum_{n=0}^{m-1}L_t(f(t,n))(s)z^{m-n}. \tag{24}$$

Proof. Using (11) and (18), we have $L_2(f(t,n+m))(s,z)=L_t(Z_n(f(t,n+m))(z))(s)=$

$$L_t\left(z^mZ_n(f(t,n))(z)-\sum_{n=0}^{m-1}f(t,n)z^{m-n}\right)(s)=$$

$$z^mL_t(Z_n(f(t,n))(z))(s)-\sum_{n=0}^{m-1}L_t(f(t,n))(s)z^{m-n}=z^mL_2(f(t,n))(s,z)-\sum_{n=0}^{m-1}L_t(f(t,n))(s)z^{m-n}.$$

Theorem 5. (Differentiation of the original) *For a natural number* $m\neq 0$, *we have*

$$L_2\left(\frac{d^mf(t,n)}{dt^m}\right)(s,z)=s^mL_2(f(t,n))(s,z)-\sum_{k=0}^{m-1}Z_n\left(\frac{d^kf(0+,n)}{dt^k}\right)(z)s^{m-k-1}, \tag{25}$$

for $\operatorname{Re}(s)\geq\sigma>\sigma(f)$ *and* $|z|\geq R>R(f)$.

Proof. Using (1) and (19), we have $L_2\left(\frac{d^mf(t,n)}{dt^m}\right)(s,z)=Z_n\left(L_t\left(\frac{d^mf(t,n)}{dt^m}\right)(s)\right)(z)=$

$$Z_n\left(s^mL_t(f(t,n))(s)-\sum_{k=0}^{m-1}\frac{d^kf(0+,n)}{dt^k}s^{m-k-1}\right)(z)=$$

$$s^mZ_n(L_t(f(t,n))(s))(z)-\sum_{k=0}^{m-1}Z_n\left(\frac{d^kf(0+,n)}{dt^k}\right)(z)s^{m-k-1}=$$

$$s^mL_2(f(t,n))(s,z)-\sum_{k=0}^{m-1}Z_n\left(\frac{d^kf(0+,n)}{dt^k}\right)(z)s^{m-k-1}.$$

Theorem 6. (First formula of differentiation of the image) *If* $m\neq 0$ *is a natural number, we have*

$$L_2\left(t^mf(t,n)\right)(s,z)=(-1)^m\frac{\partial^m}{\partial s^m}L_2(f(t,n))(s,z). \tag{26}$$

for $\operatorname{Re}(s)\geq\sigma>\sigma(f)$ *and* $|z|\geq R>R(f)$.

Proof. Using (6) and (19), one obtains $L_2\left(t^mf(t,n)\right)(s,z)=Z_n\left(L_t\left(t^mf(t,n)\right)(s)\right)(z)=$

$$Z_n\left((-1)^m \frac{d^m}{ds^m} L_t(f(t,n))(s)\right)(z) = (-1)^m \frac{\partial^m}{\partial s^m} Z_n(L_t(f(t,n))(s))(z) =$$

$$(-1)^m \frac{\partial^m}{\partial s^m} L_2(f(t,n))(s,z).$$

Theorem 7. (Second formula of differentiation of the image) *If* $m \neq 0$ *is a natural number, we have*

$$L_2(n(n+1)\cdots(n+m-1)f(t,n))(s,z) = (-1)^m z^m \frac{\partial^m}{\partial z^m} L_2(f(t,n))(s,z). \quad (27)$$

Proof. Using (13) and (18), one obtains $L_2(n(n+1)\cdots(n+m-1)f(t,n))(s,z) =$

$$L_t(Z_n(n(n+1)\cdots(n+m-1)f(t,n))(z))(s) = L_t\left((-1)^m z^m \frac{d^m}{dz^m} Z_n(f(t,n))(z)\right)(s) =$$

$$(-1)^m z^m \frac{\partial^m}{\partial z^m} L_t(Z_n(f(t,n))(z))(s) = (-1)^m z^m \frac{\partial^m}{\partial z^m} L_2(f(t,n))(s,z).$$

Theorem 8. (Initial value formula) *When the limits exist, we have*

$$\lim_{s\to\infty} \lim_{z\to\infty} sL_2(f(t,n))(s,z) = f(0+,0). \quad (28)$$

Proof. Using (7), (15) and (18), one obtains $\lim_{s\to\infty} \lim_{z\to\infty} sL_2(f(t,n))(s,z) =$

$$\lim_{s\to\infty} sL_t\left(\lim_{z\to\infty} Z_n(f(t,n))(z)\right)(s) = \lim_{s\to\infty} sL_t(f(t,0))(s) = f(0+,0).$$

Theorem 9. (Final value formula) *When the limits exist, we have*

$$\lim_{s\to 0} \lim_{z\to 1+, \operatorname{Im}(z)=0} s(z-1)L_2(f(t,n))(s,z) = \lim_{t\to\infty} \lim_{n\to\infty} f(t,n). \quad (29)$$

Proof. Using (8), (16) and (18), one obtains $\lim_{s\to 0} \lim_{z\to 1+, \operatorname{Im}(z)=0} s(z-1)L_2(f(t,n))(s,z) =$

$$\lim_{s\to 0} \lim_{z\to 1+, \operatorname{Im}(z)=0} sL_t((z-1)Z_n(f(t,n))(z))(s) = \lim_{s\to 0} sL_t\left(\lim_{z\to 1+, \operatorname{Im}(z)=0} (z-1)Z_n(f(t,n))(z)\right)(s) =$$

$$\lim_{s\to 0} sL_t\left(\lim_{n\to\infty} f(t,n)\right)(s) = \lim_{t\to\infty} \lim_{n\to\infty} f(t,n).$$

Theorem 10. (Separate variables case) If $f(t)$ and $g(n)$ are originals respectively for the Laplace and Z transformation, then the product $f(t)g(n)$ is original function for the hybrid Laplace transformation and we have

$$L_2(f(t)g(n))(s,z) = L(f(t))(s)Z(g(n))(z). \quad (30)$$

Proof. Using the definition (18), we have $L_2(f(t)g(n))(s,z) = L_t(Z_n(f(t)g(n))(z))(s) =$

$$L_t(f(t)Z_n(g(n))(z))(s) = Z_n(g(n))(z)L_t(f(t))(s) = L(f(t))(s)Z(g(n))(z).$$

Corollary 1. If the function $f(t)$ is an original for the Laplace transformation, then it is an original for the hybrid Laplace transformation and we have

$$L_2(f(t))(s,z) = \frac{z}{z-1} L(f(t))(s), \quad (31)$$

for $\operatorname{Re}(s) > \sigma(f)$ *and* $|z| > R(f) = 1$.

Proof. It follows from Theorem 10, the linearity of the Z transformation and formula $Z(1)(z) = \frac{z}{z-1}$.

Examples. Using the well-known examples of Laplace transforms and Corollary 1, we obtain for a real number $a \neq 0$ and a natural number m, the following examples of hybrid Laplace transforms: $L_2(t^a)(s,z) = \frac{z\Gamma(a+1)}{(z-1)s^{a+1}}$, where Γ is the Euler function of the first kind and particularly $L_2(t^m)(s,z) = \frac{m!z}{(z-1)s^{m+1}}$, $L_2(e^{at})(s,z) = \frac{z}{(z-1)(s-a)}$,

$L_2(\sin(at))(s,z) = \frac{az}{(z-1)(s^2+a^2)}$ and $L_2(\cos(at))(s,z) = \frac{sz}{(z-1)(s^2+a^2)}$.

Corollary 2. If $g(n)$ is an original for the Z transformation, then it is an original for the hybrid Laplace transformation and we have

$$L_2(g(n))(s,z) = \frac{1}{s} Z(g(n))(z), \tag{32}$$

for $\operatorname{Re}(s) \geq \sigma > \sigma(g) = 0$ *and* $|z| \geq R > R(g)$.

Proof. It follows from Theorem 10, the linearity of the Laplace transformation and formula $L(1)(s) = \frac{1}{s}$.

Examples. Using the well-known examples of Z transforms and the Corollary 1, we obtain for a real number $a \neq 0$ and a complex number $b \neq 0$, the following examples of hybrid Z transforms: $L_2(e^{an})(s,z) = \frac{z}{s(z-e^a)}$ and particularly $L_2(b^n)(s,z) = \frac{z}{s(z-b)}$,

$L_2(\sin(an))(s,z) = \frac{z\sin a}{s(z^2 - 2z\cos a + 1)}$, $L_2(\cos(an))(s,z) = \frac{z(z-\cos a)}{s(z^2 - 2z\cos a + 1)}$.

4. EXAMPLE

We give another example of hybrid Laplace transform of an original whose variables are not separated, and which will be applied in the Section 6.

Theorem11. *For* $\operatorname{Re}(s) \geq \sigma > 0$, $|z| \geq R > 0$ *and* $|s| > \frac{1}{R}$, *we have*

$$L_2\left(\frac{t^n}{n!}\right) = \frac{z}{sz-1}. \tag{33}$$

Proof. Using (19) and formulas $L(t^n)(s) = \frac{n!}{s^{n+1}}$, for $\operatorname{Re}(s) > 0$ and $Z(a^n)(z) = \frac{z}{z-a}$ for $|z| > |a|$, with $a = \frac{1}{s}$, one obtains

$$L_2\left(\frac{t^n}{n!}\right)(s,z) = Z_n\left(\frac{1}{n!}L_t(t^n)(s)\right)(z) = Z_n\left(\frac{1}{s^{n+1}}\right)(z) = \frac{1}{s}Z_n\left(\left(\frac{1}{s}\right)^n\right)(z) = \frac{1}{s}\frac{z}{z-\frac{1}{s}} = \frac{z}{sz-1}.$$

5. Hybrid Convolution

The 2D *hybrid* or the *continuous-discrete convolution* of the functions $f(t,n)$ and $g(t,n)$ is the function denoted and defined by the formula

$$(f *_2 g)(t,n) = \int_0^t \sum_{k=0}^{n} f(x,k)g(t-x,n-k)dx\ , \tag{34}$$

for $t \geq 0$ and $n \geq 0$, and $(f *_2 g)(t,n) = 0$, if $t < 0$ or $n < 0$.

Theorem 12. (Hybrid Laplace transform of the hybrid convolution) *If the functions* $f(t,n)$ *and* $g(t,n)$ *are originals for hybrid Laplace transformation, then the hybrid convolution* $(f *_2 g)(t,n)$ *is also an original and the following equality holds*

$$L_2((f *_2 g)(t,n))(s,z) = L_2(f(t,n))(s,z)L_2(g(t,n))(s,z), \tag{35}$$

for $\operatorname{Re}(s) > \max(\sigma(f),\sigma(g))$ *and* $|z| > \max(R(f),R(g))$.

Proof. We will apply formulas (18) and (34), will reverse the order of integration and summation and we will make the change of variables $y = t - x$ and $m = n - k$. Taking into account that $g(t,n) = 0$ if $t < 0$ or $n < 0$, one obtains

$$L_2((f *_2 g)(t,n))(s,z) = \int_0^\infty \sum_{n=0}^{\infty} \int_0^t \sum_{k=0}^{n} f(x,k)g(t-x,n-k)dxz^{-n}e^{-st}dt =$$

$$\int_0^\infty \sum_{n=0}^{\infty} \int_0^\infty \sum_{k=0}^{\infty} f(x,k)g(t-x,n-k)z^{-n}e^{-st}dxdt = \int_0^\infty \sum_{k=0}^{\infty} f(x,k) \int_0^\infty \sum_{n=0}^{\infty} g(t-x,n-k)z^{-n}e^{-st}dtdx =$$

$$\int_0^\infty \sum_{k=0}^{\infty} f(x,k) \int_0^\infty \sum_{m=0}^{\infty} g(y,m)z^{-(m+k)}e^{-s(x+y)}dydx = \int_0^\infty \sum_{k=0}^{\infty} f(x,k)z^{-k}e^{-sx}dx \int_0^\infty \sum_{m=0}^{\infty} g(y,m)z^{-m}e^{-sy}dy =$$

$$L_2(f(t,n))(s,z)L_2(g(t,n))(s,z).$$

Remark 1. *Not all hybrid Laplace transforms can be derived using the composition between the usual Laplace and Z transformations, such as those in Section 3. The formula which gives the hybrid Laplace transform of the hybrid convolution, given in Theorem 12, is such a result. As shown above, to prove it was necessary to use the effective definition of the hybrid transformation, namely the first equality from (18).*

6. Application: Solving by Hybrid Laplace Transformation a Recurrence Integral-Algebraic Equation with Hybrid Auto-Convolution

Using the hybrid Laplace transformation, in this Section we solve a recurrence integro-algebraic equation with hybrid auto-convolution solved in [4] by the usual Laplace transformation method, which reduces the equation to a recurrence differential-algebraic equation with auto-convolution. This later equation was solved in [4] by the method given in the earlier paper [5].

Theorem13. Let $a, b \neq 0$ with $\operatorname{Re}(a) > 0$, and s_0 be given complex numbers. The hybrid transformable solution $u(t,n)$ of the recurrence integral-algebraic equation with hybrid auto-convolution

$$tu(t,n) + \int_0^t \sum_{k=0}^{n} u(x,k)u(t-x,n-k)dx = 0\ ,\ \forall t \geq 0\ ,\ n = 0,1,2,\ldots\ , \tag{36}$$

which can be written as

$$tu(t,n)+(u*_2 u)(t,n)=0, \tag{37}$$

with the initial condition

$$\int_0^\infty e^{-s_0 t}u(t,n)dt=-ab^n,\ n=0,1,2,\ldots, \tag{38}$$

is given by formula

$$u(t,n)=-\frac{1}{n!}\left(\frac{bt}{a}\right)^n e^{\frac{as_0-1}{a}t},\ \forall t\ge 0,\ n=0,1,2,\ldots. \tag{39}$$

Proof. We apply the hybrid Laplace transformation L_2 on equation (36) and we denote $U(s,z)=L_2(u(t,n))(s,z)$, where s and z are complex variables with $\operatorname{Re}(s)\ge\sigma>\sigma(u)$ and $|z|\ge R>R(u)$. According to formulas (26) and (35), we obtain the differential equation

$$-\frac{dU(s,z)}{ds}+U^2(s,z)=0, \tag{40}$$

or $\frac{dU}{U^2}=ds$. Integrating, we obtain $-\frac{1}{U(s,z)}=s+C(z)$, hence the differential equation (40) has the solution

$$U(s,z)=-\frac{1}{s+C(z)}, \tag{41}$$

where $C(z)$ is an arbitrary differentiable function. By applying the hybrid Laplace transformation to the initial condition (38), according to (19), this becomes

$$U(s_0,t)=Z_n(L_t(u(t,n))(s_0))(z)=Z_n\left(\int_0^\infty e^{-s_0 t}u(t,n)dt\right)(z)=Z_n(-ab^n)(z).$$

Using the linearity of the Z transformation and the formula $Z(b^n)(z)=\frac{z}{z-b}$ for $|z|>|b|$, the above relation becomes

$$U(s_0,z)=-aZ_n(b^n)(z)=-\frac{az}{z-b}, \tag{42}$$

for $|z|>\max(R,|b|)$. From (41) and (42) one obtains

$$U(s_0,z)=-\frac{1}{s_0+C(z)}=-\frac{az}{z-b}$$

hence

$$C(z)=\frac{z(1-as_0)-b}{az}. \tag{43}$$

From (41) and (43) it follows that the solution of the differential equation (40) with the initial condition (42) is

$$U(s,z)=-\frac{az}{z(as-as_0+1)-b}. \tag{44}$$

On the other hand, taking into account Theorems 2 and 10, we get

$$L_2\left(\frac{e^{\lambda t}b^n t^n}{n!}\right)(s,z)=\frac{\frac{z}{b}}{(s-\lambda)\frac{z}{b}-1}=\frac{z}{(s-\lambda)z-b},\tag{45}$$

with $\lambda = as_0 - 1$. Using Theorem 1, formula (45) becomes

$$L_2\left(\frac{1}{n!}e^{\frac{as_0-1}{a}t}b^n\left(\frac{t}{a}\right)^n\right)(s,z)=\frac{az}{z(as-as_0+1)-b}.\tag{46}$$

From relations (44) and (46) we obtain that the solution of the equation (36) with the initial condition (38) is given by formula (39).

Remark 2. As it can be seen, the solution found here is the same as that obtained in the paper [4], if the number a is replaced with $-a$.

Corollary 3. The recurrence integro-algebraic equation with combinatorial hybrid auto-convolution

$$tv(t,n)+\sum_{k=0}^{n}\binom{n}{k}\int_0^t v(x,k)v(t-x,n-k)dx = 0,\ n=0,1,2,\ldots,$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ are the binomial coefficients, with the initial condition

$$\int_0^\infty e^{-s_0 t}v(t,n)dt = -n!ab^n,\ n=0,1,2,\ldots$$

has the solution

$$v(t,n)=-\left(\frac{bt}{a}\right)^n e^{\frac{as_0-1}{a}t},\ n=0,1,2,\ldots.$$

Proof. The equation and its initial condition are reduced to equation (36) and initial condition (38) by the change of function $v(t,n)=n!u(t,n),\ n=0,1,2,\ldots$, therefore the above solution is obtained by formula (39).

7. VERIFICATION.

In addition to the proof of Theorem 12, we will show that the obtained solution (39) verifies both the hybrid equation (36) and the initial condition (38). Performing the hybrid auto-convolution of the function $u(t,n)$ given by formula (39) and using the Newton's binomial formula, we obtain

$$u(t,n)*_2 u(t,n)=\int_0^t\sum_{k=0}^{n}u(x,k)u(t-x,n-k)dx=$$

$$\int_0^t\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{b}{a}\right)^k x^k e^{\frac{as_0-1}{a}x}\frac{1}{(n-k)!}\left(\frac{b}{a}\right)^{n-k}(t-x)^{n-k}e^{\frac{as_0-1}{a}(t-x)}dx=$$

$$\frac{1}{n!}\left(\frac{b}{a}\right)^n e^{\frac{as_0-1}{a}t}\int_0^t\sum_{k=0}^{n}\binom{n}{k}x^k(t-x)^{n-k}dx=\frac{1}{n!}\left(\frac{b}{a}\right)^n e^{\frac{as_0-1}{a}t}\int_0^t t^n dx=$$

$$\frac{1}{n!}\left(\frac{b}{a}\right)^n e^{\frac{as_0-1}{a}t}t^{n+1}=-tu(t,n),,$$

hence the function $u(t,n)$ given by formula (39) verifies equation (36). Let us check the initial condition:

$$\int_0^t e^{-s_0 t} u(t,n)dt = -\int_0^t e^{-s_0 t} \frac{1}{n!}\left(\frac{b}{a}\right)^n t^n e^{\frac{as_0-1}{a}t} dt = -\frac{1}{n!}\left(\frac{b}{a}\right)^n \int_0^t e^{-\frac{t}{a}} t^n dt = -\frac{1}{n!}\left(\frac{b}{a}\right)^n L(t(n))\left(\frac{1}{a}\right) =$$

$$-\frac{1}{n!}\left(\frac{b}{a}\right)^n n! a^{n+1} = -ab^n,\ n = 0,1,2,\ldots .$$

8. CONCLUSION

In this paper some theorems concerning the 2D hybrid Laplace transformation were derived from the corresponding properties of the usual Laplace and Z transformations. These properties can be extended to hybrid Laplace transformation which operates on original functions with an arbitrary finite number of continuous and discrete variables. The hybrid Laplace transformation can be applied to mathematical models described by multiple differential-difference and integro-difference equations, as well as to the study of multidimensional hybrid control systems. As an example, in this paper we solve an integro-difference equation, which has been solved by other methods in [4], achieving the same result. Comparing the two solutions, it may be noted that, in the situations in which it can be applied, the method of the hybrid Laplace transformation is simpler, because it works simultaneously on both continuous and discrete variables of the unknown.

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