International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 11, November 2014, PP 862-867
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Operational Calculus on Generalized Fourier-Laplace Transform

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Abstract: *Communication is all based on Mathematics, be it digital, wired or wireless using Fourier Transform analysis. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. The Laplace transform is a mathematical tool based on integration that has a number of applications. It particular, it can simplify the solving of many differential equation, also it is used in FM/AM stereo, 2-way radio sets, cellular phones etc. On combining these Fourier and Laplace transform the resultant Fourier-Laplace transform may be used in signal processing, solving ordinary, partial differential equations as well as integral equations.*

In this paper we described the various properties of Fourier-Laplace transform which will be useful for solving differential and integral equations. Also this paper presents Generalization of Fourier-Laplace Transform in the distributional sense. Some Testing function spaces are also defined by Gelfand-Shilov technique.

Keywords: *Fourier transform, Laplace transform, Fourier- Laplace transform, generalized function, Testing function space.*

1. Introduction

Mathematics is everywhere in every phenomenon, technology, observation, experiment etc. All we need to do is to understand the logic hidden behind [1]. Fourier transform are use in many areas of geophysics, such as image processing, time series analysis and antenna design. The Fourier transform allows you to convert between time domain and frequency. It is used in digital communication. Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytical tool for solving a broad range of partial differential equations. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CD's and DVD's, cell phones, movies, computer graphics, image processing and fingerprint analysis and storage, are in one way or another, founded upon the many ramification of Fourier theory [1].

The Laplace transform is crucial for the study of control systems, hence they are used for the analysis of HVAC (Heating, Ventilation and Air-conditioning) control systems, which are used in all modern buildings and constructions. Laplace transform also have many Engineering applications like system modeling Analysis of Electrical circuits, Analysis of Electronic circuits and digital signal processing. It also used in X-ray computed tomography (CT scan) this is a medical application of Laplace transform.

Fourier and Laplace Transform have various properties like Shifting, Scaling, Conjugate, Translation, Duality etc. The Shifting property of Fourier transform identifies the fact that a linear displacement in time corresponds to a linear phase factor in the frequency domain. This becomes useful and important when we discuss filtering and the effects of the phase characteristic of a filter in the time domain. The differentiation property for Fourier transform is very useful. In the time domain we recognize that differentiation will emphasize these abrupt changes, and this property states that consistent with this result, the high frequencies are amplified in relation to the low frequencies [2]. By all the above properties of Fourier and Laplace transform we can solve various problems like heat equations, wave equations etc. [3], [4].

We focus on the joint Fourier-Laplace transform which is essential a characteristic function in the first variable and a moment generating function in the second. And this joint Fourier-Laplace transform may have various applications in various fields like Engineering, Medical, Physics etc. and the formula for this joint Fourier-Laplace transform is given as,

$$FL\ \ f(t,x)\ = F(s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(t,x)K(t,x)dtdx \tag{1.1}$$

where, $K(t,x) = e^{-i(st-ipx)}$.

In the present paper, first we have defined the testing function spaces which are given in section 2.The definition of Distributional generalized Fourier-Laplace transform is given by section 3. In section 4. We have proved the various properties for generalized Fourier-Laplace transform. Lastly conclusions are given in section 5.

Notations and terminology as per Zemanian [5]

2. Testing Function Spaces

2.1. The Space $FL_{a,\alpha}^{\beta}$

This space is given by

$$FL_{a,\alpha}^{\beta} = \left\{ \phi : \phi \in E_{+} \,/\, \rho_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C A^k k^{k\alpha} B^l l^{l\beta} \right\} \tag{2.1}$$

where, $k,l,q = 0,1,2,3,..........$, and the constants A, B depends on the testing function ϕ .

2.2. The Space $FL_{a,\gamma}$

It is given by

$$FL_{a,\gamma} = \left\{ \phi : \phi \in E_{+} \,/\, \xi_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \right\} \tag{2.2}$$

where, $k,l,q = 0,1,2,3,..........$, and the constants depend on the testing function ϕ .

The space $FL_{a,\alpha}^{\beta}$ and $FL_{a,\gamma}$ are equipped with their natural Hausdorf locally converse topologies $T_{a.\alpha}^{\beta}$ and $T_{a,\gamma}$. These topologies are respectively generated by the total families of semi norms $\rho_{a,k,l,q}$ and $\xi_{a,k,l,q}$ given by (2.1) and (2.2).

3. Distributional Generalized Fourier-Laplace Transform

For $f(t,x) \in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\ \ f(t,x)\ = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle , \tag{3.1}$$

where, for each fixed t $\ 0<t<\infty$, x $\ 0<x<\infty$, $s>0$ and $p>0$, the right hand side of (3.1) has a sense as an application of $f(t,x) \in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)} \in FL_{a,\alpha}^{\beta}$

4. SOME PROPERTIES OF FOURIER-LAPLACE TRANSFORM

4.1. First Shifting Operator

$$FL\ \ e^{-ax-bt} f(t,x)\ \ = F(s-ib, p+a)$$

Proof:

$$FL\ \ e^{-ax-bt} f(t,x)\ \ (s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-ax-bt} e^{-i(st-ipx)} f(t,x)dtdx$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-(p+a)x} e^{-i(s-ib)t} f(t,x)dtdx$$

Put $p+a = z$ and $s-ib = r$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-zx} e^{-irt} f(t,x)dtdx$$

$$= F(r,z) = F(s-ib, p+a)$$

4.2. Scaling Property

$$FL\ \ f(at,bx)\ \ (s,p) = \frac{1}{ab} F(\frac{s}{a},\frac{p}{b})$$

Proof: $FL\ \ f(at,bx)\ \ (s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f(at,bx)dtdx$

Put $at = z$ and $bx = y$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(s\frac{z}{a}-ip\frac{y}{b})} f(z,y)\frac{dz}{a}\frac{dy}{b}$$

$$= \frac{1}{ab}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(rz-imy)} f(z,y)dzdy \qquad \because \frac{s}{a} = r\,,\quad \frac{p}{b} = m$$

$$= \frac{1}{ab} F(r,m) = \frac{1}{ab} F(\frac{s}{a},\frac{p}{b})$$

4.3. Differential Property

4.3.1. $FL\ \ f_x(t,x)\ \ (s,p) = pFL\ \ f(t,x)\ \ -k$

Proof: $FL\ \ f_x(t,x)\ \ (s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f_x(t,x)dtdx$

$$= \int_{-\infty}^{\infty} e^{-ist}dt \int_{0}^{\infty} e^{-px} f_x(t,x)dtdx$$

$$= \int_{-\infty}^{\infty} e^{-ist}dt\left[\ \ e^{-px} f(t,x)\ \ _{0}^{\infty} - \int_{0}^{\infty}(-p)e^{-px} f(t,x)dx\right]$$

$$= \int_{-\infty}^{\infty} e^{-ist}dt\left[-f(t,0) + p\int_{0}^{\infty} e^{-px} f(t,x)dx\right]$$

$$= -\int_{-\infty}^{\infty} e^{-ist} f(t,0)dt + p\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-ist} e^{-px} f(t,x)dtdx$$

$$= pFL\;\; f(t,x)\;\; - \int_{-\infty}^{\infty} e^{-ist} f(t,0)dt$$

$$= pFL\;\; f(t,x)\;\; -k\;\; , \qquad \text{where } k = \int_{-\infty}^{\infty} e^{-ist} f(t,0)dt$$

$$\therefore FL\;\; f_x(t,x)\;\; = pFL\;\; f(t,x)\;\; -k \qquad (4.3.1)$$

4.3.2. $FL\;\; f_{xx}(t,x)\;\; = p^2 FL\;\; f(t,x)\;\; - pk$

Proof: $FL\;\; f_{xx}(t,x)\;\; = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f_{xx}(t,x)dtdx$

$$= \int_{-\infty}^{\infty} e^{-ist}dt \int_{0}^{\infty} e^{-px} f_{xx}(t,x)dx$$

$$= \int_{-\infty}^{\infty} e^{-ist}dt \left[-f_x(t,0) + p\int_{0}^{\infty} e^{-px} f_x(t,x)dx \right]$$

$$= -\int_{-\infty}^{\infty} e^{-ist} f_x(t,0)dt + p\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f_x(t,x)dtdx$$

$$= pFL\;\; f_x(t,x)\;\; - \int_{-\infty}^{\infty} e^{-ist} f_x(t,0)dt = pFL\;\; f_x(t,x)\;\; -0$$

Where, $\int_{-\infty}^{\infty} e^{-ist} f_x(t,0)dt = 0$ by DUIS (Differentiation under integral sign) and it is zero for infinite integral or it is ignore.

$$= p\;\; pFL\;\; f(t,x)\;\; -k\;\; = p^2 FL\;\; f(t,x)\;\; - pk \text{ by (4.3.1)}$$

$$\therefore FL\;\; f_{xx}(t,x)\;\; = p^2 FL\;\; f(t,x)\;\; - pk \qquad (4.3.2)$$

$$FL\;\; f_{xxx}(t,x)\;\; = p^3 FL\;\; f(t,x)\;\; - p^2 k \qquad (4.3.3)$$

$$FL\;\; f_x^{(n)}(t,x)\;\; = p^n FL\;\; f(t,x)\;\; - p^{n-1} k \qquad (4.3.4)$$

4.4. Second Shifting Property

If $FL\;\; f(t,x)\;\; (s,p)$ is generalized Fourier-Laplace transform of $f(t,x)$, then

$FL\;\; f(t-a,x)\;\; (s,p) = e^{-isa}F(s,p)$

Proof: $FL\;\; f(t-a,x)\;\; (s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(st-ipx)} f(t-a,x)dtdx$

Let $t-a=z$, $\qquad = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(s(a+z)-ipx)} f(z,x)dzdx$

$$= e^{-isa}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(sz-ipx)} f(z,x)dzdx$$

$$= e^{-isa} FL\;\; f(t,x)\;\; (s,p) = e^{-isa}F(s,p)$$

4.5. Multiplication by $e^{iax}.e^{ibt}$

$$FL\{e^{iax}.e^{ibt}f(t,x)\}(s,p)=F(s-b,p-ia)$$

Proof: $FL\{e^{iax}.e^{ibt}f(t,x)\}(s,p)=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{iax}e^{ibt}e^{-i(st-ipx)}f(t,x)dtdx$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(st-bt)}e^{-i(-ipx-ax)}f(t,x)dtdx$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(s-b)t}.e^{-i(-ip-a)x}f(t,x)dtdx$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i[(s-b)t+(-ip-a)x]}f(t,x)dtdx$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i[(s-b)t-i(p-ia)x]}f(t,x)dtdx$$

Let $s-b=z$ and $p-ia=r$ then

$$FL\{e^{iax}.e^{ibt}f(t,x)\}(s,p)=\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-i(zt-irx)}f(t,x)dtdx$$

$$=FL\{f(t,x)\}(z,r)=F(z,r)=F(s-b,p-ia)$$

4.6. Generalized Fourier-Laplace Transform

Sr.No.	$f(t,x)$	$FL\{f(t,x)\}(s,p)$
1.	$f(at,bx)$	$\frac{1}{ab}F(\frac{s}{a},\frac{p}{b})$
2.	$f(t,x)\cos at\cos bx$	$\frac{1}{4}\left\{\begin{matrix}F(s-a,p-ib)+F(s-a,p+ib)+F(s+a,p-ib)+\\F(s+a,p+ib)\end{matrix}\right\}$
3.	$f(t,x)\sin at.\sin bx$	$-\frac{1}{4}\left\{\begin{matrix}F(s-a,p-ib)-F(s-a,p+ib)-F(s+a,p-ib)+\\F(s+a,p+ib)\end{matrix}\right\}$
4.	$C_1f(t,x)+C_2g(t,x)$	$C_1FL\{f(t,x)\}(s,p)+C_2FL\{g(t,x)\}(s,p)$
5.	$e^{-ax-bt}f(t,x)$	$F(s-ib,p+a)$
6.	$f_x(t,x)$	$pFL\{f(t,x)\}-k$
7.	$f_{xx}(t,x)$	$p^2FL\{f(t,x)\}-pk$
8.	$f_x^{(n)}(t,x)$	$p^nFL\{f(t,x)\}-p^{n-1}k$
9.	$f(t-a,x)$	$e^{-isa}F(s,p)$
10.	$e^{iax}.e^{ibt}f(t,x)$	$F(s-b,p-ia)$

5. CONCLUSION

This paper presents the Generalization of Fourier-Laplace transform in the distributional sense. Some testing function spaces are given. Also some properties of Fourier-Laplace transform are proved, which will be useful when this transform will be used to solve differential and integral equations.

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